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Relative Entropy Applied to Optimal Control of Stochastic Uncertain Systems on Hilbert Space

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Abstract— This paper considers minimax problems, in which the control minimizes the pay-off induced by a measure which maximizes the pay-off over the class of measures described by a relative entropy set between the uncertain and the true measure. We present several basic properties of the relative entropy on infinite dimensional spaces, and then we apply them to an uncertain system described by a Stochastic Differential inclusion on Hilbert space.

I. Introduction

The theory and contributions of this paper are developed at two levels of generality; the abstract level and the application level. At the abstract level, a general framework is put forward in which the basic ideas are explained, and the fundamental results are derived. At this level, systems are represented by measures on measurable spaces, energy signals by functionals on the space of measures, and uncertainty by sets described by bounded relative entropy between the true measure and the nominal measure. The objective of the abstract level formulation is the derivation of existence of the minimax strategies on general spaces. At the application level, the results are applied to infinite dimensional systems described by evolution equations involving unbounded operators. In this set up, an explicit computation of the maximizing measure is derived in terms of the nominal measure, which is described by an evolution equation on Hilbert spaces.

The problem formulation is related to the one considered in [5] and [6,7,8]. However, the aim of this paper as described above and the result obtained are fundamentally different from the results found in these papers, in both the abstract level and at the application level. Specifically, at the abstract level, existence and of minimax strategies is shown for nonlinear functionals of the uncertain measure, while at the application level the results are applied to stochastic evolution equations on Hilbert spaces with unbounded operators. Moreover, at the application level, the worst case measure is computed explicitly, as a function of the a priori conditional nominal measure. Most papers dealing with partially observable problems are concerned with linear-Quadratic-Gaussian problems in which the disturbance is a stochastic process.

Let X be a Polish space, in particular a complete separable metric space and let $\mathcal{B}_X$ the sigma algebra of Borel sets generated by the metric topology. Let $\mathcal{M}_1(X)$ denote the space of countably additive regular probability measures defined on $\mathcal{B}_X$. We assume throughout the paper that $\mathcal{M}_1(X)$ is furnished with the weak topology. It is well known that the weak topology is metrizable with the Prohorov metric which makes it a complete metric space. We shall denote this metric by d^P. Thus a net $\mu^\alpha \xrightarrow{w} \mu^o$ if and only if $d^P(\mu^\alpha, \mu^o) \longrightarrow o$. Since the weak topology and the metric topology are equivalent we note the following facts. A set $\Gamma \subset \mathcal{M}_1(X)$ is compact with respect to the weak topology, if and only if, it is compact with respect to the metric topology d^P. A functional $g : \mathcal{M}_1(X) \longrightarrow R$ is weakly continuous,if and only if, it is continuous with respect to the Prohorov metric d^P.

Let $\nu, \mu \in \mathcal{M}_1(X)$ and suppose ν is absolutely continuous with respect to the measure μ, to be denoted by $\nu \prec \mu$. Then the Radon-Nikodym derivative of ν with respect to μ exists and this is given by an $h \in L_1(X, \mu)$ such that

$$d\nu = hd\mu.$$

We note that, given that $\nu \prec \mu$, the RND is unique. If further, $h \log h \in L_1(X, \mu)$, then the entropy of ν relative to the measure μ, known as relative entropy, is defined and it is given by

$$H(\nu|\mu) \equiv \int_X h \log h d\mu = \int_X \big\{\log(d\nu/d\mu)\big\} d\nu. \qquad (1)$$

The basic problem we wish to study can be stated as follows.

Basic Problem: Let $\eta : \mathcal{M}_1(X) \longrightarrow [-\infty, \infty)$, be an extended real valued continuous function. Let $\mathcal{M}_0 \subset \mathcal{M}_1(X)$ denote the set of measures induced by a controlled stochastic system under the assumptions that the system is perfectly known (system operators and other parameters and coefficients etc. are all perfectly known). In other words $\mathcal{M}_0$ denotes the set of attainable measures on X associated with an unambiguously defined (perfectly known) controlled stochastic system. However in the real world situation, the system parameters are not entirely known to the controller. This introduces some

