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Coprime factorizations and stabilizability of infinite-dimensional linear systems

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Abstract— **It has been known that a matrix-valued transfer function is dynamically stabilizable iff it has a doubly coprime factorization. We extend this to operator-valued functions and also to controllers with internal loop. We then present several other equivalent conditions, such as having a stabilizable and detectable realization. Our results lead to the extension of the classical results on dynamic stabilization and dynamic partial stabilization to general proper operator-valued functions. Part of the results are new even for scalar-valued functions.**

I. INTRODUCTION

A proper rational matrix-valued function P has the following properties (among others):

(i) P has a stabilizing dynamic controller.
(ii) P has a doubly coprime factorization.
(iii) P has a right coprime factorization.
(iv) P has a stabilizable and detectable realization.
(v) The LQR Riccati equation and its dual equation for some realization of P have nonnegative solutions.

(The definitions will be given later in this article.) Not all meromorphic functions on $\mathbb{C}_0^+$ have these properties.

We show that these conditions and some others are equivalent for general operator-valued transfer functions.

However, (v) will not be treated explicitly here.

A. Notation

U, X, Y:	Hilbert spaces of arbitrary dimensions.
$\mathcal{B}(U,Y)$:	Bounded linear maps $U \to Y$.
$\mathbb{R}_+$:	$\{t \in \mathbb{R} \,\big|\, t \geq 0\}$; $\mathbb{R}_- := \{t \in \mathbb{R} \,\big|\, t \leq 0\}$.
$\mathbb{C}_\omega^+$:	$\{s \in \mathbb{C} \,\big|\, \operatorname{Re} s > \omega\}$.
$\mathrm{H}_\omega^\infty(U,Y)$:	the set of bounded holomorphic functions $\mathbb{C}_\omega^+ \to \mathcal{B}(U,Y)$.
$\mathrm{H}_\omega^2(U)$:	The Hilbert space of holomorphic functions $h : \mathbb{C}_\omega^+ \to U$ for which.

$$\|h\|_{\mathrm{H}_\omega^2} := \sup_{r>\omega} \int_{-\infty}^{\infty} \|h(r+it)\|_U^2 \, dt < \infty.$$

$\mathrm{H}^2 := \mathrm{H}_0^2$: $\quad \mathrm{H}^\infty := \mathrm{H}_0^\infty$, $\mathrm{H}_\omega^\infty(U) := \mathrm{H}_\omega^\infty(U,U)$.

By a *proper function* we mean a bounded holomorphic operator-valued function on some right half-plane ($\mathbb{C}_\omega^+$). We denote this class by $\mathrm{H}_\infty^\infty := \cup_{\omega \in \mathbb{R}} \mathrm{H}_\omega^\infty$.

The ones in H^∞ are called *stable*.

We identify a holomorphic function on a right half-plane $\mathbb{C}_\omega^+$ with its restriction to any open subset of $\mathbb{C}_\omega^+$.

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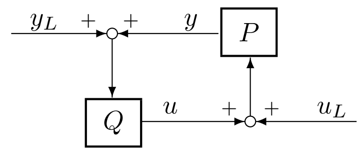


Fig. 1. Controller Q for the plant P

II. DYNAMIC STABILIZABILITY

Before going into infinite-dimensional realizations, we show that a function has a stabilizing controller iff it has a coprime factorization.

We say that $f, g \in \mathrm{H}^\infty$ are *r.c.* (right coprime) if $Gg - Ff \equiv I$ on $\mathbb{C}_0^+$ for some $G, F \in \mathrm{H}^\infty$, i.e., if $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ is left-invertible. If P is a $\mathcal{B}(U,Y)$-valued function defined on a neighborhood of $\alpha \in \mathbb{C}_0^+$, then we say that P has a *r.c.f.* (right coprime factorization) iff $P = fg^{-1}$, where $f, g \in \mathrm{H}^\infty$ are r.c. and $g(\alpha)^{-1}$ exists ($\in \mathcal{B}(\mathtt{U})$).

