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Connecting nonlinear incremental Lyapunov stability with the linearizations Lyapunov stability

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Abstract— **In this paper, we reveal new connections between the incremental Lyapunov properties of a nonlinear system and the Lyapunov properties of its linearizations. We focus on (incremental) asymptotic and exponential stability. In contrast with other works on the incremental Lyapunov properties of nonlinear systems, our approach is based on extended spaces, Gâteaux derivative and the mean value theorem in norm.**

I. Introduction

Two different approaches, based on incremental stability, have recently emerged for the nonlinear system analysis. The first one, the *incremental Lyapunov stability approach* focuses on the analysis of the nonlinear system trajectories associated to a given set of initial conditions, that is, qualitative properties. The second one, the *(weighted) incremental norm approach*, focuses on the input-output properties of Lipschitz continuous nonlinear systems. As in the first approach, the second approach allows qualitative property analysis. In contrast with, it allows to analyze desensitivity, robustness and more generally many quantitative properties. Note that nonlinear control specifications include both qualitative and quantitative properties. It emphasizes the strong advantage of the weighted incremental norm approach, which encompasses in a single mathematical framework both kind of properties (see [18, 15] for illustrative examples).

In this paper, we first prove that the incremental Lyapunov asymptotic stability of a nonlinear system for a convex set $\mathcal{U}$ of initial conditions is implied by the Lyapunov asymptotic stability of all its linearizations associated to each trajectory with an initial condition in $\mathcal{U}$. A stronger result is then obtained when Lyapunov exponential stability is considered. The incremental Lyapunov exponential stability of a nonlinear system for a convex set $\mathcal{U}$ of initial conditions is proved to be equivalent to the Lyapunov exponential stability of all its linearizations associated to each trajectory with an initial condition in $\mathcal{U}$. We then prove that the incremental Lyapunov exponential stability on a convex set $\mathcal{U}$ is equivalent to the Lyapunov exponential stability on $\mathcal{U}$ at any initial condition in $\mathcal{U}$. In other words, the exponential stability of the trajectories associated to a convex set $\mathcal{U}$ of initial conditions is equivalent to the exponential stability of the equilibrium trajectories with respect to $\mathcal{U}$. This result is close to a recent one in [2] derived with a different proof.

Our paper is a contribution in the scope of the incremental Lyapunov stability approach. In this approach, many contributions focus on the analysis of trajectories associated to different systems or associated to the same system but with two different initial conditions. Two arguments are usually applied. The first one is based on "incremental" Lyapunov functions, see *e.g.* [36, 3, 30]. The second one focuses on the analysis of the (time dependent) distance between two trajectories (see [24] for a survey and also [23]). The obtained conditions involve the nonlinear system linearizations (see [29, 25, 21] and more recently the contraction analysis [26, 27]). Note that related problems were considered for the error analysis of numerical integration schemes (see [4, 5] and for related problems [7]).

In contrast with these results, our proof is based on the machinery of (weighted) incremental norm approach. The weighted incremental approach focuses on the properties of incrementally stable systems, *i.e.* Lipschitz continuous systems, defined as causal operators from a normed functional space to another one. This approach clearly roots in the input-output approach, more precisely in the Zames' and Sandberg's pioneering works. The (Lipschitz) continuity was early pointed out as a natural extension to the nonlinear systems of the linear bounded input bounded output stability (see [37] and also [35]). Nevertheless, in the input-output approach of nonlinear system analysis, most of the results focus on ensuring only the input-output boundedness *e.g.*, the $\mathcal{L}_2$ gain stability, see [6]. Few works investigated the properties of incrementally stable systems, see the book [35], chapter 7 on linearizations. Some results were obtained in the sixties for a restricted class of nonlinear systems (interconnections between an LTI system and a memoryless nonlinearity), see [32, 33]. Recently, the (weighted) incremental norm approach emerged as a fruitful extension of the linear H_∞ approach to nonlinear systems, see [9, 12] and [19]. The major interest is to propose a quantitative evaluation of the robustness and the performance of closed-loop nonlinear systems. This evaluation reduces to the computation of the weighted incremental norm of some closed-loop nonlinear functions [12]. In addition, in [10, 11, 8, 13], it was also proved that the (weighted) incremental norm approach ensures many interesting incremental Lyapunov properties as the uniqueness of the steady-state, the uniform Lyapunov stability of all the unperturbed trajectories and many other ones, see [2] in the ISS context.

Based on this approach, in [16, 14], we reveal the connection between the incremental norm of a nonlinear system with the induced norm of its linearizations with an emphasis on the input-output properties. In this paper, we complete the picture by focusing on the Lyapunov properties.

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Remember that connecting a nonlinear system properties to its linearization ones is the theoretical foundation of a nonlinear control approach which is a practically credible and rigorous alternative to the widespread (and heuristic) gain scheduling control, see [16] for a detailed discussion. As *e.g.* in [16], the results are here derived using the extended spaces, the Gâteaux derivatives (linearizations) and the mean value theorem in norm, that is, the arguments of the (weighted) incremental norm approach. In contrast with [16], we are interested in the nonlinear system response to initial conditions. In [16], we investigate the properties of the nonlinear operator which associates to the nonlinear system input signals, the nonlinear system output signals. Both signals belong to infinite dimensional Banach spaces. We here investigate the properties of the nonlinear operator which associates to the initial conditions, the nonlinear system state. In contrast with [16], the nonlinear operator input is a finite dimensional Banach space, which is simpler. The (weighted) incremental norm approach, applied in [16] can be then applied to our problem. To complete the picture, in the last part of the paper, we discuss the connections between the (weighted) incremental norm analysis and the contraction analysis which was developed in the incremental Lyapunov stability approach. We actually prove that uniform exponential incremental stability implies the existence of a contraction metric. Furthermore, the equivalence between the exponential stability of the trajectories associated to a convex set $\mathcal{U}$ of initial conditions and the exponential stability of the equilibrium trajectories with respect to $\mathcal{U}$ can be interpreted as an infinitesimal test ensuring the contraction.

Missing proofs are presented in the technical report [17].

