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Control Over Wireless Communication Channel for Continuous-Time Systems

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Abstract— **This paper is concerned with the control of one dimensional continuous time linear Gaussian systems over additive white noise wireless fading channels subject to capacity constraints. Necessary and sufficient conditions are derived, for bounded asymptotic and asymptotic observability and stabilizability in the mean square sense, for controlling such systems. For the case of a noiseless time-invariant system controlled over a continuous time additive white Gaussian channel, the sufficient condition for stabilizability and observability states that the capacity of the channel, C_a, satisfies, $C_a > [A]^+$, where A is the system coefficient and $[a]^+ = a$, if $a \geq 0$ and $[a]^+ = 0$, if $a < 0$. Moreover, the necessary condition states that the channel capacity must satisfy $C_a \geq [A]^+$. It is shown that a separation principle holds between the design of the communication and the control sub-systems, implying that the controller that would be optimal in the absence of the communication channel is also optimal for the problem of the controlling the system over the communication channel.**

I. Introduction

In recent years, there has been a significant activity in addressing issues associated with the control of systems over limited capacity communication channels. A typical example is given in Fig. 1. In such control/communication systems, the controlled system output is analogous to the source that generates information, which has to be transmitted over a communication channel, with feedback, for reliable communication and control. Typical examples are applications in which a single dynamical system sends information to a distant controller via a communication link with finite capacity.

Previous work on this subject focuses on the stabilizability of discrete time systems, controlled over a discrete time communication channel with finite capacity. Fundamental results for stabilizability of such systems are derived in [1]-[8].

The objective of this paper is to extend the above line of research to continuous time systems driven by Brownian motion. In particular, when the information is communicated to the controller via a finite capacity communication channel. First, we consider the problem of stabilizability of a control/communication system, which consists of a linear continuous time-invariant noiseless plant, which is controlled over a continuous time AWGN channel with memory. By using the Bode integral formula [9], a necessary condition for the existence of a stabilizing controller is given by the condition

$$C_{ah} \geq [A]^+, \tag{1}$$

where C_{ah} is the capacity of AWGN channel with memory. In the special case of AWGN channel (e.g., channel impulse response $h(t) = \delta(t)$), condition (1) is reduced to the following condition

$$C_a \geq [A]^+, \tag{2}$$

where C_a is the AWGN channel capacity. We then consider a time-varying one dimensional linear stochastic Gaussian plant driven by Brownian motion, which is controlled over a flat fading wireless channel. Here, we assume complete knowledge of the channel throughout the transmission, at the transmitter and the receiver ends [10]. Under such assumption, we derive optimal encoding and decoding strategies which minimize the mean square decoding error, and achieve the channel capacity. We further show that under certain conditions, the proposed encoding and decoding strategies yield bounded asymptotic, and asymptotic observability, in the mean square sense. Furthermore, a sufficient condition for bounded asymptotic and asymptotic stabilizability in the mean square sense is derived. For the case of noiseless, time-invariant systems controlled over continuous time AWGN channel, the sufficient condition for asymptotic observability and stabilizability is given by condition $C_a > [A]^+$.

In this paper, it is also shown that a separation principle holds between the design of the control and the communication sub-systems.

The paper is organized as follows. In Section II, the problem formulation is given. In Section III, a necessary condition for stabilizability of control/communication system consisting a linear continuous time-invariant noiseless plant controlled over a continuous time AWGN channel with memory is presented. In Section IV, the optimal encoding/decoding scheme that ensures bounded asymptotic and asymptotic observability in the mean square sense is presented. In Section V, a stabilizing controller is designed by using a linear quadratic pay-off.

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II. PROBLEM FORMULATION

Consider the block diagram of Fig. 1. As in any typical communication system, the source which corresponds to the controlled plant output is communicated via a flat fading wireless AWGN channel. The encoder in addition of observing the plant output and the state of the channel, also observes the control signals. This kind of encoder has been discussed in [1], [2]. Thus, the encoder indirectly observes the output of the decoder. That is, the information pattern of the encoder, F, at time t is $\{(x(s),\tilde{x}(s),z(s,\theta(s)));0\leq s\leq t\}\rightarrow F(t,x,\tilde{x},\theta)(=F(t)$ in compact notation), where $\{z(s,\theta(s));0\leq s\leq t\}$ is the Channel State Information (CSI) (which dependents on the random process $\{\theta(s),\ 0\leq s\leq t\}$). Moreover, the decoder has access to the output of the channel, and to the state of the channel. That is, the information pattern of the decoder $\tilde{x}$ at time t is $\{(y(s),z(s,\theta(s)));0\leq s\leq t\}\rightarrow\tilde{x}(t,y,\theta)(=\tilde{x}(t)$ in compact notation). Finally, the controller u at time t has access to the output of the decoder at that time and the state of the channel up to time t.