Similarly, $P = \tilde{g}^{-1}\tilde{f}$ is a *l.c.f.* if $\tilde{g}\tilde{G} - \tilde{f}\tilde{F} \equiv I$ for some $\tilde{g}, \tilde{f}, \tilde{G}, \tilde{F} \in \mathrm{H}^\infty$, $\tilde{g}(\alpha)^{-1}$ exists and $P = \tilde{g}^{-1}\tilde{f}$. Given a r.c.f. and a l.c.f., we can redefine $\tilde{G}$ and $\tilde{F}$ so as to have

$$\begin{bmatrix} g & F \\ f & G \end{bmatrix} \begin{bmatrix} \tilde{G} & -\tilde{F} \\ -\tilde{f} & \tilde{g} \end{bmatrix} = \begin{bmatrix} I_U & 0 \\ 0 & I_Y \end{bmatrix} = \begin{bmatrix} \tilde{G} & -\tilde{F} \\ -\tilde{f} & \tilde{g} \end{bmatrix} \begin{bmatrix} g & F \\ f & G \end{bmatrix}. \tag{1}$$

We call (1) a *d.c.f.* (doubly coprime factorization) of P.

If P and Q are holomorphic functions on a neighborhood of some $\alpha \in \mathbb{C}_0^+$ and $\left[\begin{smallmatrix} I & -P \\ -Q & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty$ (this means that there exists $E \in \mathrm{H}^\infty(Y \times U)$ such that $E\left[\begin{smallmatrix} I & -P \\ -Q & I \end{smallmatrix}\right] = I = \left[\begin{smallmatrix} I & -P \\ -Q & I \end{smallmatrix}\right]E$ near α), then we say that P is *dynamically stabilizable* and that *Q is a stabilizing controller for P*.

This is the case iff the maps $\left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in Figure 1 are stable (i.e., H^2 is mapped into H^2).

We now extend a result that is well known in the matrix-valued case [Vid85] [Smi89]:

Theorem II-A: A function is dynamically stabilizable iff it has a d.c.f.

Further equivalent conditions will be given in Theorem V-A.

If P is matrix-valued (i.e., $\dim U, \dim Y < \infty$) and has an r.c.f., then it can be stabilized by some $Q \in \mathrm{H}^\infty$, by [Qua04].

III. WELL-POSED LINEAR SYSTEMS

We have now treated the conditions (i)–(iii) to some extent. The remaining two conditions and some others are given in terms of realizations. This means that we interpret the given function P as the transfer function of a system. To cover all proper functions, the realizations must be taken from a very general class of infinite-dimensional systems.

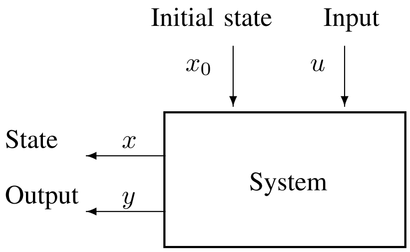


Fig. 2. Input, state and output of a system

Our choice is the Salamon–Weiss class, or the class of well-posed linear systems (WPLSs, or abstract linear systems), developed in, e.g., [Sal87] [Sal89] [Wei94b] [Wei94a]. The readers not interested in state-space results may proceed directly to Section V.

These systems are generalizations of the following type of systems:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{cases} \qquad (t \geq 0) \tag{2}$$
$$x(0) = x_0.$$

In the simplest possible case, the *generators* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(X \times U, X \times Y)$ are bounded linear operators between the Hilbert spaces U, X and Y. Given an *initial state* $x_0 \in X$ and an *input* $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; U)$, the *state* trajectory $x : \mathbb{R}_+ \to X$ and the *output* $y : \mathbb{R}_+ \to Y$ of the system (2) are obviously given by

$$\begin{cases} x(t) &= \mathscr{A}^t x_0 + \mathscr{B}^t u \\ y &= \mathscr{C} x_0 + \mathscr{D} u, \end{cases} \qquad \text{where} \tag{3}$$

$$\begin{aligned} \mathscr{A}^t &= \mathrm{e}^{At}, & \mathscr{B}^t u &= \int_0^t \mathscr{A}^{t-s} B u(s)\, ds, \\ \mathscr{C} x_0 &= C \mathscr{A}^{\cdot} x_0, & \mathscr{D} u &= C \mathscr{B}^{\cdot} u + Du. \end{aligned} \tag{4}$$

It is possible to extend the above theory to the case where A is a generator of a strongly continuous semigroup $\mathscr{A}$, and $B : U \to (\mathrm{Dom}(A^*))^*$ and $C : \mathrm{Dom}(A) \to Y$.

In this setting we can, e.g., require u to be smooth and allow y to be a distribution.

