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A New Strategy to the Multi-Objective Control of Linear Systems

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***Abstract*— A new approach to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of linear systems is presented. The proposed methodology is based on a new stabilization condition leading the control parametrization to be independent of the Lyapunov function. This nice property allows the use of multiple Lyapunov matrices to the multi-objective control design in a numerical and tractable way. A numerical example illustrates the approach as a tool for $\mathcal{H}_2/\mathcal{H}_\infty$ control synthesis.**

I. INTRODUCTION

Nowadays, there are elegant solutions to several single objective control specifications formulated in terms of convex optimization problems such as the $\mathcal{H}_2$ (or LQG) and $\mathcal{H}_\infty$ control. Generally speaking, the $\mathcal{H}_2$ approach is applied to meet performance specifications and/or impulsive disturbance rejection while guaranteeing closed-loop stability. On the other hand, the $\mathcal{H}_\infty$ framework has the ability to improve the closed-loop robustness against system uncertainties and deterministic disturbance signals [1]. However, in many practical applications it is common to appear conflicting design specifications such as simultaneous rejection of disturbances with different characteristics (white noise, bounded energy, persistent); good tracking of classes of inputs or satisfaction of bounds on the energy of the control or output signals, cannot always be satisfied by a single norm form. So, a mixed performance specification is naturally considered in such cases [2].

Certainly, the linear matrix inequality (LMI) framework is one of the most powerful formulation to the stability analysis and control design for a wide variety of linear control problems [3]. Such formulation offers a numerically tractable manner to deal with problems where there is no analytical solution. Therefore, a control problem recast in terms of LMIs can be efficiently solved by powerful numerical tools. Recently, LMI solutions to the $\mathcal{H}_2/\mathcal{H}_\infty$ multi-objective have been proposed such as the references [4], [5]. At least in the continuous-time case, the LMI solutions for $\mathcal{H}_2/\mathcal{H}_\infty$ control consider a common Lyapunov function to compute a bound on both norms at the cost of conservativeness [6], [7]. More precisely, the individual $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems are solved using a control parametrization in terms of the Lyapunov matrix. Therefore, a convex solution to the multi-objective control problem is only obtained if a common Lyapunov matrix is considered. One can cite several examples of iterative procedures for multi-objective control to consider multiple Lyapunov matrices such as [8], [9].

Aiming at finding a robust solution in terms of parameter-dependent Lyapunov functions, several researchers have proposed alternative LMI characterizations to certain multi-objective control problems leading to a controller parametrization that does not involve the Lyapunov matrix. See for instance the references [10]–[12] for the continuous-time case and [13], [14] to the discrete-time counterpart. However, in the continuous-time case, there are some shortcomings because either, the quadratic stability case is not recovered [10], or there is an extra parameter to be determined through a gridding (non-convex) approach [11].

Using an alternative control parametrization, not involving the Lyapunov matrix, convex characterizations for both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems are proposed in the present paper. It will be shown that whenever there exists a solution to the standard case (quadratic stability condition) one can find a feasible solution for the improved $\mathcal{H}_2$ and $\mathcal{H}_\infty$ conditions. Then, a solution to the multi-objective control problem is proposed in terms of LMIs considering multiple Lyapunov matrices.

The rest of this paper is as follows. Section II establishes the problem to be addressed, Section III proposes a new stabilization condition for linear systems, and the extension for multi-objective control design is given in Section IV. A numerical example, in Section V, illustrates the approach and Section VI ends the paper.

The notation used in this paper is standard. $\mathbb{R}^n$ denotes the set of n-dimensional real vectors, $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices, I_n is the $n\times n$ identity matrix, $0_{n\times m}$ is the $n\times m$ matrix of zeros and 0_n is the $n\times n$ matrix of zeros. For a real matrix S, S' denotes its transpose and $S>0$ means that S is symmetric and positive-definite. For a symmetric block matrix, the symbol $\star$ stands for the symmetric block outside the main diagonal. Matrix and vector dimensions are omitted whenever they can be inferred from the context.