Next, we give the precise definition of the signals, and blocks associated with Fig. 1. Let $(\Omega,\mathcal{F},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and time $t\in[0,T]$, $T>0$. Let $x\triangleq\{x(s);0\leq s\leq t\leq T\}$, $x(t)\in\Re$ denote the output of the controlled plant (transmitted signal), $u\triangleq\{u(s);0\leq s\leq t\leq T\}$, $u(t)\in\Re$ the control signal, $y\triangleq\{y(s);0\leq s\leq t\leq T\}$, $y(t)\in\Re$ the output of the communication channel, $z\triangleq\{z(s,\theta(s));0\leq s\leq t\leq T\}$, $z(t,\theta(t))\in\Re$ the channel state information, $v\triangleq\{v(s);0\leq s\leq t\leq T\}$, $v(t)\in\Re$ the channel noise, and $w\triangleq\{w(s);0\leq s\leq t\leq T\}$, $w(t)\in\Re$ the plant process noise. Here, it is assumed that z depends on the random process $\theta(t)\in\Re^q$. The plant noise w, and the channel noise v are independent standard Brownain motions, which are independent of the initial state $x(0)$. Let $\{\mathcal{F}^x_{0,t}\}_{t\geq 0}$, $\{\mathcal{F}^{\tilde{x}}_{0,t}\}_{t\geq 0}$, $\{\mathcal{F}^y_{0,t}\}_{t\geq 0}$, $\{\mathcal{F}^z_{0,t}\}_{t\geq 0}$, and $\{\mathcal{F}^\theta_{0,t}\}_{t\geq 0}$ denote the complete filtration generated by $\mathcal{F}^x_{0,t}\triangleq\sigma\{x(s);0\leq s\leq t\}$, $\mathcal{F}^{\tilde{x}}_{0,t}\triangleq\sigma\{\tilde{x}(s);0\leq s\leq t\}$, $\mathcal{F}^y_{0,t}\triangleq\sigma\{y(s);0\leq s\leq t\}$, $\mathcal{F}^z_{0,t}\triangleq\sigma\{z(s);0\leq s\leq t\}$, and $\mathcal{F}^\theta_{0,t}\triangleq\sigma\{\theta(s);0\leq s\leq t\}$, respectively, which are sub-sigma fields of $\{\mathcal{F}_t\}_{t\geq 0}$. Here, $\mathcal{F}^x_{0,t}$, $\mathcal{F}^{\tilde{x}}_{0,t}$, $\mathcal{F}^y_{0,t}$, $\mathcal{F}^z_{0,t}$, and $\mathcal{F}^\theta_{0,t}$ are the Borel σ-algebras on the space of continuous functions $C([0,T];\Re),C([0,T];\Re),C([0,T];\Re)$, $C([0,T];\Re)$, and $C([0,T];\Re^q)$ respectively. Next, the blocks of Fig. 1 are defined.

Plant. The state of the plant is described by an one dimensional continuous time, controlled diffusion process given by the *Itô* equation

$$dx(t)=A(t)x(t)dt+B(t)u(t)dt+G(t)dw(t),\ x(0),\quad (3)$$

where $A:[0,T]\rightarrow\Re$, $B:[0,T]\rightarrow\Re$, and $G:[0,T]\rightarrow\Re$ are Borel measurable and bounded, and $x(0)$ is Gaussian random variable $x(0)\sim N(\bar{x}_0,\bar{V}_0)$, which is independent of w. The control u is square integrable, and $\{\mathcal{F}_{0,t}\}_{t\geq 0}$ adapted.

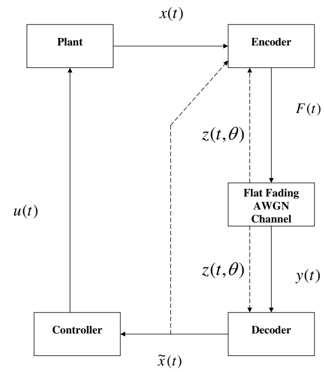

Fig. 1. Control/communication system over flat fading AWGN channel

Throughout, we assume there exists a unique solution of (3), such that $x\in\mathcal{X}$, where $\mathcal{X}\triangleq\Big\{x\in C([0,T];\Re);x$ is $\{\mathcal{F}_{0,t}\}_{t\geq 0}$ adapted and $E\int_0^T|x(t)|^2dt<\infty\Big\}$ (see [11]).