An important special case is the one where y is a function and the maps $x_0, u \mapsto x(t), y$ are continuous in $X \times \mathrm{L}^2_{\mathrm{loc}}$, for some (hence all) $t > 0$, equivalently, that

$$\|x(t)\|_X^2 + \int_0^t \|y(s)\|_Y^2\, ds \leq K_t\big(\|x_0\|_X^2 + \int_0^t \|u(s)\|_U^2\, ds\big)$$

for some (equivalently, all) $t > 0$, where K_t does not depend on x_0 nor u. Such systems are called **well-posed**.

Throughout this article, all systems are assumed to be well-posed.

Set $\mathrm{L}^2_\omega := \mathrm{e}^{\cdot\omega}\mathrm{L}^2$, with $\|u\|^2_{\mathrm{L}^2_\omega} := \int_{\mathbb{R}} \|u(t)\|_U^2 \mathrm{e}^{-2\omega t}\, dt$; $\pi_+ f := \chi_{\mathbb{R}_+} f$, $\pi_- f := \chi_{\mathbb{R}_-} f$, $(\tau^t f)(r) := f(r+t)$.

An equivalent definition of WPLSs is as follows:

Definition III-A: Let $\omega \in \mathbb{R}$. An *ω-stable well-posed linear system (WPLS) on* (U, X, Y) is a quadruple $\Sigma = \left[\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array}\right]$, where $\mathscr{A}^t$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ are bounded linear operators of the following type:

(1.) $\mathscr{A}^t : X \to X$ is a strongly continuous semigroup of bounded linear operators on X satisfying $\sup_{t \in \mathbb{R}_+} \|\mathrm{e}^{-\omega t}\mathscr{A}^t\| < \infty$;
(2.) $\mathscr{B} : \mathrm{L}^2_\omega(\mathbb{R}; U) \to X$ satisfies $\mathscr{A}^t \mathscr{B} u = \mathscr{B} \tau^t \pi_- u$ for all $u \in \mathrm{L}^2_\omega(\mathbb{R}; U)$ and $t \in \mathbb{R}_+$;
(3.) $\mathscr{C} : X \to \mathrm{L}^2_\omega(\mathbb{R}; Y)$ satisfies $\mathscr{C}\mathscr{A}^t x = \pi_+ \tau^t \mathscr{C} x$ for all $x \in X$ and $t \in \mathbb{R}_+$;
(4.) $\mathscr{D} : \mathrm{L}^2_\omega(\mathbb{R}; U) \to \mathrm{L}^2_\omega(\mathbb{R}; Y)$ satisfies $\tau^t \mathscr{D} u = \mathscr{D} \tau^t u$, $\pi_- \mathscr{D} \pi_+ u = 0$, and $\pi_+ \mathscr{D} \pi_- u = \mathscr{C}\mathscr{B} u$ for all $u \in \mathrm{L}^2_\omega(\mathbb{R}; U)$ and $t \in \mathbb{R}$.

(The above $\mathscr{B}$ corresponds to the $\mathscr{B}^t : u_{|[0,t]} \mapsto x(t)$ in (3)–(4) through $\mathscr{B}^t := \mathscr{B}\tau^t\pi_+ = \mathscr{B}\tau^t\pi_{[0,t]}$. By x and y we always denote the state and output, through (3).)

By $\widehat{u}$ we denote the *Laplace transform* of u:

$$\widehat{u}(s) := \int_{\mathbb{R}} \mathrm{e}^{-st} u(t)\, dt \qquad (s \in \mathbb{C}^+_\omega). \tag{5}$$

The Laplace transform is an isometric (modulo $\sqrt{2\pi}$) isomorphism of $\mathrm{L}^2_\omega := \mathrm{e}^{\cdot\omega}\mathrm{L}^2$ onto H^2_ω.

Proposition III-B: Let $\omega \in \mathbb{R}$. For any ω-stable WPLS $\left[\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array}\right]$ there exists a unique *transfer function* $\hat{\mathscr{D}} \in \mathrm{H}^\infty_\omega$ s.t. $\widehat{\mathscr{D}u} = \hat{\mathscr{D}}\widehat{u}$ on $\mathbb{C}^+_\omega$ for any $u \in \mathrm{L}^2_\omega(\mathbb{R}_+; U)$.

Conversely, any $\hat{\mathscr{D}} \in \mathrm{H}^\infty_\omega(U, Y)$ is the transfer function of some ω-stable WPLS (which is called the *realization* of $\hat{\mathscr{D}}$).