II. PROBLEM STATEMENT

Consider the following linear time-invariant (LTI) system:

$$\begin{cases} \dot{x}(t) &= Ax(t)+Bu(t)+B_1w_1(t)+B_2w_2(t), \\ z_1(t) &= C_1x(t)+D_1u(t), \\ z_2(t) &= C_2x(t)+D_2u(t)+Ew_2(t), \\ u(t) &= Kx(t),\ x(0)=x_0, \end{cases} \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}^{n_u}$ the control signal, $w_1\in\mathbb{R}^{n_1}$ are impulsive disturbances, $w_2\in\mathbb{R}^{n_2}$ are

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disturbances with bounded energy, $z_1 \in \mathbb{R}^{m_1}$ the output channel for $\mathcal{H}_2$ performance, $z_2 \in \mathbb{R}^{m_2}$ the output channel for $\mathcal{H}_\infty$ performance, $K \in \mathbb{R}^{n_u \times n}$ the control gain, and $A, B, B_1, \ldots, E$ are constant matrices with appropriate dimensions.

Typically, the closed-loop performance is characterized in terms of input-to-output experiments by means of the $\mathcal{H}_2$- and (or) $\mathcal{H}_\infty$-norms. For random or impulsive disturbance signals, the $\mathcal{H}_2$-norm is more appropriated to quantify the system performance. On the other hand, the $\mathcal{H}_\infty$-norm is adequate to enforce robustness and to deal with square integrable signals [5].

Let $\mathcal{T}_i$ denote the closed-loop transfer functions of system (1) from w_i to z_i, $i = 1, 2$. For mixed performance specifications, each objective is formulated in terms of appropriate input-to-output (I/O) channels. In this paper, the transfer function $\mathcal{T}_1$ represents the I/O channel for the $\mathcal{H}_2$ performance and $\mathcal{T}_2$ to the $\mathcal{H}_\infty$ case. For completeness, the following definitions of system norms are given.

Definition 1: The $\mathcal{H}_2$-norm of system (1) is given by

$$\|\mathcal{T}_1\|_2^2 \triangleq \sum_{i=1}^{m_1} \|\bar{z}_{1i}(t)\|_2^2 \tag{2}$$

where $\bar{z}_{1i}(t)$ is the system response to a unitary impulse in the $\mathcal{H}_2$ input channels with $x(0) = 0$ and $w_2 \equiv 0$. ■

If the disturbance signal $w_1(t)$ is a white noise with zero mean value and unitary power density spectra, the $\mathcal{H}_2$-norm may be interpreted as follows

$$\|\mathcal{T}_1\|_2^2 \triangleq \mathcal{E}(z_1(t)' z_1(t))$$

where $\mathcal{E}(z_1(t)' z_1(t))$ denotes the mathematical expectation of the random variable $z_1(t)' z_1(t)$.

For zero initial conditions, the greatest energy gain that can be obtained from the disturbance signal $w_2(t) \in \mathcal{L}_2$ to the output $z_2(t)$ corresponds to the $\mathcal{H}_\infty$-norm of system (1) leading to the following definition.

Definition 2: The $\mathcal{H}_\infty$-norm of system (1) is given by

$$\|\mathcal{T}_2\|_\infty \triangleq \sup_{0 \neq w_2 \in \mathcal{L}_2} \frac{\|z_2(t)\|_2}{\|w_2(t)\|_2} \tag{3}$$

where $x(0) = 0$, $w_1 \equiv 0$, and $\mathcal{L}_2$ denotes the space of square integrable vector functions on $[0, \infty)$. ■

The computation of both norms is a standard LMI result [3]. The next two lemmas characterize LMI solutions to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems.