Encoder. The encoder, F, is a non-anticipative functional of the state of the plant x, the decoder output $\tilde{x}$, and the channel state information z. Define $\mathcal{F}^{x,\tilde{x},\theta}_{0,t}\triangleq\mathcal{F}^x_{0,t}\bigvee\mathcal{F}^{\tilde{x}}_{0,t}\bigvee\mathcal{F}^\theta_{0,t}$. The set of admissible encoders is defined by $\mathcal{F}_{ad}\triangleq\Big\{F:[0,T]\times C([0,T];\Re)\times C([0,T];\Re)\times C([0,T];\Re^q)\rightarrow\Re;F$ is $\{\mathcal{F}^{x,\tilde{x},\theta}_{0,t}\}_{t\geq 0}$ adapted, and $E\int_0^T|F(t,x,\tilde{x},\theta)|^2dt<\infty\Big\}$.

Channel. The communication channel is an AWGN, flat fading, wireless channel z. The channel output y is defined by the following stochastic differential equation

$$\begin{aligned}&dy(t)=z(t,\theta(t))F(t,x,\tilde{x},\theta)dt+dv(t),\\&y(0)=0,\ 0\leq t\leq T,\end{aligned}\quad (4)$$

where v is the Gaussian standard Brownian motion. Throughout, we shall assume that (4) has a unique solution [11], and for a fixed sample path θ

$$\text{Prob}\Big\{\int_0^T z^2(t,\theta(t))|F(t,x,\tilde{x},\theta)|^2dt<\infty\Big\}=1.$$

and

$$E_\theta\int_0^T z^2(t,\theta(t))|F(t,x,\tilde{x},\theta)|^2dt<\infty,\quad (5)$$

where $E_\theta[.]$ denote expectation with respect to sample path θ.

Decoder. The decoder denoted by $\tilde{x}$ is adapted to $\{\mathcal{F}^{y,\theta}_{0,t}\}_{t\geq 0}$, where $\mathcal{F}^{y,\theta}_{0,t}\triangleq\mathcal{F}^y_{0,t}\bigvee\mathcal{F}^\theta_{0,t}$. The set of admissible decoders is denoted by $\mathcal{D}_{ad}$. The decoder plays the role of a state estimator.

Controller. The controller u is a square integrable non-anticipative functional of the output of the decoder and the channel state information, e.g., u is $\{\mathcal{F}^{y,\theta}_{0,t}\}_{t\geq 0}$ adapted. The set of admissible controller is denoted by $\mathcal{U}_{ad}$.

The objective of this paper is to find necessary and sufficient conditions for bounded asymptotic and asymptotic

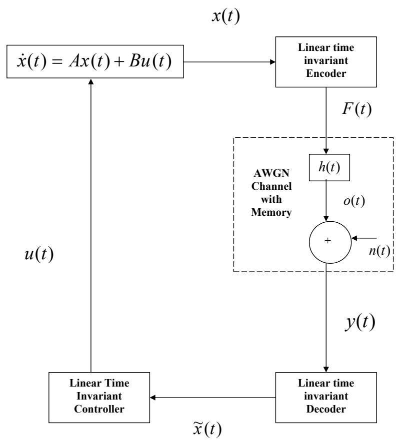

Fig. 2. Control/communication system over AWGN

observability and stabilizability of system (3), in the mean square sense, over a continuous time (flat fading) AWGN communication channel, defined as follows.

Definition 2.1: (Bounded Asymptotic and Asymptotic Observability in the Mean Square Sense). Define $\mathcal{E}(t) \triangleq E[(x(t) - \tilde{x}(t, y, \theta))^2 \Big| \mathcal{F}_{0,t}^{y,\theta}]$. System (3), (4) is bounded asymptotically (resp. asymptotically) observable, in the mean square sense, if there exists an encoder and decoder such that $\lim_{t\to\infty} \mathcal{E}(t) < \infty$, P-a.s. (resp. $\lim_{t\to\infty} \mathcal{E}(t) = 0$, P-a.s.).

Definition 2.2: (Bounded Asymptotic and Asymptotic Stabilizability in the Mean Square Sense). System (3), (4) is bounded asymptotically (resp. asymptotically) stabilizable, in the mean square sense, if there exits a controller, encoder and decoder, such that $\lim_{t\to\infty} E[|x(t)|^2 \Big| \mathcal{F}_{0,t}^{\theta}] < \infty$, P-a.s.,(resp. $\lim_{t\to\infty} E[|x(t)|^2 \Big| \mathcal{F}_{0,t}^{\theta}] = 0$, P-a.s).