If $B \in \mathcal{B}(U, X)$ or $C \in \mathcal{B}(X, Y)$, then

$$\hat{\mathscr{D}}(s) = D + C(s - A)^{-1}B. \tag{6}$$

By the resolvent equation, we get

$$\hat{\mathscr{D}}(s) = \hat{\mathscr{D}}(z) + (z - s)C(s - A)^{-1}(z - A)^{-1}B. \tag{7}$$

The formula (7) is valid for arbitrary WPLSs, so a WPLS can be defined by giving suitable A, B, C and $\hat{\mathscr{D}}(z)$ for some z in a suitable right half-plane.

Proposition III-B explains why WPLSs are the correct choice of realizations for our purposes. Another important property is that the dual system of a WPLS is also a WPLS. The *dual* system means the one generated by $\left(\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right)$ in place of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ (if B or C is bounded; similarly in the general case).

The WPLSs have also been called abstract linear systems or Salamon–Weiss systems. The Lax–Phillips scattering systems [LP67] [AN96] and the continuous-time version of the operator-based models of Béla Sz.-Nagy and Ciprian Foiaş [SF70] can be interpreted as special cases of WPLSs (see [Sta05, Sections 2.7 and 2.9 and Chapter 11]).

IV. STATE FEEDBACK

Bounded state feedback means that we use $u(t) = Kx(t)$ as the input (for some $K \in \mathcal{B}(X, U)$). Substituted into (2), this leads to the *closed-loop system*

$$\dot{x}(t) = (A + BK)x(t), \qquad y(t) = (C + DK)x(t), \tag{8}$$

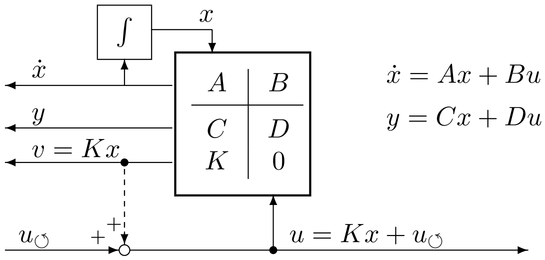


Fig. 3. State-feedback connection

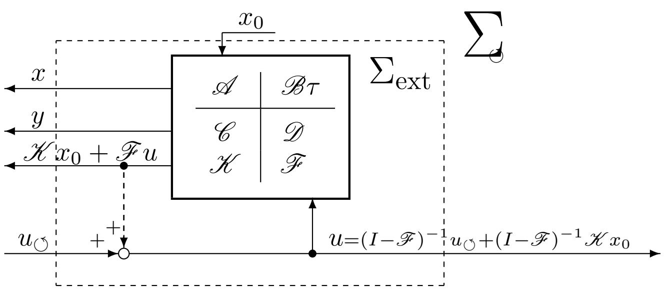


Fig. 4. State-feedback connection for a general WPLS

or, if we allow an *external input* $u_\circlearrowleft$, i.e., $u(t) = Kx(t) + u_\circlearrowleft(t)$, as in Figure 3, this leads to

$$\begin{cases} \dot{x}(t) = (A + BK)x(t) + Bu_\circlearrowleft(t), \\ y(t) = (C + DK)x(t) + Du_\circlearrowleft(t). \end{cases} \tag{9}$$

Thus, the generators $\left(\begin{array}{c|c} A & B \\ \hline C & D \\ K & 0 \end{array}\right)$ and $\left(\begin{array}{c|c} A+BK & B \\ \hline C+DK & D \\ K & 0 \end{array}\right)$, determine the open-loop and closed-loop systems, respectively, if we take into account the feedback signal $Kx(t)$.

So state feedback means adding an extra output to the system and feeding that output back to the input (like $v(t) = Kx(t)$ in Figure 3).

For a general WPLS, the definition is the same: a state feedback means a pair $[\ \mathscr{K} \mid \mathscr{F}\ ]$ such that the extended system Σ_{ext} in Figure 4 is a WPLS. (By (7), it suffices to give a suitable $K \in \mathcal{B}(\mathrm{Dom}(A), U)$ and $\hat{\mathscr{F}}(z) \in \mathcal{B}(U)$.)

The state feedback is called *admissible* if $(I - \hat{\mathscr{F}})^{-1}$ exists and is proper, or equivalently, if the map $(I - \mathscr{F}) : u \mapsto u_\circlearrowleft$ has a bounded and causal inverse on L^2_ω for some $\omega \in \mathbb{R}$. This means that the (closed) state-feedback loop is well-posed under external disturbance.