Lemma 1: ($\mathcal{H}_2$ control design) Consider system (1) and the $\mathcal{H}_2$ I/O channel. There exist matrices (P_1, Y, N) solving the following optimization problem:

$$\min \ \text{trace}(N) :$$
$$\begin{bmatrix} N & (C_1 P_1 + D_1 Y) \\ \star & P_1 \end{bmatrix} > 0 \tag{4}$$
$$\begin{bmatrix} (AP_1 + P_1 A' + BY + Y'B') & B_1 \\ B_1' & -I_{n_1} \end{bmatrix} < 0 \tag{5}$$

if and only if the system (1), in closed loop with $u = Y P_1^{-1} x$, is asymptotically stable and $\|\mathcal{T}_1\|_2^2 < \text{trace}(N)$. ■

Lemma 2: ($\mathcal{H}_\infty$ control design) Consider the $\mathcal{H}_\infty$ I/O channel of system (1) and let γ be a given positive scalar. There exist matrices (P_2, Y, N) solving the following LMI problem:

$$P_2 > 0, \tag{6}$$
$$\begin{bmatrix} (AP_2 + P_2 A' + BY + Y'B') & B_2 & \star \\ B_2' & -\gamma I_{n_2} & E' \\ (C_2 P + D_2 Y) & E & -\gamma I_{m_2} \end{bmatrix} < 0$$

if and only if the system (1), in closed loop with $u = Y P_2^{-1} x$, is asymptotically stable and$\|\mathcal{T}_2\|_\infty < \gamma$. ■

The classical multi-objective control approach, e.g.

$$\min \ \|\mathcal{T}_1\|_2 : \|\mathcal{T}_2\|_\infty < \gamma, \tag{7}$$

constraints the Lyapunov functions in Lemmas 1 and 2 to be common obtaining a convex (and potentially conservative) solution [7]. That is $P = P_1 = P_2$, where the control-gain is parameterized as $K = YP^{-1}$.

The purpose of this paper is to devise alternative conditions under which the controller parametrization does not involve the Lyapunov matrix. If this is done we can solve, for instance, (7) without the constraint $P_1 = P_2$ leading to less conservative results.

III. Improved Stabilization Conditions

Let us start with the problem of deriving a new stabilization condition where the control parametrization does not involve the Lyapunov matrix. To this end, consider the following LTI system

$$\dot{x} = Ax + Bu, \ u = Kx. \tag{8}$$

The stability condition for the closed loop system may be expressed by means of the following matrix inequality:

$$AP + PA' + BKP + PK'B' < 0, \tag{9}$$

where $P > 0$ and K are the decision variables. The above condition is convex if the control-gain is parameterized by $K = YP^{-1}$. As discussed in Section II, this parametrization is conservative for multi-objective control design.

Aiming at reducing the conservativeness, consider the following parametrization

$$K = rFS^{-1}, \tag{10}$$

where $r \in \mathbb{R}$, $F \in \mathbb{R}^{n_u \times n}$ and $S \in \mathbb{R}^{n \times n}$ (supposed to be nonsingular) are the variables to be determined such that the closed-loop system is asymptotically stable.

Taking into account (9) and (10), one get the following condition:

$$x'(AP + PA' + BFrS^{-1}P + P(S')^{-1}rF'B')x < 0,$$

for all $x \in \mathbb{R}^n$. Introducing the auxiliary variable y such that $rS^{-1}Px = x - y$, the above inequality yields

$$\begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} AP + PA' + BF + F'B' & \star \\ -F'B' & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} < 0,$$
$$\forall \begin{bmatrix} x \\ y \end{bmatrix} : \begin{bmatrix} (S - rP) & -S \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0. \quad (11)$$

From the Finsler's Lemma [15], one get the following condition

$$\begin{bmatrix} AP + PA' + BF + F'B' & \star \\ S - rP - F'B' & -S - S' \end{bmatrix} < 0, \quad (12)$$

where r, F, S, P are the decision variables.

Notice that the proposed parametrization as defined in (10) does not involve the Lyapunov matrix, and hence it appears more suitable for multi-objective control design.