Notations, considered systems $\mathcal{L}_\infty$ is the set of all essentially bounded, measurable $\mathbb{R}^n$-value functions on $[t_0,\infty)$ equipped with $\|f\|_{\mathcal{L}_\infty} \triangleq \text{esssup}_{t\in[t_0,\infty)} \|f(t)\|$. The *causal truncation* of $f \in \mathcal{L}_\infty$ at time $T \in [t_0,\infty)$, denoted by $P_T f$, is given by $P_T f(t) = f(t)$ for $t \le T$ and 0 otherwise. $\|P_T u\|_{\mathcal{L}_\infty}$ is denoted by $\|u\|_{\mathcal{L}_\infty,T}$. The *extended space* associated to $\mathcal{L}_\infty$, denoted $\mathcal{L}^e_\infty$, is composed with the functions whose causal truncations belong to $\mathcal{L}_\infty$. A real-value function $\varphi(r)$, defined from $\mathbb{R}^+$ into $\mathbb{R}^+$, belongs to class $\mathcal{K}$ if it is defined, continuous and strictly increasing and such that $\varphi(0) = 0$. A real-value function $\sigma(t)$, defined from $\mathbb{R}^+$ into $\mathbb{R}^+$, belongs to class $\mathcal{L}$ if it is defined, continuous and strictly decreasing and such that $\lim_{t\to\infty} \sigma(t) = 0$. A real-value function $\beta(t,r)$, defined form $\mathbb{R}^+ \times \mathbb{R}^+$ into $\mathbb{R}$, belongs to class $\mathcal{KL}$ if it is defined, continuous and if for each fixed t belongs to class $\mathcal{K}$ and each fixed r it is a monotone decreasing to zero as t increases. We consider nonlinear systems $y = \Sigma_{t_0}(\xi)$ defined from $\mathcal{W}$, an open (not empty) set of $\mathbb{R}^n$, into $\mathcal{L}^e_\infty$, associated to:

$$\Sigma_{t_0} \left\{ \begin{array}{rcl} \dot{x}(t) &=& f(t,x(t)) \\ y(t) &=& x(t) \\ x(t_0) &=& \xi \end{array} \right. \quad (1)$$

where $x(t) \in \mathbb{R}^n$. Σ_{t_0} is assumed well-defined from $\mathcal{W}$ into $\mathcal{L}^e_\infty$, that is, for any $T \in [t_0,\infty)$ and any $\xi \in \mathcal{W}$, the differential equation solution exists on $[t_0,T]$.

Assumption 1.1: f and $\dfrac{\partial f}{\partial x}$ are continuous functions of x uniformly for almost every $t \in [t_0,\infty)$ and are measurable functions on $[t_0,\infty)$ for every fixed value of $x \in \mathbb{R}^n$.

II. Linearizations, integral relation and the mean value theorem in norm

A. Gâteaux derivative on $\mathcal{L}^e_\infty$

Definition 2.1: [1, 34, 28] Let Σ_{t_0} be a causal operator, defined from $\mathcal{W}$ into $\mathcal{L}^e_\infty$ and let be $\xi \in W$. If, for any $T \in [t_0,\infty)$ and for any $\nu \in \mathbb{R}^n$, there exists a continuous linear operator[1] $D\Sigma_{t_0G}[\xi]$ from $\mathbb{R}^n$ into $\mathcal{L}^e_\infty$ such that

$$\lim_{\lambda\downarrow 0} \left\| \frac{\Sigma_{t_0}(\xi+\lambda\nu) - \Sigma_{t_0}(\xi)}{\lambda} - D\Sigma_{t_0G}[\xi](\nu) \right\|_{\mathcal{L}_\infty,T} = 0$$

then $D\Sigma_{t_0G}[\xi]$ is the *Gâteaux derivative* (the *linearization*) of Σ_{t_0} at ξ on $\mathcal{L}^e_\infty$.

For (1), Definition 2.1 reduces to the usual linearization one.

Lemma 2.2: Let be Σ_{t_0} given by (1) with assumption 1.1. Then, for any $\xi \in \mathcal{W}$, the system has a Gâteaux derivative that satisfies the following differential equations:

$$D\Sigma_{t_0G}[\xi](\nu) \left\{ \begin{array}{rcl} \dot{\bar{x}}(t) &=& A(t)\bar{x}(t) \\ \bar{y}(t) &=& \bar{x}(t) \\ \bar{x}(t_0) &=& \nu \end{array} \right. \quad (2)$$

with $A(t) = \dfrac{\partial f}{\partial x}(t,x(t))$ and $x(t)$ the solution of (1) with $x(t_0) = \xi$.

B. Local vs Global: exact relation

Let Σ_{t_0} be a dynamical system from $\mathcal{W}$ into $\mathcal{L}^e_\infty$. Let us associate to ξ_1 and ξ_2 in $\mathcal{U}$, a convex subset of $\mathcal{W}$,

$$\Sigma_{t_0}(\xi_2) - \Sigma_{t_0}(\xi_1) = \sum_{i=1}^{N} \Sigma_{t_0}(\xi_1 + i\delta_\xi) - \Sigma_{t_0}(\xi_1 + (i-1)\delta_\xi)$$

with $\delta_\xi = (\xi_2 - \xi_1)/N$ and where N is an integer. If Σ_{t_0} is Gâteaux differentiable on $\mathcal{W}$ then for a given $T \in [t_0,\infty)$ and a given $N \in \mathbb{N}$ large enough and for any $t \in [t_0,T]$,

$$\Sigma_{t_0}(\xi_2) - \Sigma_{t_0}(\xi_1) \approx \frac{1}{N} \sum_{i=0}^{N-1} D\Sigma_{t_0G}[\xi_1 + i\delta_\xi](\xi_2 - \xi_1) \quad (3)$$

since the Gâteaux derivative is a linear operator. The problem is to ensure that the approximation has a sense when N goes to infinity and that this sum converges to the expected integral, a Bochner integral[2] (since values are taken in a functional space (see *e.g.* [22][Chap III] and [28]).