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degree of uncertainty in the system dynamics around the nominal controlled system and hence the set of measures induced by the true system may differ from those represented by the set $\mathcal{M}_0$. For each fixed $r > 0$ and $\mu \in \mathcal{M}_0 \subset \mathcal{M}_1(X)$, define the set

$$\mathcal{A}_r(\mu) \equiv \{\nu \in \mathcal{M}_1(X) : H(\nu|\mu)) \leq r\}. \quad (2)$$

This set represents the set of true measures induced by the actual physical system given that the nominal controlled system induces the measure μ. This is the set of uncertainty about the system measured in terms of relative entropy, relative to the measure μ. Thus the set of measures induced by the uncertain controlled system is given by

$$\mathcal{M}_r \equiv \left\{\bigcup(\mathcal{A}_r(\mu)), \mu \in \mathcal{M}_0\right\}.$$

Clearly $\mathcal{M}_0 \subset \mathcal{M}_r$ for all $r \geq 0$ and $\lim_{r\downarrow 0} \mathcal{M}_r = \mathcal{M}_0$. Since the relative entropy is nonnegative, the parameter r determines the degree of uncertainty. Thus the larger the r is, the greater is the uncertainty.
The problem is to find a $\mu^o \in \mathcal{M}_0$ that minimizes the functional

$$F_r(\mu) \equiv \sup\{\eta(\nu); \nu \in \mathcal{A}_r(\mu)\} \quad (3)$$

over the set $\mathcal{M}_0$. In other words we wish to find a solution to the min-max problem

$$(P_{is}) : \inf_{\mu \in \mathcal{M}_0} \sup\{\eta(\nu) : \nu \in \mathcal{A}_r(\mu)\} \quad (4)$$

In the following section we investigate the question of existence of solution to the basic problem (P_{is}).

II. Existence of Solution to Problem (P)

Lemma 2.1 Let $\eta : \mathcal{M}_1(X) \longrightarrow [-\infty, +\infty)$ be an upper semi continuous function and strictly concave. Then, for every $\mu \in \mathcal{M}_0 \subset \mathcal{M}_1(X)$, there exists a unique $\nu^o \in \mathcal{A}_r(\mu)$ at which η attains its supremum.
Proof. Found in [9].

Lemma 2.2 Suppose the assumptions of Lemma 2.1 hold. Then the graph $\mathcal{G}(\mathcal{A}_r)$ of the multifunction $\mathcal{A}_r$ is sequentially closed and the function F_r as defined by (3) is lower semi continuous.
Proof. Here we outline the prove. By definition the function F_r is given by

$$F_r(\mu) \equiv \sup\{\eta(\nu); \nu \in \mathcal{A}_r(\mu)\}.$$

By virtue of the previous lemma, for every $\mu \in \mathcal{M}_0$, there exists a unique $\nu \in \mathcal{A}_r(\mu)$ so that

$$F_r(\mu) = \eta(\nu) \equiv \eta(\nu(\mu)),$$

that is, the maximizer ν is uniquely determined by μ. Thus F_r is a well defined single valued functional. Hence, the derivation proceeds by showing that the graph of the multifunction $\mu \longrightarrow \mathcal{A}_r(\mu)$ given by

$$\mathcal{G}(\mathcal{A}_r) \equiv \{(\nu, \mu) \in \mathcal{M}_1(X) \times \mathcal{M}_1(X) : \nu \in \mathcal{A}_r(\mu)\}$$

is closed, and that F_r is lower semi continuous.

Theorem 2.3 Consider the problem (P_{is}) and suppose that $\eta : \mathcal{M}_1(X) \longrightarrow [-\infty, +\infty)$ is upper semi continuous and strictly concave and the set $\mathcal{M}_0$ is compact with respect to the Prohorov topology. Then the problem (P_{is}) has a solution.
Proof. The proof follows from Lemma 2.2. Indeed, by virtue of Lemma 2.2, the map $\mu \longrightarrow F_r(\mu)$ is lower semi continuous and by assumption $\mathcal{M}_0$ is compact. Hence F_r attains its minimum on $\mathcal{M}_0$. •.

Example. We present an example illustrating the preceding results.