The following theorem summarizes the above result.

Theorem 1: (A new stability condition) Consider system (8). There exist matrices $P > 0, F, S$ and the scalar r satisfying (12) if and only if there exist matrices $P > 0, K$ satisfying (9). In the affirmative case, the solution of (12) is such that $P > 0$ and $K = rFS^{-1}$ also satisfy (9). ■

Proof. Suppose that there exist matrices $P > 0, F, S$ and the scalar r satisfying (12) and rewrite it as

$$\begin{bmatrix} \begin{pmatrix} AP + PA' + \\ BF + F'B' \end{pmatrix} & \star \\ -F'B' & 0 \end{bmatrix} + Q_1'Q_2 + Q_2'Q_1 < 0, \quad (13)$$

where

$$Q_1 = \begin{bmatrix} 0 & I \end{bmatrix}, \; Q_2 = \begin{bmatrix} (S - rP) & -S \end{bmatrix}.$$

Define a matrix Q_3 such that $Q_2Q_3' = 0$, e.g.

$$Q_3 = \begin{bmatrix} I & S^{-1}(S - rP) \end{bmatrix}.$$

Observe that (12) implies S is nonsingular. Multiplying (13) by Q_3 to the left and Q_3' to the right, we see that (9) is satisfied with

$$K = rFS^{-1}.$$

Suppose now there exist matrices $P > 0, K$ satisfying (9). We show in the sequel that (12) is satisfied for $F = KP$, $S = rP$ and r large enough. With the Schur complement and the above choices of F, S, the inequality (12) is equivalent to:

$$AP + PA' + BKP + PK'B' + BKPK'B'(2r)^{-1} < 0,$$

which is guaranteed to be satisfied for r large enough. □

Observe that (12) is not convex with respect to the scalar r. In the sequel we present a quasi-convex procedure to determine this scalar.

A. The r parameter

From the above proof we see that for $S = rP$ and r large enough the inequality (12) is satisfied whenever the usual quadratic stabilizability condition $AP + PA' + BF + F'B' < 0$ is satisfied. The miminum value of r leading (12) to be satisfied with $S = rP$ can be easily computed by solving the quasi-convex problem

$$\min r \;:\; \begin{bmatrix} AP + PA' + BF + F'B' & \star \\ -F'B' & -2rP \end{bmatrix} < 0$$

Once this minimum value of r, namely $r = r^*$ is computed we can increase r from r^* in (12) while removing the constraint $S = rP$. The interest of this procedure is twofold. Firstly, this problem can be recast as another quasi-convex problem with the change of variable $r = r^* + \epsilon^{-1}$ leading to: miminize ϵ such that

$$\epsilon \begin{bmatrix} AP + PA' + BF + F'B' & \star \\ S - r^*P - F'B' & -S - S' \end{bmatrix} +$$
$$- \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} < 0$$

Secondly, this procedure will never be more conservative than the usual quadratic stabilizability condition $AP + PA' + BF + F'B' < 0$ because in the limit case, where ϵ tends to zero, the scalar r tends to infinity and thus the choice $S = (r^* + \epsilon^{-1})P = rP$ renders the condition (12) equivalent to

$$AP + PA' + BF + F'B' < 0.$$

At the two extremes, namely $r = r^*$ and $r \to \infty$, the constraint $S = rP$ leads the proposed controller parametrization to be the usual $K = FP^{-1}$. However, in between these extremes situations the controller parametrization is $K = FS^{-1}r$ which does not involve the Lyapunov matrix. In summary, the problem of increasing the scalar r from r^* in (12) has nice properties under the constraint $S = Pr$. Unfortunately, it is not possible to show that these same nice properties hold when we let S to be a free decision variable. However, numerical results reported in the next section indicate that a similar behavior could be expected. These ideas are exploited in the next section to solve the mixed $\mathcal{H}_2$/$\mathcal{H}_\infty$ control problem via state feedback.