Theorem 2.3: Let Σ_{t_0} be a causal operator, defined from $\mathcal{W}$ into $\mathcal{L}^e_\infty$ and let $\mathcal{U}$ be a convex subset of $\mathcal{W}$. If Σ_{t_0} has a Gâteaux derivative on $\mathcal{W}$ then for any ξ_1 and $\xi_2 \in \mathcal{U}$,

$$P_T\left(\Sigma_{t_0}(\xi_2) - \Sigma_{t_0}(\xi_1)\right) = (B)\int_0^1 P_T D\Sigma_{t_0G}[\xi_1 + \alpha(\xi_2 - \xi_1)](\xi_2 - \xi_1)d\alpha.$$

[1] For any fixed $T \in [t_0,\infty)$, there exists a finite $\gamma_T > 0$ such that for any $\nu \in \mathbb{R}$ $\|D\Sigma_{t_0G}[\xi](\nu)\|_{\mathcal{L}_\infty,T} \le \gamma_T\|\nu\|$.

[2] The Bochner integral of an abstract function (a vector value function) f_a defined from $[a,b] \subset \mathbb{R}$ to $\mathcal{L}_\infty$ is denoted by $(B)\int_a^b f(\alpha)d\alpha$.

The main step of this theorem proof is proving the integral existence, using the continuity of the integrated abstract function (see [16] for detail).

C. The mean value theorem in norm

From Theorem 2.3, we deduce that for any $T \in [t_0, \infty)$,

$$\|\Sigma_{t_0}(\xi_2) - \Sigma_{t_0}(\xi_1)\|_{\mathcal{L}_\infty,T} = \left\|(B)\int_0^1 D\Sigma_{t_0G}[\xi_1 + \alpha(\xi_2-\xi_1)](\xi_2-\xi_1)d\alpha\right\|_{\mathcal{L}_\infty,T}$$

and by the Bochner integral definition, that:

$$\|\Sigma_{t_0}(\xi_2) - \Sigma_{t_0}(\xi_1)\|_{\mathcal{L}_\infty,T} \leq \int_0^1 \|D\Sigma_{t_0G}[\xi_1 + \alpha(\xi_2-\xi_1)](\xi_2-\xi_1)\|_{\mathcal{L}_\infty,T}\, d\alpha.$$

This inequality actually corresponds to the "mean value in norm" condition since one has $\|\Sigma_{t_0}(\xi_2) - \Sigma_{t_0}(\xi_1)\|_{\mathcal{L}_\infty,T} \leq \eta_T\|\xi_2 - \xi_1\|$ where η_T is defined by:

$$\eta_T \triangleq \sup_{\alpha\in[0,1]} \frac{\|D\Sigma_{t_0G}[\xi_1 + \alpha(\xi_2-\xi_1)](\xi_2-\xi_1)\|_{\mathcal{L}_\infty,T}}{\|\xi_2-\xi_1\|} \quad (4)$$

that is, the greatest "$\mathcal{L}_\infty$ gain" of linearizations of Σ_{t_0} for a specific set of inputs. A deeper result can be in fact obtained.

Theorem 2.4 (Mean value Theorem in norm): Let Σ_{t_0} be defined by (1), with a Gâteaux derivative at each point ξ of $\mathcal{W}$. Let $\xi_1, \xi_2 \in \mathcal{W}$ such that $[\xi_1, \xi_2] \subset \mathcal{W}$, *i.e.* for any $\lambda \in [0,1]$ $\lambda\xi_1 + (1-\lambda)\xi_2 \in \mathcal{W}$. Then, for any $T \in [t_0, \infty)$, there exists $\eta_T > 0$ such that $\|\Sigma_{t_0}(\xi_1) - \Sigma_{t_0}(\xi_2)\|_{\mathcal{L}_\infty,T} \leq \eta\|\xi_1 - \xi_2\|$ if and only if for any $\xi \in [\xi_1, \xi_2]$, one has $\|D\Sigma_{t_0G}[\xi](\nu)\|_{\mathcal{L}_\infty,T} \leq \eta\|\nu\|$ for any $\nu \in [\xi_1, \xi_2]$.

For the operators defined on a convex subset of $\mathcal{W}$, Theorem 2.4 leads to Proposition 2.5, a central result for our derivations.

Proposition 2.5: Let Σ_{t_0} be defined by (1), with a Gâteaux derivative at each point ξ of $\mathcal{W}$. Let $\mathcal{U}$ be a convex and closed subset of $\mathcal{W}$. Then, for any $T \in [t_0, \infty)$, there exists $\eta_T > 0$ such that for any $\xi_1, \xi_2 \in \mathcal{U}$, $\|\Sigma_{t_0}(\xi_1) - \Sigma_{t_0}(\xi_2)\|_{\mathcal{L}_\infty,T} \leq \eta_T\|\xi_1 - \xi_2\|$ if and only if for any $\xi \in \mathcal{U}$ and any $\nu \in \mathbb{R}^n$, one has $\|D\Sigma_{t_0G}[\xi](\nu)\|_{\mathcal{L}_\infty,T} \leq \eta_T\|\nu\|$.

All the previous results are established on $\mathcal{L}_\infty^e$. It is important to investigate under which conditions, they can be extended on $\mathcal{L}_\infty$. The following theorem, a well-known result in the input-output approach [35, 6], is central.

Theorem 2.6: [35] Let Σ_{t_0} be a causal operator, defined from $\mathcal{W}$ into $\mathcal{L}_\infty^e$ and let η be a finite constant. Then for any $T \in [t_0, \infty)$ and any $\xi_1, \xi_2 \in \mathcal{U}$, one has $\|\Sigma_{t_0}(\xi_1) - \Sigma_{t_0}(\xi_2)\|_{\mathcal{L}_\infty,T} \leq \eta\|\xi_1 - \xi_2\|$ if and only if $\|\Sigma_{t_0}(\xi_1) - \Sigma_{t_0}(\xi_2)\|_{\mathcal{L}_\infty} \leq \eta\|\xi_1 - \xi_2\|$.