III. Necessary Condition for Existence of Stabilizing Controller

In this Section, we consider the time-invariant noiseless analogue of system (3), that is the plant is given by

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{6}$$

The communication channel is a continuous time AWGN channel ($z = 1$) with memory, that is the output of the channel is given by

$$y(t) = o(t) + n(t), \quad o(t) = h(t) * F(t), \; y(t) \in \Re, \tag{7}$$

where o is a stochastic process with power spectral density $S_o(\omega)$, n is a Gaussian white noise process with the power spectral density $S_n(\omega) = N_0$, (o, n) are independent, $h(t)$ is the channel impulse response with the corresponding transfer function $H(jw)$, and " $*$ " is the convolution operator. Here the encoder, decoder and the controller are linear time-invariant with transfer functions $E(jw), D(jw)$ and $C(jw)$, respectively (see Fig. 2). Let C_{ah} denote the capacity of AWGN channel with memory. We shall show that $C_{ah} \geq [A]^+$ is a necessary condition for the existence of a stabilizing controller.

In this Section, a controller is called stabilizable if the corresponding closed loop sensitivity transfer function $S(jw) = \frac{Y(jw)}{N(jw)}$, from n to y, is strictly stable or alternatively $\lim_{t\to\infty} E|y(t)|^2 < \infty$ or $\lim_{t\to\infty} E|x(t)|^2 < \infty$.

Next, applying the Bode integral formula [9], the main result of this Section is given in the following theorem. The Proof and the extension to the vector case can be found in [12] or [13].

Theorem 3.1: Consider the control/communication system (6), (7) given in Fig. 2. A necessary condition for the existence of a stabilizing controller is given by

$$C_{ah} \geq [A]^+, \tag{8}$$

where C_{ah} is the capacity of the AWGN channel with memory.

Remark 3.2: In the case of AWGN channel (e.g., $h(t) = \delta(t)$), the necessary condition (8) is reduced to the following condition

$$C_a \geq [A]^+, \tag{9}$$

where C_a is the AWGN channel capacity.

Please note that the analogue of condition (9) for controlling discrete time systems over discrete time noisy channel (including discrete time AWGN channel) is given by [1]

$$C \geq [\log |A|]^+. \tag{10}$$

In Section V, we will recover (9) by constructing an encoder, decoder and controller which stabilize the plant.

IV. Optimal Encoding/Decoding Scheme for Observability

In this Section, we design an optimal encoder/decoder pair for the time-varying system defined by (3), (4) that guarantees the observability condition defined in the sense of Definition 2.1. The necessary and sufficient condition for the existence of such an encoder/decoder pair is given in terms of the capacity of the channel and the time-varying coefficient $A(t)$. The extension of the results of this Section to the vector case can be found in [12] or [13].

A. Optimal Encoding/Decoding Scheme for Observability

In this Section, we first define the mutual information between the state of the plant x and the channel output y, when the channel state information z is known to the transmitter and the receiver. Next, we introduce a power constraint on the admissible encoders, and we compute the flat fading continuous time AWGN channel capacity using a variant of the methodology found in [11]. Even in our case, it can be shown that the capacity is attained by a white noise [11]. Further, we design an encoder which minimizes the mean square decoding error, and achieves the channel capacity. In particular, the mean square error is bounded if $G(t)$ is nonzero, and tends to zero asymptotically if $G(t)$ is zero. This can be explained by the fact that if $G(t)$ is nonzero, the state equation is driven by Brownian motion which has unbounded variation. Thus, when $G(t)$ is zero, the mean square error is asymptotically zero, implying that the computed channel capacity represents the operational capacity. Finally, we state the necessary and sufficient conditions for bounded asymptotic and asymptotic observability in the mean square sense.

The part of the results which are concerned with construction of optimal encoding/decoding pair for the scalar case, first appeared in preliminary form in [14].

1) Conditional Mutual Information and Capacity of Feedback Systems: Using a variant of the mutual information expression of signals described by stochastic differential equations driven by Brownian motion found in [15], we derive the expression for the conditional mutual information given in the next theorem.

Theorem 4.1: [14]. Consider the model given by (3), (4), shown in Fig. 1. The mutual information between the state of plant x, and the channel output y, conditional on the state z is given by the following equivalent expressions

$$i)\ I_T(x;y|\theta) \triangleq\ E_{x,y,\theta}\Big[\log\frac{dP_{x,y|\theta}(x,y|\theta)}{dP_{x|\theta}(x|\theta)\times dP_{y|\theta}(y|\theta)}\Big]$$

$$ii)\ I_T(x;y|\theta) = \frac{1}{2}E_\theta\int_0^T z^2(t,\theta(t))E\Big[|F(t,x,\tilde{x},\theta)|^2 - |\hat{F}(t,\tilde{x},\theta)|^2\Big|\theta\Big]dt$$

$$iii)\ I_T(x;y|\theta) = \int_\theta I_T(x,y|\theta)dP_\theta(\theta) \qquad (11)$$

where $E_{x,y,\theta}[.]$ represents expectation with respect to the sample paths x, y, θ, $E_\theta[.]$ denote expectation with respect to the sample path θ, and $\hat{F}(t,\tilde{x},\theta) = E[F(t,x,\tilde{x},\theta)\Big|\mathcal{F}_{0,t}^{y,\theta}]$.