(Target seeking Problem): Let $\mu^d \in \mathcal{M}_1(X)$ be a desired measure, and suppose we wish to find a $\mu \in \mathcal{M}_0$ (the attainable set) that approximates μ^d as closely as possible. Clearly the obvious choice of the objective functional is the Prohorov metric giving

$$\eta(\nu) \equiv d^P(\mu^d, \nu).$$

Since d^P is a metric, the functional η as defined is continuous. If one disregards the uncertainty, one would be willing to minimize this functional on the attainable set of the perfect system given by $\mathcal{M}_0$. By our assumption this set is compact and since η is continuous it attains its minimum on it. Let $\mu^o \in \mathcal{M}_0$ be a minimizer. Then $\eta(\mu^o) \leq \eta(\mu), \mu \in \mathcal{M}_0$. But since the system is uncertain, the actual law (true measure) in force may be any element from $\mathcal{A}_r(\mu^o)$ and one may face the worst situation,

$$\eta(\nu^*) \equiv \sup\{\eta(\nu), \nu \in \mathcal{A}_r(\mu^o)\} = F_r(\mu^o).$$

Clearly $\eta(\nu^*) \geq \eta(\mu^o)$ with $H(\nu^*|\mu^o) \leq r$. So instead of choosing μ^o, we may choose one that minimizes the maximum distance. That is, we choose an element $\mu^* \in \mathcal{M}_0$ so that

$$F_r(\mu^*) = \inf\{F_r(\mu), \mu \in \mathcal{M}_0\}$$

where

$$F_r(\mu) \equiv \sup\{\eta(\nu) \equiv d^P(\mu^d, \nu) : \nu \in \mathcal{A}_r(\mu)\}.$$

Clearly

$$F_r(\mu^*) \leq F_r(\mu^o) = \eta(\nu^*).$$

Thus we conclude that the inf-sup strategy is closer to the desired goal than the strategy based on the deterministic dynamics with attainable set $\mathcal{M}_0$. Additional examples are found in [9].

III. Uncertain Stochastic Control System

In this section we wish to apply the results of the previous section to a class of stochastic control problems. Consider the controlled stochastic system

$$dx = Axdt + f_o(x, u)dt + B(x)dW, x(0) = x_0,$$
$$t \in I \equiv [0, T], \quad (5)$$

on a separable Hilbert space H, with A being the infinitesimal generator of a C_0 semigroup $S(t), t \geq 0$, on H, $f_0 : H \times U \longrightarrow H$ and $B : H \longrightarrow \mathcal{L}(H)$ suitable maps and W a cylindrical Brownian motion adapted to the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t \uparrow, P)$. It is assumed that all the parameters $\{A, f_0, B, x_0\}$ defining the system are perfectly known. Very often the system parameters, possibly based on noisy estimates, are not exactly known. In this case the true physical system may be described by a perturbed version of (5) as follows,

$$dx = Axdt + f_o(x,u)dt + \tilde{f}(x)dt + B(x)dW,$$
$$t \in I \equiv [0,T], x(0) = x_0, \tilde{f}(x) \in \tilde{F}(x), x \in H, \tag{6}$$

where $\tilde{f}$ is any measurable selection of a Borel measurable multifunction $\tilde{F}$ representing the uncertainty in the nominal drift f_0. Note that $\tilde{F}$ may absorb any bounded perturbation of the unbounded operator A and those associated with the nominal drift. We identify the system (5) as the nominal system and system (6) as the true physical system with parametric uncertainty as described above.

In general a non parametric model may cover a wider class of uncertain stochastic control systems provided the uncertainty is measured in terms of Prohorov metric topology. But for a fairly large class of physical systems, relative entropy (though it is not a metric) provides a good and mathematically convenient measure of uncertainty. This is the measure that we use throughout the paper. Let X denote a Polish space, $\mathcal{B}_X$ the Borel algebra of subsets of the set X, and $\mathcal{M}_1(X)$ the space of probability measures defined on $\mathcal{B}_X$. This is the state space. Let $\mathcal{U}_{ad}$ denote the class of admissible controls. The perfect system can be described by the single valued map,

$$\Phi : \mathcal{U}_{ad} \longrightarrow \mathcal{M}_1(X),$$

that assigns a unique element $\mu \in \mathcal{M}_1(X)$ corresponding to each choice of control policy u giving $\Phi(u) = \mu$. Clearly, the attainable measures, denoted by the set

$$\mathcal{M}_0 \equiv \{\Phi(u), u \in \mathcal{U}_{ad}\},$$

describes the power of the control system in the sense of its capacity to produce a wider variety of probability measures (possibly with different support properties) guaranteeing more freedom of maneuver. The model for the uncertain system may be chosen as follows: for any choice of $\mu \in \mathcal{M}_0$ and any real number $r > 0$, define the set valued map as before by