III. Incremental asymptotic stability and linearization asymptotic stability

Definition 3.1: Let $\mathcal{U}$ be a subset of $\mathcal{W}$. Σ_{t_0} is said to be incrementally asymptotically Lyapunov stable on $\mathcal{U}$ if there exists a class $\mathcal{KL}$ function β_{t_0} such that $\|\Sigma_{t_0}(\xi_1) - \Sigma_{t_0}(\xi_2)\| \leq \beta_{t_0}(t, \|\xi_1 - \xi_2\|)$ for any $t \geq t_0$, any ξ_1, ξ_2 in $\mathcal{U}$. Σ_{t_0} is said to be incrementally exponentially Lyapunov stable on $\mathcal{U}$ if there exists $a > 0$ and $b > 0$ such that $\|\Sigma_{t_0}(\xi_2)(t) - \Sigma_{t_0}(\xi_2)(t)\| \leq be^{-a(t-t_0)}\|\xi_2 - \xi_1\|$ for any $t \geq t_0$ and any $\xi_1, \xi_2 \in \mathcal{U}$.

Definition 3.2: The linearizations of Σ_{t_0} are said to be strongly asymptotically Lyapunov stable on $\mathcal{U}$ if there exist a class $\mathcal{L}$ function σ_{t_0} and $b > 0$ such that $\|D\Sigma_{t_0G}[\xi](\nu)(t)\| \leq b\|\nu\|\sigma_{t_0}(t - t_0)$ for any $t \geq t_0$, any $\xi \in \mathcal{U}$ and any $\nu \in \mathbb{R}^n$. Σ_{t_0} are said to be strongly exponentially stable on $\mathcal{U}$ if there exists $a > 0$ and $b > 0$ such that $\|D\Sigma_{t_0G}[\xi](\nu)(t)\| \leq be^{-a(t-t_0)}\|\nu\|$ for any $t \geq t_0$ and any $\xi \in \mathcal{U}$.

Proposition 3.3: Let Σ_{t_0} be defined by (1), with a Gâteaux derivative at each point ξ of $\mathcal{W}$. Let $\mathcal{U}$ be a convex and closed subset of $\mathcal{W}$. Then Σ_{t_0} is incrementally Lyapunov (resp. asymptotically Lyapunov) stable on $\mathcal{U}$ if all the linearizations of Σ_{t_0} are strongly Lyapunov (resp. asymptotically Lyapunov) stable on $\mathcal{U}$.

Proof: If the Σ_{t_0} linearizations are strongly asymptotically Lyapunov stable then there exists a class $\mathcal{L}$ function σ_{t_0} and $b > 0$ such that $\|D\Sigma_{t_0G}[\xi](\nu)(t)\| \leq b\|\nu\|\sigma_{t_0}(t-t_0)$ and thus $\|\sigma_{t_0}(t-t_0)^{-1}D\Sigma_{t_0G}[\xi](\nu)\|_\infty \leq b\|\nu\|$ for $\xi \in \mathcal{U}$ and any $\nu \in \mathbb{R}^n$. Let be $\widetilde{\Sigma}_{t_0}(\xi)(t) = \sigma_{t_0}^{-1}(t-t_0)\Sigma_{t_0}(\xi)(t)$. Since Σ_{t_0} is Gâteaux differentiable on $\mathcal{U}$, $\widetilde{\Sigma}_{t_0}$, as the composition of a Fréchet differential function with a Gâteaux one, is also Gâteaux differentiable on $\mathcal{U}$ with $D\widetilde{\Sigma}_{t_{0G}}[\xi](\nu)(t) = \sigma_{t_0}(t-t_0)^{-1}D\Sigma_{t_0G}[\xi](\nu)(t)$. The strong asymptotic Lyapunov stability ensures that for any $\xi \in \mathcal{U}$, any $\nu \in \mathbb{R}^n$, $\|D\widetilde{\Sigma}_{t_{0G}}[\xi]\|_{\infty,T} \leq b\|\nu\|$. The mean value theorem thus ensures that $\|\widetilde{\Sigma}_{t_0}(\xi_1) - \widetilde{\Sigma}_{t_0}(\xi_2)\|_{\infty,T} \leq b\|\xi_1 - \xi_2\|$. We then conclude that $\|\Sigma_{t_0}(\xi_1) - \Sigma_{t_0}(\xi_1)\|_{\infty,T} \leq b\|\xi_1 - \xi_2\|\sigma_{t_0}(t - t_0)$ for any $\xi_1, \xi_2 \in \mathcal{U}$ and any $T \in [t_0, \infty)$. ■

A more interesting result is obtained for exponential stability.

Proposition 3.4: Let Σ_{t_0} be defined by (1) with a Gâteaux derivative at each point ξ of $\mathcal{W}$. Let $\mathcal{U}$ be a convex and closed subset of $\mathcal{W}$. Σ_{t_0} is incrementally exponentially stable on $\mathcal{U}$ if and only if all the linearizations of Σ_{t_0} are strongly exponentially stable on $\mathcal{U}$.

The implication of the incremental exponential stability of Σ_{t_0} on $\mathcal{U}$ by the strong exponential Lyapunov stability of the linearizations is a direct consequence of the previous proposition. The converse implication proof is deduced from the following lemma (a corollary of the mean value theorem).

Lemma 3.5: Let Σ_{t_0} be defined by (1) with a Gâteaux derivative at each point ξ of $\mathcal{W}$. Let $\mathcal{U}$ be a convex and closed subset of $\mathcal{W}$. If there exist φ a C^1 class $\mathcal{K}$ function and σ a class $\mathcal{L}$ function such that for ξ_1, $\xi_2 \in \mathcal{U}$, any $t \in [t_0, \infty)$, one has $\|\Sigma_{t_0}(\xi_1)(t) - \Sigma_{t_0}(\xi_2)(t)\| \leq \varphi(\|\xi_1 - \xi_2\|)\sigma(t - t_0)$ then for any $\xi \in [\xi_1, \xi_2]$, one has $\|\sigma^{-1}D\widetilde{\Sigma_{t_0}}[\xi](\nu)\|_{\infty,T} \leq \varphi'(0)\|\nu\|$ for any $T \in [t_0, \infty)$ and any $\nu \neq 0$.