Next, the definition of the channel capacity for a Gaussian flat fading channel, when the CSI is fully known is given. Thereafter, an upper bound on the mutual information is introduced, and subsequently it is shown that this upper bound is the channel capacity (i.e., the upper bound is attained by a certain transmitted signal).

Definition 4.2: Consider the model given by (3), (4). The channel capacity (often called information capacity), when the fading process z or θ is completely known to the transmitter, and receiver, subject to the instantaneous power constraint

$$E\Big[|F(t,x,\tilde{x},\theta)|^2\Big|\theta\Big] \leq P \qquad (12)$$

is defined by

$$C_f \triangleq \lim_{T\to\infty}\sup_{(x,F)\in\mathcal{X}\times\mathcal{F}_{ad}}\frac{1}{T}I_T(x;y|\theta). \qquad (13)$$

Here, the supremum is taken over all state processes $x\in\mathcal{X}$ which give strong solutions to (3) and over all encoding functions $F\in\mathcal{F}_{ad}$ that satisfy (12) (see [10], [11], [16]).

Lemma 4.3: [14]. Consider the model given by (3), (4). Then,

$$\frac{1}{T}I_T(x;y|\theta) \leq \frac{1}{2T}P\int_0^T E_\theta[z^2(t,\theta(t))]dt. \quad (14)$$

Moreover, the channel capacity is given by

$$C_f = \lim_{T\to\infty}\frac{P}{2T}\int_0^T E_\theta[z^2(t,\theta(t))]dt. \qquad (15)$$

Following the same methodology as in [11], it is shown that the above upper bound determines the channel capacity. Namely, the signal x that reaches the channel capacity is a white Gaussian noise.

2) Optimal Encoding and Decoding: In this Section, we design an encoding/decoding strategy that achieves the channel capacity C_f. The proposed encoding/decoding strategy is optimal in the sense that among all admissible encoding/decoding schemes that satisfy condition (12), it minimizes the mean square decoding error and at the same time achieves the channel capacity. We then employ the expression for the minimum mean square decoding error to obtain necessary and sufficient conditions for bounded asymptotic and asymptotic observability. In the subsequent development, only linear encoders are considered, because along the same lines of [11] it can be shown that linear encoders achieve the channel capacity and the minimum mean square decoding error.

Definition 4.4: The set of linear admissible encoders $\mathcal{L}_{ad}$, where $\mathcal{L}_{ad}\subset\mathcal{F}_{ad}$, is the set of linear non-anticipative functionals F with respect to $(x,\tilde{x},\theta)$, which have the following form

$$F(t,x,\tilde{x},\theta) = F_0(t,\tilde{x},\theta) + F_1(t,\tilde{x},\theta)x(t), \qquad (16)$$

in which for a fixed sample path of θ

$$\text{Prob}\Big\{\int_0^T|F_0(t,\tilde{x},\theta)|^2dt<\infty\Big\} = 1$$
$$\text{and}\quad \sup_{\{\xi\in C([0,T];\Re),0\leq t\leq T\}}|F_1(t,\xi,\theta)|<\infty.$$

Using linear encoders, the received signal y is given by

$$dy(t) = z(t,\theta(t))[F_0(t,\tilde{x},\theta)+F_1(t,\tilde{x},\theta)x(t)]dt + dv(t),\quad y(0)=0. \qquad (17)$$

Decoding. Because the decoded signal $\tilde{x}$ is a function of the received signal y, and the channel z, the optimal decoder minimizing the mean square decoding error is the conditional expectation given by

$$\tilde{x}_{opt}(t,y,\theta) = \hat{x}(t,y,\theta) = E[x(t)\Big|\mathcal{F}_{0,t}^{y,\theta}]. \quad (18)$$

The conditional error variance for the decoder defined by (18) is

$$V(t,y,\theta) = E[(x(t)-\hat{x}(t,y,\theta))^2\Big|\mathcal{F}_{0,t}^{y,\theta}]. \qquad (19)$$