$$\mathcal{A}_r(\mu) \equiv \{\nu \in \mathcal{M}_1(X) : H(\nu|\mu) \leq r\}.$$

Clearly this set is always non empty,since it contains μ itself. Then the uncertain system is governed by the multi valued map

$$\Phi_r(u) = \mathcal{A}_r(\Phi(u)),$$

with range given by

$$\mathcal{M}_r \equiv \bigcup\{\Phi_r(u), u \in \mathcal{U}_{ad}\} = \bigcup\{\mathcal{A}_r(\mu); \mu \in \mathcal{M}_0\}.$$

The min-max problems studied in section II directly applies to this later class of uncertain systems. In this section we consider a control problem for the system (6) which is described by parametric uncertainty.

Admissible Controls: Let U be a closed subset of another Polish space and $\mathcal{U}_{ad}$ the class of admissible controls taking values from U. More detailed information on the structure of the class of admissible controls will be presented later.

Existence of Solutions, Properties of Attainable Measures: Standard assumptions imposed on $\{f_0, B\}$ guaranteeing the existence of mild solutions are as stated below [3] . There exist constants $K, L > 0$, such that for all $v \in U$, f_0 and B satisfy the following Lipschitz and growth properties:

$$\| f_0(x,v) \|^2 + \| B(x) \|^2_{H.S} \leq K(1+ \| x \|^2), \forall\ x \in H$$
$$\| f_0(x,v) - f_0(y,v) \|^2 + \| B(x) - B(y) \|^2_{H.S}$$
$$\leq L \| x - y \|^2, \forall\ x, y \in H, \tag{7}$$

where $\| B(x) \|_{H.S}$ denotes the Hilbert-Schmidt norm of the operator $B(x)$. Under the above assumptions, it is well known [3] that for each control $u \in \mathcal{U}_{ad}$ the (nominal) system (5) has a unique $\mathcal{F}_t$-adapted mild solution x with the properties

(p1): $\sup_{t \in I} E \| x(t) \|^2_H < \infty$ and (p2): $x \in C(I,H)\ \ P - a.s.$

In case controls are based on state feedback, the Lipschitz property imposed on the drift f_0 is rather restrictive and may be lost. Thus it is necessary to generalize this. This can be done by accepting martingale solution (weak solution).

Lemma 3.1 Consider the system

$$dx = Axdt + B(x)dW, x(0) = x_0, \tag{8}$$

and suppose A generates a C_0-semigroup $S(t), t \geq 0$, on H, B satisfy the assumptions (7) and that $E \| x_0 \|^2 < \infty$. Then (8) has a unique mild solution with sample paths $x \in C(I,H)$ P-a.s satisfying $E\{\sup_{t \in I} \| x(t) \|^2_H\} < \infty$.

Proof See Da Prato- Zabczyk [3, Chapter 7].

Lemma 3.2 Consider the control system

$$dx = Axdt + f_0(x,u)dt + B(x)dW, x(0) = x_0. \tag{9}$$

Suppose A and B satisfy the assumptions of Lemma 3.1 and further there exists a finite positive number γ such that

$$\sup\{\| B^{-1}(x) f_0(x,v) \|_H, (x,v) \in H \times U\} \leq \gamma \tag{10}$$

and f_0 satisfy the growth assumption (7), and that $E \| x_0 \|^2 < \infty$. Then for any given control $u \in \mathcal{U}_{ad}$, the system (9) has a martingale solution uniquely determined by the solution of (8) and the control $u \in \mathcal{U}_{ad}$.