The link between the incremental and the non incremental Lyapunov stability on $\mathcal{U}$ is now revealed by an interesting result, close to a recent one [2] derived with a completely different proof. Indeed, Proposition 3.6 explains that the incremental exponential stability on $\mathcal{U}$ is equivalent to the exponential stability of Σ_{t_0} on $\mathcal{U}$ at any $\xi_0 \in \mathcal{U}$.

Proposition 3.6: Let Σ_{t_0} be defined by (1) with a Gâteaux derivative at each point ξ of $\mathcal{W}$. Let $\mathcal{U}$ be a convex and closed subset of $\mathcal{W}$. The following properties are equivalent.
1. Σ_{t_0} is incrementally exponentially Lyapunov stable on $\mathcal{U}$;
2. Σ_{t_0} is exponentially Lyapunov stable on $\mathcal{U}$ at any $\xi \in \mathcal{U}$;
3. Σ_{t_0} is exponentially Lyapunov stable on $\mathcal{U}$ at a $\xi \in \mathcal{U}$.

Proof: **A.** The implication between incremental exponential Lyapunov stability and the exponential Lyapunov stability on $\mathcal{U}$ at any $\xi \in \mathcal{U}$ is a consequence of the definition. **B.** If Σ_{t_0} is exponentially Lyapunov stable on $\mathcal{U}$ at ξ_0 then there exist $a > 0$, $b > 0$ such that for any $\xi_{0p} \in \mathcal{U}$, one has $\|\Sigma_{t_0}(\xi_{0p}) - \Sigma_{t_0}(\xi_0)\| \le be^{-a(t-t_0)}\|\xi_{0p} - \xi_0\|$. We thus deduce by the mean value theorem in norm and the convexity of $\mathcal{U}$ that $\|D\Sigma_{t_0 G}[\xi](\nu)(t)\| \le be^{-a(t-t_0)}\|\nu\|$ for any $\xi \in \mathcal{U}$ and thus the announced result. ∎

At this point, we only consider non-uniform Lyapunov stability. In the last part of this section, the extension of our results to the uniform Lyapunov stability is discussed. Note that, in the Lyapunov stability, uniformity can recover various aspects (see *e.g.* [20][p. 172]). Since global or semi-global stability properties is enlightened in this paper, we define:

Definition 3.7: Σ_{t_0} is said to be uniformly incrementally asymptotically Lyapunov stable on $\mathcal{U} \subset \mathcal{W}$ if there exists a class $\mathcal{KL}$ function β such that $\|\Sigma_{t_i}(\xi_1) - \Sigma_{t_i}(\xi_2)\| \le \beta(t - t_i, \|\xi_1 - \xi_2\|)$ for any $t_i \ge t_0$, any $t \ge t_i$, any ξ_1, ξ_2 in $\mathcal{U}$.

Extending our results to the uniform case is straightforward with a stronger assumption on the linearizations of Σ_{t_0}. It is then an exercise to prove that the uniform strong asymptotical stability of the linearizations of Σ on $\mathcal{U}$ implies the uniform incremental asymptotical Lyapunov stability on $\mathcal{U}$. The uniform incremental exponential stability on $\mathcal{U}$ can be proved to be equivalent to the uniform strong exponential stability on $\mathcal{U}$ of the linearizations of Σ_{t_0}. Finally, in the same way, Proposition 3.6 can be easily extended to the uniform case. The equivalence between the uniform incremental exponential stability on $\mathcal{U}$ and the uniform exponential stability on $\mathcal{U}$ for any initial condition in $\mathcal{U}$ is then obtained.

IV. Connection with the contraction approach

We now reveal the link between the mean value theorem approach and the one based on the analysis of the time dependent distance between the two extreme curves. We first recall the definition of the length of a curve in $\mathbb{R}^n$ and a fundamental theorem concerning the length of a curve with absolutely continuous components.

A. Background about the length of rectifiable curves

Let be $c(\alpha)$, a curve in $\mathbb{R}^n$, defined by n functions c_i, defined from $[0,1]$ into $\mathbb{R}$, $c(\alpha) \triangleq (c_1(\alpha), \cdots, c_n(\alpha))$. The length of the curve $c(\alpha)$ between its two extreme points $c(0)$ and $c(1)$ is defined as the upper bound of $\sigma(c, \{\alpha_i\}) = \sum_{i=1}^{N} \|c(\alpha_{i-1}) - c(\alpha_i)\|$ for any sequences of finite number of points $\{\alpha_i\}$ belonging to $[0,1]$ and such that $0 = \alpha_0 \le \alpha_1 \le \cdots \le \alpha_N = 1$. If the upper bound is finite, its length between $c(0)$ and $c(1)$, denoted by L, is then defined as this upper bound. Tonelli's Theorem 4.1 is fundamental [31].

Theorem 4.1: If all the components of $c(\alpha)$ are absolutely continuous functions of α then

$$L = \int_0^1 \sqrt{\frac{dc_1}{d\alpha}^2(\alpha) + \cdots + \frac{dc_n}{d\alpha}^2(\alpha)} d\alpha.$$

Let us now compared two trajectories associated respectively to $\xi_1 \in \mathcal{U}$ and $\xi_2 \in \mathcal{U}$. For any fixed $t \in [t_0, \infty)$, a curve $c_t(\alpha)$ associated to Σ_{t_0} is defined by $c_t(\alpha) = \Sigma_{t_0}(\xi_1 + \alpha(\xi_2 - \xi_1))(t)$. It is not difficult to prove that for any fixed $t \in [t_0, T]$, $c_t(\alpha)$ is rectifiable and absolutely continuous. We deduce by Theorem 4.1 that the length $L(t)$ of $c_t(\alpha)$ between $c_t(0)$ and $c_t(1)$ is equal to

$$L(t) = \int_0^1 \|D\Sigma_{t_0 G}[\xi_1 + \alpha(\xi_2 - \xi_1)](\xi_2 - \xi_1)(t)\| d\alpha.$$