Moreover, the decoder $\hat{x}(t,y,\theta)$ and the corresponding conditional error variance $V(t,y,\theta)$ satisfy the following Generalized Kalman Filtering equations.

$$\begin{aligned}d\hat{x}(t,y,\theta) &= A(t)\hat{x}(t,y,\theta)dt + B(t)u(t)dt\\ &+z(t,\theta(t))V(t,y,\theta)F_1(t,\tilde{x},\theta)\\ &.[dy(t)-z(t,\theta(t))(F_0(t,\tilde{x},\theta)+F_1(t,\tilde{x},\theta)\hat{x}(t,y,\theta))\\ &.dt],\end{aligned} \qquad (20)$$

$$\dot{V}(t,y,\theta) = 2A(t)V(t,y,\theta) - z^2(t,\theta(t))F_1^2(t,\tilde{x},\theta)V^2(t,y,\theta) + G^2(t), \quad (21)$$

with initial conditions $\hat{x}(0) = \bar{x}_0$, and $V(0) = \bar{V}_0$.

Encoding. From the point of view of the coding theorem, an encoder is optimal if it operates near the channel capacity, while ensuring a decoding error that tends to zero exponentially fast, as the codeword length tends to infinity. In our case, the choice of an optimal encoder (F_0, F_1) is directly related to the expression for the conditional error variance (21). By choosing (F_0, F_1) appropriately, the conditional error variance is minimized, and the channel capacity C_f is achieved.

Theorem 4.5: (Coding Theorem)[14]. Suppose the received signal is defined by (4), and the source by (3). Then the encoder, which achieves the channel capacity, the optimal decoder, and the corresponding error variance, are respectively, given by

$$F^*(t,x,\hat{x}^*,\theta) = \sqrt{\frac{P}{V^*(t,y,\theta)}}(x(t) - \hat{x}^*(t,y,\theta)) \quad (22)$$

$$d\hat{x}^*(t,y,\theta) = A(t)\hat{x}^*(t,y,\theta)dt + B(t)u(t)dt + z(t,\theta(t))\sqrt{PV^*(t,y,\theta)}dy(t), \quad (23)$$

$$V^*(t,y,\theta) = V^*(0)\exp\{2\int_0^t A(s)ds - \int_0^t z^2(s,\theta(s))Pds\} + \int_0^t G^2(s)\exp\{2\int_s^t A(u)du - \int_s^t z^2(u,\theta(u))Pdu\}ds, \quad (24)$$

where $\hat{x}^*(0) = \bar{x}_0$, and $V^*(0) = \bar{V}_0$.
From (24), it follows that by employing the proposed optimal encoding/decoding scheme the mean square estimation error $V^*(t,y,\theta)$ is independent of the control signal. Hence, under certain conditions, the decoding error can be made arbitrary small, regardless of control signals. This suggests that the encoder and decoder, can be design independent of the controller, or in other words, a separation principle holds between the control, and the communication part of the design.
Next, in the following theorem, we present a sufficient condition for bounded asymptotic observability and asymptotic observability, as a direct result of Theorem 4.5.

Theorem 4.6: i) When $G(t) \neq 0$, a sufficient condition for bounded asymptotic observability in the mean square sense is

$$\frac{P}{2}z^2(t,\theta(t)) > [A(t)]^+, \quad a.e. - t \geq 0, \quad P - a.s. \quad (25)$$

Moreover, a necessary condition for bounded asymptotic observability is given by

$$\frac{P}{2}z^2(t,\theta(t)) \geq [A(t)]^+ \quad a.e - t \geq 0, \ P - a.s. \quad (26)$$

ii) When $G(t) = 0$, (25) is a sufficient condition for asymptotic observability in the mean square sense, while, when $\bar{V}_0 \neq 0$, condition (26) is a necessary condition for asymptotic observability in the mean square sense.

Remark 4.7: i) When $G(t) = 0$ and the channel is the continuous time AWGN channel ($z = 1$), for which the channel capacity is $C_a = \frac{P}{2}$, it is easily shown that another sufficient condition for asymptotic observability is

$$C_a = \frac{P}{2} > \limsup_{t\to\infty} \frac{1}{t}\int_0^t A(s)ds. \quad (27)$$

Moreover, a necessary condition for asymptotic observability is

$$C_a \geq \limsup_{t\to\infty} \frac{1}{t}\int_0^t A(s)ds. \quad (28)$$

ii) In the case of continuous time AWGN channel ($z = 1$), conditions (25) and (26) are reduced to the following conditions respectively.

$$C_a > [A(t)]^+, \quad a.e. - t \geq 0, \quad (29)$$
$$C_a \geq [A(t)]^+, \quad a.e. - t \geq 0. \quad (30)$$

while, for time-invariant plants, (29) and (30) are reduced to $C_a > [A]^+$ and $C_a \geq [A]^+$, respectively.