IV. Multi-Objective Control Design

In the following, the new stabilization condition is extended to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design. Then, the main result of this paper ends this section.

A. $\mathcal{H}_\infty$ Control

From Lemma 2 it follows that the system (1), in closed loop with $u = Kx$, is asymptotically stable and $\|\mathcal{T}_2\|_\infty < \gamma$, if and only if there exist matrices P_2, K such that:

$$\xi_2' \Psi_2 \xi_2 < 0, \; \forall \; \xi_2 \neq 0,$$

where $\xi_2 = \begin{bmatrix} x' & z_2' & w_2' \end{bmatrix}'$ and Ψ_2 is as follows

$$\Psi_2 = \begin{bmatrix} \begin{pmatrix} (A+BK)P_2 \\ +P_2(A+BK)' \end{pmatrix} & B_2 & \star \\ B_2' & -\gamma I_{n_2} & E' \\ (C_2+D_2K)P_2 & E & -\gamma I_{m_2} \end{bmatrix} \quad (14)$$

Considering the controller parametrization $K = rFS^{-1}$, as in the previous section, and introducing the auxiliary variable $g_2 = x - S^{-1}Prx$, we get the identity

$$\xi_2'\Psi_2\xi_2 = \phi_2'\Gamma_2\phi_2,$$

where $\phi_2 = \begin{bmatrix} x' & z_2' & w_2' & g_2' \end{bmatrix}'$ and Γ_2 is given by

$$\Gamma_2 = \begin{bmatrix} \Pi_2 & B_2 & \star & \star \\ B_2' & -\gamma I_{n_2} & E' & 0 \\ C_2P_2 + D_2F & E & -\gamma I_{m_2} & \star \\ -F'B' & 0 & -F'D_2' & 0 \end{bmatrix}, \quad (15)$$

with

$$\Pi_2 = AP_2 + P_2A' + BF + F'B'. \quad (16)$$

Observing that the auxiliary variable g_2 is such that

$$\begin{bmatrix} S - rP_2 & 0 & 0 & -S \end{bmatrix}\phi_2 = 0,$$

and using the Finsler's lemma, as in the previous section, it follows that

$$\phi_2'\Gamma_2\phi_2 < 0, \ \forall\ \phi_2 \neq 0 : \begin{bmatrix} S - rP_2 & 0 & 0 & -S \end{bmatrix}\phi_2 = 0,$$

if the inequality $\Omega_2 < 0$ is satisfied, where

$$\Omega_2 = \begin{bmatrix} \Pi_2 & B_2 & \star & \star \\ B_2' & -\gamma I & E' & 0 \\ C_2P_2 + D_2F & E & -\gamma I & -D_2F \\ S - F'B' - rP_2 & 0 & \star & -S - S' \end{bmatrix} \quad (17)$$

The above results may be summarized as follows.

Theorem 2: (A new $\mathcal{H}_\infty$ condition) There exist matrices $P_2 > 0, F, S$ and the scalar r satisfying $\Omega_2 < 0$ if and only if the system (1), in closed loop with $u = rFS^{-1}x$, is asymptotically stable and $\|\mathcal{T}_2\|_\infty < \gamma$.

■

The proof is similar to the proof of Theorem 1 and will be omitted.

B. $\mathcal{H}_2$ Control

From the lemma 1 it follows that the system (1), in closed loop with $u = Kx$, is asymptotically stable and $\|\mathcal{T}_1\|_2^2 < \mathrm{trace}(N)$ if and only if there exist matrices (P_1, K, N) solving the following optimization problem.

$$\min\ \mathrm{trace}(N):$$
$$\Psi_{1a} = \begin{bmatrix} N & C_1P_1 + D_1KP_1 \\ \star & P_1 \end{bmatrix} > 0$$
$$\Psi_{1b} = \begin{bmatrix} AP_1 + P_1A' + BKP_1 + P_1K'B' & B_1 \\ B_1' & -I_{n_1} \end{bmatrix} < 0$$