Proof Under the given assumption (10) it follows from Theorem 10.14 (see also Proposition 10.17) [3] that the process

$$\hat{W}(t) \equiv W(t) - \int_0^t B^{-1}(x(s)) f_0(x(s), u_s)\ ds \qquad (11)$$

evaluated along the mild solution of equation (8) is a cylindrical Brownian motion on an extended (Skorohod extension) probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t \uparrow, \hat{P})$. Thus on this probability space the mild solution of (8) is in fact the martingale solution of (9) driven by the Brownian motion $\hat{W}$. By Girsanov theorem [3, Da Prato-Zabczyk, Theorems 10.14, 10.18, p290], the probability measure induced by the controlled process governed by (9) is absolutely continuous with respect to that induced by the process governed by (8). More precisely, let $X \equiv C(I, H)$ denote the space of continuous functions on I with values in H. Furnished with the standard sup norm topology this is a Banach space. Let $\mathcal{B}_X$ denote the Borel algebra of subsets of the set X making $(X, \mathcal{B}_X)$ a measurable space. This is the canonical sample space, a Borel measurable space $(X, \mathcal{B}_X)$, on which we may define different measures. Let μ^0 denote the measure induced by the system (8) on $\mathcal{B}_X$ and μ^u the one induced by the system (9). Then by virtue of Girsanov theorem as mentioned above, we have $\mu^u \prec \mu^0$ and the RND is given by

$$\begin{aligned}&(d\mu^u/d\mu^0) = \\ &exp\left\{\int_0^T (B^{-1}f_0, dW) - (1/2)\int_0^T \parallel B^{-1}f_0 \parallel_H^2\ ds\right\} \\ &\equiv \Lambda(u).\end{aligned} \qquad (12)$$

This means that the martingale (or weak) solution of (9), described in terms of the probability measure it induces on the path space X, is uniquely determined by the measure induced by (8) and the control $u \in \mathcal{U}_{ad}$ through the relation

$$d\mu^u = \Lambda(u) d\mu^0. \qquad (13)$$

This completes the proof. •

Remark The assumption (10) is satisfied if f_0 satisfies the growth condition (7) and the operator $B(x)$ is surjective satisfying

$$\parallel B(x)z \parallel_H \geq c(1 + \parallel x \parallel^q) \parallel z \parallel_H, q \geq 1, \qquad (14)$$

for a constant $c > 0$.

Theorem 3.3 Consider the control system (9) and suppose the assumptions of Lemma 3.2 hold, and the set $\mathcal{D}_{rnd} \equiv \left\{\Lambda(u), u \in \mathcal{U}_{ad}\right\}$ is a closed bounded subset of $L_1(X, \mu^0)$ satisfying

$$\lim_{\mu^0(\Gamma) \to 0} \int_\Gamma \Lambda(u) d\mu^0 = 0 \text{ uniformly with respect } u \in \mathcal{U}_{ad}.$$

Then the set of attainable measures $\mathcal{M}_0$, induced by the control system (9), is compact in the Prohorov topology.

Proof. The proof is a direct consequence of the celebrated Dunford-Pettis theorem [4, pp.93]. •

Perturbed System. The uncertain system described by (6) is essentially a stochastic differential inclusion

$$\begin{aligned}&dx \in Axdt + f_0(x, u)dt + \tilde{F}(x)dt + B(x)dW, \\ &x(0) = x_0, t \in I.\end{aligned} \qquad (15)$$

We assume throughout that $\tilde{F} : H \longrightarrow 2^H \setminus \emptyset$ is a measurable multifunction in the sense that for every $\Gamma \in \mathcal{B}_H$ the set $\{x \in H : \tilde{F}(x) \cap \Gamma \neq \emptyset\} \in \mathcal{B}_H$.
Let $c(H) \subset 2^H \setminus \emptyset$ denote the class of nonempty closed subsets of H. The following result relates the uncertain system (15) to the nominal system (9) in terms of relative entropy (information distance).

Theorem 3.4 Consider the system (15) and suppose $\{A, f_0, B\}$ satisfy the assumptions of Lemma 3.2. Further, suppose that $\tilde{F}$ is a measurable multifunction mapping H to $c(H)$ and there exists a finite positive number β such that

$$\sup\left\{\parallel B^{-1}(x)h \parallel_H, h \in \tilde{F}(x)\right\} \leq \beta\ \forall\ x \in H. \qquad (16)$$

Then for each fixed $u \in \mathcal{U}_{ad}$, the system has a nonempty set of martingale solutions denoted by $\mathcal{M}_{\tilde{F},u} \subset \mathcal{M}_1(X)$ and that each member ν of $\mathcal{M}_{\tilde{F},u}$ is absolutely continuous with respect to the corresponding martingale solution μ^u of the nominal (or perfect) system (9). Further, there exists a finite positive number r such that

$$H(\nu|\mu^u) \leq r,\ \forall\ \nu \in \mathcal{M}_{\tilde{F},u}\ u \in \mathcal{U}_{ad}.$$