V. Optimal Controller, Sufficient Condition for Stabilizability

In this Section, we propose an output feedback controller that minimizes a quadratic pay-off. The extension of the results of this Section to the vector case can be found in [12] or [13].
For a fixed sample path $\{z(s,\theta(s)); 0 \leq s \leq T\}$, the output feedback controller is chosen to minimize the quadratic pay-off

$$J = \frac{1}{T}E\{\int_0^T [|x(t)|^2Q(t) + |u(t)|^2R(t)]dt\}, \quad (31)$$

in the limit as $T \to \infty$, while at the same time it stabilizes the control/communication system given in Fig. 1. Here, the scalar $Q(t) > 0$ and $R(t) > 0$, $\forall t \in [0,T)$. We assume that the noiseless analogue of the system (3) (e.g., when $w(t) = 0$) is completely controllable (controllable for the time-invariant analogue) or exponentially stable. According to the classical separation theorem of estimation and control, the optimal controller that minimizes (31) subject to a flat fading AWGN communication channel and the linear encoder (16) is separated into a state estimator and a certainty equivalent controller given by

$$u^*(t) = -\bar{K}(t)\hat{x}(t,y,\theta), \quad \bar{K}(t) = R^{-1}(t)B(t)\bar{P}(t), \quad (32)$$

where $\hat{x}(t,y,\theta)$ is the solution of (20) with the corresponding observer Ricatti equation (21) and $\bar{P}(t)$ is the steady-state solution of the following regulator Riccati equation

$$-\dot{P}(t) = Q(t) - P^2(t)B^2(t)R^{-1}(t) + 2A(t)P(t), \quad \lim_{T\to\infty} P(T) = 0. \quad (33)$$

For a fixed sample path of the channel, it follows that if the observer and regulator Ricatti equations have steady state solution $\bar{V}(t)$ and $\bar{P}(t)$, respectively, the averaged criterion

$$\bar{J} = \lim_{T\to\infty} \frac{1}{T} E\{\int_0^T [|x(t)|^2 Q(t) + |u^*(t)|^2 R(t)]dt\} \quad (34)$$

can be expressed in the alternative form

$$\bar{J} \;=\; \lim_{T\to\infty} \frac{1}{T}\{\int_0^T [\bar{P}(t)G^2(t) + \bar{V}(t) .\bar{K}^2(t)R(t)]dt\}, \quad (35)$$

where for the time-invariant case, (e.g., $G(t) = G, Q(t) = Q, R(t) = R$), (35) is reduced to

$$\bar{J} \;=\; \lim_{T\to\infty} \frac{1}{T} E\{\int_0^T [|x(t)|^2 Q + |u^*(t)|^2 R]dt\} \;=\; \bar{P}G^2 + \bar{V}\bar{K}^2 R. \quad (36)$$

From (24), it follows that the observer Riccati equation (21) has a steady-state solution $\bar{V}$ as $t \to \infty$, if the optimal encoding/decoding scheme proposed in Theorem 4.5 is used and condition (25) holds. Moreover, the regulator Riccati equation (33) has the steady state solution $\bar{P}(t)$ as $t \to \infty$, if the noiseless analogue of system (3) is completely controllable or exponentially stabilizable.

Next, for a time-invariant analogue of system (3), we have the following proposition for stabilizability defined in the sense of Definition 2.2.

Proposition 5.1: [12]. Consider the time-invariant analogue of system (3). Assume that the time invariant noiseless analogue of the system (3) is controllable or exponentially stabilizable, and that the sample path of the channel z is completely known.

Then, for a fixed sample path of the channel, we have the followings.

i) Assuming $G \neq 0$ and $V(t, y, \theta) \to \bar{V}$ as $t \to \infty$, by using the optimal policy (32), $E|x(t)|^2 < \infty$ and $E|u(t)|^2 < \infty$, as $t \to \infty$.

ii) Assuming $G = 0$ and $V(t, y, \theta) \to 0$ as $t \to \infty$, by using the optimal policy (32), $E|x(t)|^2 = 0$ and $E|u(t)|^2 = 0$, as $t \to \infty$.

Next, using Theorem 4.6 and Proposition 5.1 the following theorem for bounded asymptotic and asymptotic stabilizability in the mean square sense is derived.