Since, for almost every $t \in [t_0, T]$ one has

$$\|D\Sigma_{t_0 G}[\xi_1 + \alpha(\xi_2 - \xi_1)](\xi_2 - \xi_1)(t)\| \le \|D\Sigma_{t_0 G}[\xi_1 + \alpha(\xi_2 - \xi_1)](\xi_2 - \xi_1)\|_{\mathcal{L}_\infty, T}$$

we then deduce that for almost every $t \in [t_0, T]$, one has:

$$L(t) \le \int_0^1 \|D\Sigma_{t_0 G}[\xi_1 + \alpha(\xi_2 - \xi_1)](\xi_2 - \xi_1)\|_{\mathcal{L}_\infty, T} d\alpha.$$

This previous inequality and the fact that the length of a curve $c_t(\alpha)$ is necessarily greater than the length of the straight line between $c_t(0)$ and $c_t(1)$ allow to conclude that:

$$\|\Sigma_{t_0}(\xi_2) - \Sigma_{t_0}(\xi_2)\|_{\mathcal{L}_\infty, T} \le \sup_{t \in [t_0, T]} L(t) \le \eta_T \|\xi_2 - \xi_1\| \quad (5)$$

where η_T is given by (4).

We thus conclude that the length approach and the mean value theorem approach are strongly related. We now investigate the relations between the section III results and the length context ones. We specially focus our interest on the results of [29] (and also [25, 21]) and more recently in the contraction analysis [26, 27].

B. Time derivative of the length for asymptotic results

In the sequel, $\mathcal{U}$ is a convex and closed subset of $\mathcal{W}$. Actually, if the linearizations are strongly exponentially stable on $\mathcal{U}$ then there exists $a > 0$ and $b > 0$ such that for any $t \ge t_0$, one has $\|D\Sigma_{t_0 G}[\xi](\nu)(t)\| \le be^{-a(t-t_0)}\|\nu\|$ and then for any $t \ge t_0$, $L(t) \le be^{-a(t-t_0)}\|\xi_2 - \xi_1\|$. Inequality (5) allows to deduce that $\|\Sigma_{t_0}(\xi_2)(t) - \Sigma_{t_0}(\xi_2)(t)\| \le be^{-a(t-t_0)}\|\xi_2 - \xi_1\|$. We then have an alternative proof to the implication of the incremental exponential stability on $\mathcal{U}$ by the (strong) exponential stability of linearizations on $\mathcal{U}$, see Proposition 3.4. It is crystal clear that the same arguments allow to deduce the previous section results on the Lyapunov linearization stability and the system one.

In order to deepen our understanding on the connection between the mean value theorem and the length approach results, we now investigate the obtained results when the time derivative of the length is considered. Let us define: $x(t, \alpha) \triangleq \Sigma_{t_0}(\xi_1 + \alpha(\xi_2 - \xi_1))(t)$ and $\bar{x}(t, \alpha) \triangleq D\Sigma_{t_0 G}[\xi_1 +$

$\alpha(\xi_2-\xi_1)](\xi_2-\xi_1)(t)$. If the $\mathbb{R}^n$ norm is the Euclidean one then the time derivative of L is given by:

$$\frac{dL(t)}{dt} = \frac{d}{dt}\int_0^1 \sqrt{\bar{x}^T(t,\alpha)\bar{x}(t,\alpha)}d\alpha.$$

By classical results on derivation of integrals, we have

$$\begin{aligned}\frac{dL(t)}{dt} &= \frac{1}{2}\int_0^1 \frac{\bar{x}^T(t,\alpha)\dot{\bar{x}}(t,\alpha)+\dot{\bar{x}}^T(t,\alpha)\bar{x}(t,\alpha)}{\sqrt{\bar{x}^T(t,\alpha)\bar{x}(t,\alpha)}}d\alpha \\ &= \frac{1}{2}\int_0^1 \frac{\bar{x}^T(t,\alpha)[A(t,\alpha)+A^T(t,\alpha)]\bar{x}(t,\alpha)}{\sqrt{\bar{x}^T(t,\alpha)\bar{x}(t,\alpha)}}d\alpha\end{aligned}$$

since by definition $\dot{\bar{x}}(t,\alpha) = A(t,\alpha)\bar{x}(t,\alpha)$ where $A(t,\alpha) = \frac{\partial f}{\partial x}(t,x(t,\alpha))$ (notation pointed out the dependence of $A(t)$ on α though $x(t,\alpha)$). Since the matrix $A(t,\alpha)+A(t,\alpha)^T$ is not generally sign definite, nothing can be concluded on the evolution of $L(t)$ with respect to time. This is a classical problem since the asymptotic stability does not generally implies that the norm of the system states is a decreasing time function at each instant. This problem is classically bypassed by introducing a "weighted" norm or more generally by the definition of a Lyapunov function which is is now decreasing at each instant and which is obviously strongly related to the norm of system states. In the length context, such idea leads to introduce suitable Riemann metrics or more generally Finsler metrics, see [25, 29, 21] and also more recently [26]. A related condition is here considered since we assume that there exist $\epsilon > 0$, $\beta_1 > 0, \beta_2 > 0$ and a symmetric matrix $M(t,x)$ defined from $[t_0,\infty)\times\mathbb{R}^n$ into $\mathbb{R}^{n\times n}$ such that for any $\xi \in \mathcal{U}$, any $t \in [t_0,\infty)$, with $x(t) = \Sigma_{t_0}(\xi)(t)$, one has:

(i) $\quad \beta_1^2 I_n \leq M(t,x(t)) \leq \beta_2^2 I_n$

(ii) $\quad \frac{dM}{dt}(t,x(t)) + M(t,x(t))\frac{\partial f}{\partial x}(t,x(t)) + \cdots + \frac{\partial f}{\partial x}^T(t,x(t))M(t,x(t)) \leq -\epsilon I_n$

The previous conditions ensure that the Σ_{t_0} linearizations are strongly exponentially stable on $\mathcal{U}$, *i.e.* for any $\xi \in \mathcal{U}$. By Proposition 3.4, Σ is thus incrementally exponentially stable on $\mathcal{U}$, *i.e.*,

$$\|\Sigma_{t_0}(\xi_2)(t) - \Sigma_{t_0}(\xi_2)(t)\| \leq \beta_2\beta_1^{-1}e^{-\frac{\epsilon}{2\beta_1^2}(t-t_0)}\|\xi_2-\xi_1\|.$$

In order to obtain the uniform strong exponential stability on $\mathcal{U}$, it is necessary to assume that condition (ii) is now satisfied uniformly by Σ_{t_i} for any $t_i \in [t_0,\infty)$.