Proof. Since H is a separable Hilbert space, it is also a Polish space with respect to its standard topology. Clearly $(H, \mathcal{B}_H)$ is a measurable space and by our assumption $\tilde{F} : H \longrightarrow c(H)$ is measurable. Thus it follows from standard selection theorem [see 2, Theorem 2.1,pp.154] that $\tilde{F}$ has a nonempty set of measurable selections which we may denote by $\mathcal{S}_{\tilde{F}}$. Let $\tilde{f} \in \mathcal{S}_{\tilde{F}}$ and consider the system

$$\begin{aligned}&dx = Axdt + f_0(x, u)dt + \tilde{f}(x)dt + B(x)dW, \\ &x(0) = x_0, t \in I.\end{aligned} \qquad (17)$$

For any $u \in \mathcal{U}_{ad}$, let $\mu^u \in \mathcal{M}_1(X)$ be the measure corresponding to the martingale solution of (15). Then following identical arguments as in the proof of Lemma 3.2, one can verify that $\tilde{\rho}$ given by

$$\tilde{\rho} \equiv Exp\left\{\int_I (B^{-1}\tilde{f}, dW) - (1/2)\int_I \parallel B^{-1}\tilde{f} \parallel_H^2\ dt\right\} \qquad (18)$$

is the RND of the measure $\tilde{\nu}^u \in \mathcal{M}_1(X)$, associated with the martingale solution of (17), with respect to the measure μ^u, associated with the martingale solution of

(9). Computing the entropy of $\tilde{\nu}^u$ relative to μ^u, we find that

$$H(\tilde{\nu}^u|\mu^u) = (1/2)\int_X \tilde{\varphi}(\xi)d\nu^u(\xi) \quad (19)$$

where

$$\tilde{\varphi}(x) \equiv \int_I \| B^{-1}(x(t))\tilde{f}(x(t)) \|_H^2 \, dt \ , x \in X \equiv C(I,H).$$

By our assumption (16), it follows from (19) that

$$H(\tilde{\nu}^u|\mu^u) \leq (1/2)\beta^2 T \equiv r. \quad (20)$$

Since (20) holds for all selections $\tilde{f} \in \mathcal{S}_{\tilde{F}}$, and every admissible control u, we conclude that $H(\tilde{\nu}^u|\mu^u) \leq r$ for all $u \in \mathcal{U}_{ad}$. •

Pay-Off Functional. Let (U,d) be a compact metric space and $B_\rho(H,U)$ denote the class of Borel measurable functions from H to U furnished with the metric topology

$$\rho(f,g) \equiv \sup_{x\in H} d(f(x),g(x)), f,g \in B_\rho(H,U).$$

Let $\mathcal{U}_o$ denote the class of measurable functions on I with values in the metric space $B_\rho(H,U)$. We may introduce a metric on it as follows.

$$\gamma(u,v) \equiv \lambda\{t \in I, \rho(u_t,v_t) \neq 0\},$$

where λ denotes the Lebesgue measure on I. With respect to this topology, $(\mathcal{U}_0,\gamma)$ is a complete metric space. Let $\mathcal{U}_{ad} \subset \mathcal{U}_0$ denote the class of admissible controls assumed to be compact with respect to the γ topology. Thus this class of controls represent a class of compact Markovian feedback controls. Now consider the classical control problem (P_c) : Find a control $u^o \in \mathcal{U}_{ad}$ at which the following inf-sup is attained

$$J^o \equiv \inf_{u\in\mathcal{U}_{ad}} \sup_{\nu\in\mathcal{M}_{\tilde{F},u}} E^\nu\left(\int_I \ell(t,x,u)dt + \Psi(x(T))\right), \quad (21)$$

where $E^\nu(\cdot)$ denotes integration with respect to the measure ν, a martingale solution of (15) corresponding to a given Markovian control u and a given measurable selection $\tilde{f} \in \mathcal{S}_{\tilde{F}}$. Defining

$$\psi_u(x) \equiv \int_I \ell(t,x(t),u(t,x(t)))dt + \Psi(x(T)),$$

our problem can be compactly presented as follows: Find $u \in \mathcal{U}_{ad}$ that minimizes the functional $J^o(u)$ given by

$$J^o(u) \equiv \sup_{\nu\in\mathcal{M}_{\tilde{F},u}} \int_X \psi_u(x)d\nu(x). \quad (22)$$