Theorem 5.2: Consider the time-invariant analogue of the system (3). Assume the time-invariant noiseless analogue of system (3) is controllable or exponentially stabilizable. Then

i) For the case $G \neq 0$, a sufficient condition for bounded asymptotic stabilizability in the mean square sense is given by

$$\frac{P}{2} z^2(t, \theta(t)) > [A]^+, \; a.e. - t \geq 0, \; P - a.s. \quad (37)$$

ii) For the case $G = 0$, (37) is also a sufficient condition for asymptotic stabilizability in the mean square sense.

Remark 5.3: In the case of continuous time AWGN channel ($z = 1$), condition (37) is reduced to

$$C_a > [A]^+, \quad (38)$$

while from [3], it follows that for stabilizability of a discrete time analogue system over a discrete time AWGN channel, the sufficient condition is given by

$$C_a > [\log |A|]^+. \quad (39)$$

Remark 5.4: As it was shown in this Section, a separation principle holds between the design of the communication and the control subsystems. The efficient encoding/decoding scheme that minimizes the mean square estimation error and achieves the channel capacity is given in Section IV, while the optimal certainty equivalent controller that optimizes a quadratic cost functional is given in (32). Although we design optimal encoder/decoder pair and controller separately, the whole system is optimal since the separation principle holds. The communication system also sends information at capacity.

REFERENCES

[1] S. Tatikonda and S. Mitter, Control over Noisy Channels, *IEEE Transaction on Automatic Control*, vol. 49, No. 7, pp. 1196-1201, July 2004.

[2] S. Tatikonda and S. Mitter, Control under Communication Constraints, *IEEE Transaction on Automatic Control*, vol. 49, No. 7, pp. 1056-1068, July 2004.

[3] S. Tatikonda, A. Sahai and S. Mitter, Stochastic Linear Control over a Communication Channel, *IEEE Transaction on Automatic Control*, vol. 49, No. 9, pp. 1549-1561, September 2004.

[4] Nicola Elia, When Bode Meets Shannon: Control-Oriented Feedback Communication Schemes, *IEEE Transaction on Automatic Control*, vol. 49, No. 9, pp. 1477-1488, September 2004.

[5] Nicola Elia,"Control-Oriented Feedback Communication Schemes", in *Proceedings of the 42nd IEEE Conference on Decision and Control* , Hawaii USA, pp. 3161-3166, December 2003.

[6] G. N. Nair and R. J. Evans, "Mean Square Stabilisability of Stochastic Linear Systems with Data Rate Constraints", in *Proceedings of the 41st IEEE Conference on Decesion and Control*, Las Vegas, USA, pp. 1632-1637, December 2002.

[7] G. N. Nair, R. J. Evans, I. M. Y. Mareels and W. Moran, Topological Feedback Entropy and Nonlinear Stabilization, *IEEE Transation on Automatic Control*, vol. 49, No. 9, pp. 1585-1597, September 2004.

[8] G. N. Nair, S. Dey and R. J. Evans, "Infimum Data Rates for Stabilising Markov Jump Linear Systems", in *Proceedings of the 42nd IEEE Conference on Decision and Control*, Hawaii USA, pp. 1176-1181, December 2003.

[9] B. F. Wu, and E. A. Jonckheere, A Simplified Approach to Bode's Theorem for Continuous-Time and Discrete-Time Systems, *IEEE Transactions on Automatic Control*, vol. 37, No. 11, pp. 1797-1802, November, 1992.

[10] E. Biglieri, J. Proakis, and S. Shamai (Shitz), Fading Channels: Information-Theoretic and Communications Aspects, *IEEE Transactions on Information Theory*, vol. 44, no. 6, pp. 2619-2692 October, 1998.

[11] R. S. Liptser and A. N. Shiryayev, *Statistics of Random Processes-Applications II*, Springer-Verlang, new york, 1978.

[12] C. D. Charalambous and Alireza Farhadi, Control of Continuous-Time Systems over Continuous-Time Wireless Channels, 2005, (*Preprint*).

[13] C. D. Charalambous, Stojan Denic, and Alireza Farhadi, Control of Continous-Time Linear Gaussian Systems over Additive Gaussian Wireless Fading Channels: A Seperation Principle, *Submitted to the IEEE Transactions on Automatic Control*, July 2005.

[14] C. D. Charalambous and Stojan Denic, "On the Channel Capacity of Wireless Fading Channels", in *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, Nevada USA, December 2002.

[15] T. Duncan, On the Calculation of Mutual Information, *SIAM J. Appl. Math*, vol. 19, pp. 215-220, July 1970.

[16] S. Ihara, Coding Theorems for Continuous-Time Gaussian Channels with Feedback, *IEEE Transactions on Information Theory*, vol. 40, No. 6, pp. 2041-2045, November 1994.