Remark 4.2: In order to recast the contraction definition in context of conditions (i) and (ii), it is then necessary to define the set $\mathcal{U}$ in order to be compatible with $M(t,x)$ and with (ii), *i.e.* $\mathcal{U}$ is the set of initial conditions which is such all the associated trajectories of Σ_{t_0} satisfied (ii). ■

In order to derive the length approach result, we define:

$$\tilde{L}(t) = \int_0^1 \sqrt{\bar{x}(t,\alpha)M(t,x(t,\alpha))\bar{x}(t,\alpha)}d\alpha$$

which is, by the right hand side of condition (i), such that $L(t) \leq \beta_1^{-1}\tilde{L}(t)$. We now compute the derivative of $\tilde{L}(t)$:

$$\frac{d\tilde{L}}{dt}(t) = \frac{1}{2}\int_0^1 \frac{\bar{x}(t,\alpha)^T\Pi(t,\alpha)\bar{x}(t,\alpha)}{\sqrt{\bar{x}^T(t,\alpha)M(t,x(t,\alpha))\bar{x}(t,\alpha)}}$$

where $\Pi(t,\alpha) = \frac{dM}{dt}(t,x(t,\alpha)) + M(t,x(t,\alpha))A(t,\alpha) + M(t,x(t,\alpha))A^T(t,\alpha)$. From (ii) and (i), one deduces that

$$\begin{aligned}\frac{d\tilde{L}}{dt}(t) &\leq -\frac{\epsilon}{2\beta_1^2}\int_0^1 \frac{\bar{x}(t,\alpha)^T M(t,x)\bar{x}(t,\alpha)}{\sqrt{\bar{x}^T(t,\alpha)M(t,x(t,\alpha))\bar{x}(t,\alpha)}} \\ &\leq -\frac{\epsilon}{2\beta_1^2}\tilde{L}(t)\end{aligned}$$

and thus $\tilde{L}(t) \leq e^{-\frac{\epsilon}{2\beta_1^2}(t-t_0)}\tilde{L}(t_0)$. In order to conclude, $\tilde{L}(t_0)$ is evaluated as a function of the initial condition increment, *i.e.* $\|\xi_2-\xi_1\|$. It is straightforward to prove that for $t = t_0$, one has $L(t_0) = \|\xi_2-\xi_1\|$. Moreover, by condition (i), $M(t,x)$ has an upper bound and then for any $t \in [t_0,\infty)$, one has $\tilde{L}(t) \leq \beta_2 L(t_0)$ and then $\|\Sigma_{t_0}(\xi_2)(t) - \Sigma_{t_0}(\xi_2)(t)\| \leq \beta_2\beta_1^{-1}e^{-\frac{\epsilon}{2\beta_1^2}(t-t_0)}\|\xi_2-\xi_1\|$ which corresponds exactly to the result already obtained using condition (i) and (ii) and Proposition 3.4 (see above). Let us finally note that the definition of new length under the used of a metric "$M(t,x)$" allows to deduce stability result if we are able to relate this new length to the norm associated to Euclidean space. In the previous developments, it is clear that such link is ensured by the both sides of inequality (i).

C. From uniform incremental exponential stability to the contraction

In section III, a necessary and sufficient condition for the incremental exponential Lyapunov stability on $\mathcal{U}$ was presented in term of exponential Lyapunov stability of Σ_{t_0} linearizations. In the previous subsection, we present a sufficient condition involving the existence of a suitable matrix. We now prove that the uniform incremental exponential stability implies the existence of the suitable matrix $M(t,x)$. The conclusion is that previous subsection conditions (i) and (ii) are, in fact, as the uniform strong exponential stability, a necessary and sufficient condition for the uniform incremental exponential stability on $\mathcal{U}$. This point has already been investigated in [27] where the authors point out that exponential stability implies the existence of a suitable metric where the considered system is contractive. To our best understanding, the converse result proposed in [27] is only local. We now present a converse result for the uniform incremental exponential stability on $\mathcal{U}$. For simplifying the derivations, the function $f(t,x)$ is now assumed at least a C^3 function of its first arguments; $\frac{\partial f}{\partial x}$ is assumed to have an upper bound for the set of possible trajectories (or f is uniformly Lipschitz continuous of its second argument on $\mathbb{R}^n$).

Theorem 4.3: Let Σ_{t_0} be defined by (1) with a Gâteaux derivative at each point ξ of $\mathcal{W}$. Let $\mathcal{U}$ be a convex and closed subset of $\mathcal{W}$. If Σ_{t_0} is uniformly incrementally exponentially stable on $\mathcal{U}$ then there exists a symmetric matrix $M(t,x)$ such that for any $t_i \geq t_0$, any $\xi \in \mathcal{U}$ and any $t \geq t_i$, with $x(t) = \Sigma_{t_i}(\xi)(t)$, one has:

(i) $\quad \beta_1^2 I_n \leq M(t,x(t)) \leq \beta_2^2 I_n$

(ii) $\quad \frac{dM}{dt}(t,x(t)) + M(t,x(t))\frac{\partial f}{\partial x}(t,x(t)) + \cdots + \frac{\partial f}{\partial x}^T(t,x(t))M(t,x(t)) \leq -\epsilon I_n.$

To prove theorem 4.3, we first show that the uniform incremental exponential stability guarantees the existence of incremental Lyapunov function which satisfies suitable "incremental" inequalities on $\mathcal{U}$. To conclude the proof, it then remains to prove that the theorem inequalities correspond to the infinitesimal counterpart of these "incremental" inequalities on $\mathcal{U}$ (see [17] for details).