In other words,

$$J^o \equiv \inf_{u\in\mathcal{U}_{ad}} J^o(u). \quad (23)$$

Clearly this is a particular case of the general problem studied in section II. In view of the discussions of section III, this problem can be formulated as the original problem studied in section II. For this problem one has to choose $\eta, \mathcal{M}_0$, and the multifunction $\mathcal{A}_r$ as given below:

$$\begin{aligned}
&\eta(\nu) \equiv \int_X \psi_u(x)d\nu(x), \nu \in \mathcal{M}_{\tilde{F},u} \\
&\mathcal{M}_0 \equiv \{\mu \in \mathcal{M}_1(X) : d\mu = gd\mu^o, g \in \mathcal{D}_{rnd}\} \\
&\mathcal{A}_r(\mu) \equiv \{\nu \in \mathcal{M}_1(X) : H(\nu|\mu) \leq (1/2)\beta^2 T \equiv r\}, \\
&\mu \in \mathcal{M}_0.
\end{aligned} \quad (24)$$

The min-max problem (P_c) has a solution as stated in the following corollary.

Corollary 3.5 Consider the control problem (P_c) for the uncertain system described by the SDI(stochastic differential inclusion) (15) and suppose the assumptions of Theorem 3.3 and 3.4 hold and that ℓ and Ψ are bounded away from $-\infty$. Let $\{\eta, \mathcal{M}_0, \mathcal{A}_r\}$ be as described by (24). Then the problem (P_c) has a min-max strategy $u^o \in \mathcal{U}_{ad}$.

Proof. The proof is similar to that of Theorem 2.3.

The Worst Case Measure. This section is concerned with reformulating the constrained optimization of (22), as an unconstrained optimization using duality theory (Lagrange functionals), and then showing the equivalence of the two problems.
For every $s \in \Re$, define the Lagrangian

$$J^{s,r}(u,\nu^u) = E^{\nu^u}(\psi_u(x)) - s\Big(H(\nu^u|\mu^u) - r\Big), \quad s \in \Re$$

and its associated dual functional

$$J^{s,r}(u,\nu^{*,u}) = \sup_{\nu^u\in\mathcal{M}_1(X)} J^{s,r}(u,\nu^u) \quad (25)$$

In addition, define the quantity

$$\varphi^{s^*}(u,r) = \inf_{s\geq 0} J^{s,r}(u,\nu^{*,u}) \quad (26)$$

Next, the equivalence between the unconstrained functional and the constrained problem is established using [8], pp. 224-225.

Theorem 3.6 Suppose $u \in \mathcal{U}_{ad}$ is given, $\psi_u(x)$ a measurable function bounded from below.
Then

$$\begin{aligned}
J(u,\nu^{*,u}) &= \sup_{\left\{\nu^u\in\mathcal{M}_1(X); H(\nu^u|\mu^u)\leq r\right\}} \int_X \psi_u(x)d\nu^u \\
&= \inf_{s\in\Re} \sup_{\nu^u\in\mathcal{M}_1(X)} \left\{E^{\nu^u}\psi_u - s\Big(H(\nu^u|\mu^u) - r\Big)\right\} \\
&= \inf_{s\in\Re} J^{s,r}(u,\nu^{*,u})
\end{aligned} \quad (27)$$

Proof. Follows from [8], page 224-225.

Lemma 3.7 For a given $u \in \mathcal{U}_{ad}$, for some $s \in \Re$ such that $\frac{\psi_u(x)}{s}$ a measurable function bounded from below, $e^{\frac{\psi_u(x)}{s}} \in L_1(\mu^u)$, $H(\nu^u|\mu^u) < \infty$ the following statements hold.