Considering the same controller parametrization used in the $\mathcal{H}_\infty$ control, i.e.

$$K = rFS^{-1},$$

and introducing the auxiliary variable

$$g_1 = x - S^{-1}Prx,$$

we get the identity

$$\xi_1'\Psi_{1b}\xi_1 = \phi_1'\Gamma_{1b}\phi_1,$$

where $\xi_1 = [\ x' \quad w_1'\ ]'$, $\phi_1 = [\ x' \quad w_1' \quad g_1'\ ]'$, and

$$\Gamma_{1b} = \begin{bmatrix} \Pi_1 & B_1 & \star \\ B_1' & -I_{n_1} & \star \\ -F'B' & 0 & 0 \end{bmatrix}, \quad (18)$$

with

$$\Pi_1 = AP_1 + P_1A' + BF + F'B'. \quad (19)$$

Observing that the auxiliary variable g_1 is such that

$$\begin{bmatrix} S - rP_1 & 0 & -S \end{bmatrix}\phi_1 = 0$$

and using the Finsler's lemma, it follows that

$$\phi_1'\Gamma_{1b}\phi_1 < 0, \ \forall\ \phi_1 \neq 0 : \begin{bmatrix} S - rP_1 & 0 & -S \end{bmatrix}\phi_1 = 0,$$

if the inequality $\Omega_{1b} < 0$ is satisfied, where

$$\Omega_{1b} = \begin{bmatrix} \Pi_1 & B_1 & \star \\ B_1' & -I_{n_1} & \star \\ -F'B' + S - rP_1 & 0 & -S - S' \end{bmatrix}. \quad (20)$$

On the other hand, introducing the auxiliary variables

$$\xi_3 = \begin{bmatrix} z' & x' \end{bmatrix}', \ \phi_3 = \begin{bmatrix} z' & x' & g_3' \end{bmatrix}',$$

and $g_3 = x - S^{-1}Prx$, and following the same above steps, we get

$$\phi_3'\Psi_{1a}\phi_3 > 0, \ \forall\ \phi_3 \neq 0,$$

if the inequality $\Omega_{1a} > 0$ is satisfied, where

$$\Omega_{1a} = \begin{bmatrix} N & C_1P + D_1F & -D_1F \\ \star & P & * \\ \star & rP - S & S + S' \end{bmatrix}. \quad (21)$$

To sum up, we introduce the following result.

Theorem 3: (A new $\mathcal{H}_2$ condition) There exist matrices $N, P_1 > 0, F, S$ and the scalar r satisfying $\Omega_{1b} < 0$, $\Omega_{1a} > 0$ if and only if the system (1), in closed loop with $u = rFS^{-1}x$, is asymptotically stable and $\|\mathcal{T}_1\|_2^2 < \mathrm{trace}(N)$.

■

The proof is similar to the proof of Theorem 1 and will be omitted for brevity.

C. Main Result

By lumping together the results in the previous subsections we can state the main result of the paper, namely, a sufficient condition for designing a state feedback controller such that the system (1), in closed loop with this control law, minimizes an upper bound of $\|\mathcal{T}_1\|_2$ subject to $\|\mathcal{T}_2\|_\infty < \gamma$ for a given γ.

Theorem 4: (A new $\mathcal{H}_2/\mathcal{H}_\infty$ condition) For a given scalar $\gamma > 0$, if there exist matrices $N, P_1, P2 > 0, F, S$ and a scalar r solving the optimization problem below:

$$\min\ \mathrm{trace}(N) : \Omega_{1b} < 0, \Omega_{1a} > 0, \Omega_2 < 0.$$

Then, system (1), in closed loop with $u = rFS^{-1}x$, is asymptotically stable, $\|\mathcal{T}_1\|_2^2 < \mathrm{trace}(N)$ and $\|\mathcal{T}_2\|_\infty < \gamma$. ■

The above optimization problem must be solved iteratively on r. Based on the result proposed in Section III-A, the idea is to find the minimum r, namely r^*, solving the problem under the following constraint

$$S = rP_1 = rP_2, \tag{22}$$

This is a quasi-convex problem having a solution whenever the usual $\mathcal{H}_2/\mathcal{H}_\infty$ control has a solution. Next, solve the problem iteratively on r by removing the constraint (22) and increasing r from r^*.