REFERENCES

[1] V. Alexéev, V. Tikhomirov, and S. Fomine. *Commande optimale*. Edition MIR, Moscou, 1982.

[2] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Trans. Auto. Contr.*, 47:410–422, 2002.

[3] L. O. Chua and D. N. Green. A qualitative analysis of the behavior of dynamic nonlinear networks: Steady-state solutions of nonautonomous networks. *IEEE Trans. Circ. and Syst.*, 23(9):530–550, 1976.

[4] G. Dahlquist. Stability and error bounds in the numerical integration of ordinary differential equations. *Transactions R. Inst. Tech. Stockholm*,130:1–82, 1959.

[5] C. A. Deseor and H. Haneda. The measure of matrix as a tool to analyze computer algorithms for circuit analysis. *IEEE Trans. Circ. Theo.*, 19(5):480 – 486, Sept. 1972.

[6] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Academic Press, 1975.

[7] M. Elwakkad-Zaghloul and P. R. Bryant. Error bounds on solutions of nonlinear networks when using approximate element characteristics. *IEEE Trans. Circ. Theo.*, 27(1):20 – 29, Jan. 1980.

[8] V. Fromion. Some results on the behavior of Lipschitz continuous systems. In *Proc. European Cont. Conf.*, 1997.

[9] V. Fromion, S. Monaco, and D. Normand-Cyrot. A possible extension of H_∞ control to the nonlinear context. In *CDC*, 1995.

[10] V. Fromion, S. Monaco, and D. Normand-Cyrot. Asymptotic properties of incrementally stable systems. *IEEE Trans. Aut. Contr.*, 41:721–723, 1996.

[11] V. Fromion, S. Monaco, and D. Normand-Cyrot. A link between input-output stability and Lyapunov stability. *Syst. Contr. Lett.*, 27(4):243–248,1996.

[12] V. Fromion, S. Monaco, and D. Normand-Cyrot. The weighting incremental norm approach: from linear to nonlinear H_∞ control. *Automatica*, 37:1585–1592, 2001.

[13] V. Fromion and G. Scorletti. The behavior of incrementally stable discrete time systems. *Syst. Contr. Lett.*, 46(4):289–301, 2002.

[14] V. Fromion and G. Scorletti. Characterization of nonlinear systems by their linearizations. In *Proc. CDC*, 2002.

[15] V. Fromion and G. Scorletti. Performance and robustness analysis of nonlinear closed loop systems using μ_{nl} analysis: Applications to nonlinear PI controllers. In *Proc. CDC*, 2002.

[16] V. Fromion and G. Scorletti. A theoritical framework for gain scheduling. *Int. J. Robust. Nonlin. Contr.*, 13(6):951–982, 2003.

[17] V. Fromion and G. Scorletti. Connecting nonlinear incremental Lyapunov stability with the linearizations Lyapunov stability. *GREYC Technical Report*, 2005.

[18] V. Fromion, G. Scorletti, and G. Ferreres. Nonlinear performance of a PID controlled missile : an explanation. *Int. J. Robust Nonlin. Contr.*, 9(8):485–518, 1999.

[19] T. T. Georgiou. Differential stability and robust control of nonlinear systems. *MCSS*, 6:485–518, 1993.

[20] W. Hahn. *Stability of Motion*. Springer-Verlag, Berlin, 1967.

[21] P. Hartman. On stability in the large for systems of ordinary differential equations. *Canadian J. Math.*, 13:480–492, 1961.

[22] E. Hille and R. S. Phillips. *Functional Analysis and Semi-Groups*, Vol. 31 American Mathematical Society, 1957.

[23] J. Jouffroy. *Stabilité et systémes non linéaires : Réfêxions sur l'analyse de contraction*. PhD thesis, Université de Savoie, 2002.

[24] D. Kristiansen. *Modeling of cylinder gyroscopes and observer design for nonlinear oscillations*. PhD thesis, University of Science and Technology, Trondheim, Norway, 2000.

[25] D. C. Lewis. Metric properties of differential equations. *Americ. J. Math.*, 73:294 – 312, 1949.

[26] W. Lohmiller. *Contraction analysis for nonlinear systems*. PhD thesis, Cambridge, Massachusetts, 1999.

[27] W. Lohmiller and J. J. Slotine. On contraction analysis for nonlinear systems. *Automatica*, 34(6):683–696, 1998.

[28] M. Z. Nashed. *Nonlinear Functional Analysis and Application*, chapter Differentiability and related properties of nonlinear operators: Some aspects of the role of differentials in nonlinear functional analysis, pages 103–309. L.B. Rall (Ed.), Academic Press, New York, 1971.

[29] Z. Opial. Sur la stabilité asymptotique des solutions d'un système d'équations différentielles. *Annales Polonici Mathematici*, 7:259 – 267, 1960.

[30] A. Pavlov, A. Pogromsky, N. van de Wouw, and H. Nijmeijer. Convergent dynamics, a tribute to Boris Pavlovich Demidovich. *Syst. Contr. Lett.*, 52:257 – 261, 2004.

[31] S. Saks. *Théorie de L'intégrale*. Warszawa, Institut Mathématique de l'Université de Moscou, 1933.

[32] I. W. Sandberg. Some results in the theory of physical systems governed by nonlinear functional equations. *Bell Syst. Tech. J.*, 44:871–898, 1965.

[33] Y. Z. Tsypkin. Absolute stability and process absolute stability in nonlinear sampled-data automatic control systems. *Aut. Remote Contr.*, 24(12):4–12, 1963.

[34] M. M. Vainberg. *Variational methods for the study of nonlinear operators*. Holden-Day, INC., 1964.

[35] J. C. Willems. *The Analysis of Feedback Systems*, volume 62 of *Research Monographs*. MIT Press 1971.

[36] T. Yoshizawa. *Stability Theory and the Existence of Periodic and Almost Periodic Solutions*, volume 14 of *Applied Mathematical Sciences*. Springer-Verlag, 1973.

[37] G. Zames. On the input-output stability of time-varying nonlinear feedback systems–Part I, II. *IEEE Trans. Auto. Contr.*, 11:228–238, 465–476, 1966.