1) The dual functional $J^{s,r}(u,\nu^{*,u})$ is related to the cumulant generating function of $\psi_u(x)$ with respect to $\mu^u \in \mathcal{M}_1(X)$ via

$$J^{s,r}(u,\nu^{*,u}) = s \sup_{\left\{\nu^u \in \mathcal{M}_1(X); H(\nu^u|\mu^u)<\infty\right\}} \left\{\frac{1}{s}\int_X \psi_u(x)d\nu^u - H(\nu^u|\mu^u)\right\} + sr$$
$$= s\log\int_X e^{\frac{\psi_u(x)}{s}}d\mu^u + sr = s\Psi_{\mu^u}(\frac{1}{s}) + sr \quad (28)$$

Moreover, if $\psi_u(x)e^{\frac{\psi_u(x)}{s}} \in L_1(\mu^u)$ the supremum in is attained at $\nu^{*,u} \in \mathcal{M}_1(X)$ and it is given by

$$d\nu^{*,u} = \frac{e^{\frac{\psi_u(x)}{s}}d\mu^u}{\int_X e^{\frac{\psi_u(x)}{s}}d\mu^u} \quad (29)$$

In addition,

$$\int_X \psi_u(x)d\nu^{*,u}(x) = s\log\int_X e^{\frac{\psi_u(x)}{s}}d\mu^u$$
$$+sH(\nu^{*,u}|\mu), \quad s\in(0,\infty) \quad (30)$$

2) If for any $\eta > 0$, $\psi_u(x)e^{\eta\psi_u(x)} \in L_1(\mu^u)$ and $(\psi_u(x))^2e^{\eta\psi_u(x)} \in L_1(\mu^u)$ then the infimum of the functional $J^{s,r}(u,\nu^{*,u})$ over $s>0$ is uniquely attained at

$$H(\nu^{*,u}|\mu^u)|_{s=s^*} = r \quad (31)$$

Moreover,

$$\frac{d}{ds}s\log\int_\Sigma e^{\frac{\psi_u(x)}{s}}d\mu^u = \log\int_X e^{\frac{\psi_u(x)}{s}}d\mu^u$$
$$-\frac{1}{s}E^{\nu^{*,u}}\{\psi_u(x)\} = -H(\nu^{*,u}|\mu^u) \quad (32)$$

3) Under the assumptions of 2), the relative entropy $H(\nu^{*,u}|\mu^u)$ is a non-increasing function of $s>0$, that is,

$$0 \le H(\nu^{*,u}|\mu^u)|_{s=s_2} \le H(\nu^{*,u}|\mu^u)|_{s=s_1}$$
$$\le H(\nu^{*,u}|\mu^u)|_{s=s^*} = r, \quad 0<s^*\le s_1\le s_2 \quad (33)$$

Proof. This is similar to the one found in [7].

Evolution of the Worst Case Measure. As stated above, in the linear functional case the worst case measure is given by

$$d\nu^{u,*} = \frac{exp\{\frac{1}{s}\psi_u\}d\mu^u.}{\int_X exp\{\frac{1}{s}\psi_u\}d\mu^u}$$

However, this measure is defined on the path space $C(I,H)\equiv X$.

It will be desirable to have this measure defined on H. This is considered in [9], where it is shown that

$$\sup_{\nu^u\in M_1(X)}\left\{\int_X \psi_u(x)d\nu^u - s\Big(H(\nu^u|\mu^u)-r\Big)\right\}$$
$$= \sup_{\chi^u\in M_1(H)}\left\{\int_H \Psi(z)d\chi^u(T,z|x)\right.$$
$$\left.-s\Big(H(\chi^u(T,z|x)|\pi^u(T,z|x))-R\Big)\right\}$$

where R is a real number which depends on the control and $\{\pi^u_t(\phi)(x); 0\le s\le T\}$ is a solution of the evolution equation

$$\pi^u_t(\phi)(x) = \pi^u_s(\phi)(x) + \int_s^t \pi^u_\tau(L(u(\tau))\phi)(x)d\tau$$
$$+\frac{1}{s}\int_s^t \pi^u_\tau(\ell(u(\tau))\phi)(x)d\tau \quad (34)$$

where $L(u)$ is the operator associated with the nominal model.

Hence, we observe that the problem can be transformed into an uncertain problem in the space on measures on H, in which the nominal measure is the conditional measure on H, namely, $\pi^u(T,z|x)$ which satisfies (34).

Clearly, again by the duality, the worst case measure is given by

$$d\chi^{u,*}(T,z|x) = \frac{exp\{\frac{1}{s}\Psi(z)\}d\pi^u(T,z|x).}{\int_H exp\{\frac{1}{s}\Psi(z)\}d\pi^u(T,z|x)}$$

Additional details and results are given in [9].

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