For some $r > r^*$, the above procedure will never lead to a more conservative result if compared to the usual $\mathcal{H}_2/\mathcal{H}_\infty$ control.

V. Illustrative Example

To demonstrate the approach, consider a 4-th order LTI system as described in (1) with the following matrices [16]:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix};$$

$$\begin{matrix} B_2 = B_1; \\ D_2 = D_1; \end{matrix}\ D_1 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}; E = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

where $k = 0.245$ and $f = 0.0219$.

To check the performance of the proposed methodology, the results obtained from Theorem 4, considering the interval $r^* \le r < \infty$, are compared to the standard $\mathcal{H}_2/\mathcal{H}_\infty$ multi-objective control design (Lemmas 1 and 2 with a single Lyapunov function).

Figure 1 shows a curve of the system $\mathcal{H}_2$-norm, i.e. $\|\mathcal{T}_1\|_2^2 = \mathrm{trace}(N)$ as a function of the scalar r, where the r parameter is varying from $r^* = 3.3$ to $r \to \infty$ for a given $\gamma = 0.7$. The value of r^* was obtained by taking (22), and the procedure specified at the end of Section IV.

For comparison purposes, the standard approach has led to $\|\mathcal{T}_1\|_2^2 < 35$ (the straight line in Figure 1) and Theorem 4 has obtained a less conservative estimate, i.e. $\|\mathcal{T}_1\|_2^2 < 30$, for an optimal $r_{opt} = 10.3$.

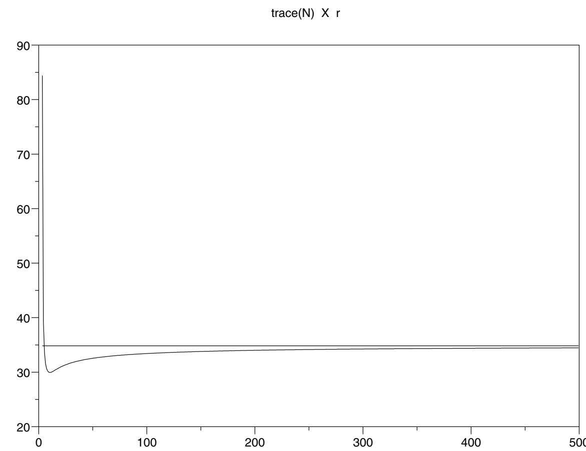

Fig. 1. $\|\mathcal{T}_1\|_2^2$ estimates for $r^* \le r < \infty$.

VI. Conclusion and Future Work

A. Concluding Remarks

This paper has introduced a new approach to the $\mathcal{H}_2/\mathcal{H}_\infty$ multi-objective control design of linear systems in terms of LMIs considering multiple Lyapunov functions.

The key point of the proposed result is that the controller parametrization that does not involve the Lyapunov matrix. This interesting feature allows for the use of multiple Lyapunov functions in contrast with the (potentially conservative) standard technique to the multi-objective control design that considers a single Lyapunov function.

The control design conditions depend on tuning a parameter r through an iterative technique. Even though it is non-convex, some interesting properties of this problem are shown and a simple strategy to solve it is also proposed in the paper. In particular, it is shown that, under certain conditions, it is possible to transform the non-convex conditions over r into a pair of quasi-convex problems that can be solved by standard LMI packages.

B. Future Work

A point that remains open is to study the geometry of $\mathrm{trace}(N)$ as a function of r. If this function has always the same nice properties as exhibited in Figure 1, the line search required for tuning the scalar r may be considerably simplified.

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