Any $K \in \mathcal{B}(X, U)$ determines an admissible state feedback (with $\hat{\mathscr{F}}(s) := K(s - A)^{-1}B$, so that $v(t) = Kx(t) + 0u(t)$), but so do also some unbounded operators.

An admissible state feedback is called *stabilizing* if the resulting closed-loop system is stable, i.e., if there exists $M < \infty$ such that for each $x_0 \in X$ and $u_\circlearrowleft \in \mathrm{L}^2$ the state x and output $\left[\begin{smallmatrix} y \\ v \end{smallmatrix}\right]$ of $\Sigma_\circlearrowleft$ satisfy

$$\|x(t)\|_X + \left\| \left[\begin{smallmatrix} y \\ v \end{smallmatrix}\right] \right\|_2 \le M \left(\|x_0\|_X + \|u_\circlearrowleft\|_2\right) \quad (t \ge 0). \tag{10}$$

If $\dim X < \infty$, then an equivalent condition is that $\sigma(A + BK)$ is contained in the open left half-plane $\mathbb{C}^- := \{\mathrm{Re}\, s < 0\}$. If there exists a stabilizing state feedback, then we call the system *stabilizable*. We call the system *detectable* if its dual system is stabilizable.

An admissible state-feedback pair is called *output-stabilizing* if there exists $M < \infty$ such that

$$\left\| \left[\begin{smallmatrix} y \\ v \end{smallmatrix}\right] \right\|_2 \le M\|x_0\|_X \quad (x_0 \in X) \tag{11}$$

(when $u_\circlearrowleft = 0$; note that then $u = v$). Obviously, a necessary condition for this is the *output-FCC* (output-Finite Cost Condition): For each $x_0 \in X$, there exists $u \in \mathrm{L}^2(\mathbb{R}_+; U)$ such that $y \in \mathrm{L}^2(\mathbb{R}_+; Y)$. That is, some *stable* (i.e., L^2) input makes the output stable. This condition is also necessary:

Theorem IV-A: There exists an output-stabilizing state-feedback pair iff the output-FCC holds.

If we drop the admissibility requirement, then Theorem IV-A can be found in [Zwa96] (or already in [FLT88] for systems having a bounded output operator $C \in \mathcal{B}(X, Y)$). As such, the theorem was proved in [Mik05c], where it was also shown that if the output-FCC holds, we can actually satisfy

$$\left\| \left[\begin{smallmatrix} y \\ v \end{smallmatrix}\right] \right\|_2 \le M(\|x_0\|_X + \|u_\circlearrowleft\|_2) \quad (x_0 \in X,\ u_\circlearrowleft \in \mathrm{L}^2). \tag{12}$$

Moreover, the state-feedback can be chosen so that, in addition, the map (say, $\left[\begin{smallmatrix} \mathscr{R} \\ \mathscr{S} \end{smallmatrix}\right]$) $u_\circlearrowleft \mapsto \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ is *weakly right coprime (w.r.c.)*, which means that $u_\circlearrowleft \in \mathrm{L}^2 \Leftrightarrow \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathrm{L}^2$ for any $u_\circlearrowleft \in \mathrm{L}^2_\omega(\mathbb{R}_+; U)$, $\omega \in \mathbb{R}$. (This defines a w.r.c.f. $\mathscr{R}\mathscr{S}^{-1} : u \mapsto y$ of the transfer function; even in the scalar case it need not have a r.c.f. For rational functions, w.r.c. and r.c. are equivalent. Our definition of "w.r.c." is equivalent to the standard matrix-valued-case definition "no common (square) right factors" used in [Fuh81] and [Smi89], by [Mik05e].)

V. Equivalent conditions

In Theorem V-A we list several equivalent conditions for a proper function. The terminology will be explained later in this section (see [Mik05a] or [Mik02] for more details or further equivalent conditions).

Theorem V-A (D.c.f.⇔...): The following are equivalent for any $P \in \mathrm{H}^\infty_\infty(U, Y)$:

- (dcf) P has a d.c.f.
- (rcf) P has a r.c.f. or a l.c.f.
- (SC) P has a stabilizing controller.
- (CC) P has a stabilizing canonical controller.
- (IL) P has a stabilizing controller with internal loop.
- ($\left[\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right]$) $\left[\begin{smallmatrix} P & 0 \\ 0 & I_Z \end{smallmatrix}\right]$ has a d.c.f. for some Hilbert space Z.
- (FCC) P has a realization Σ s.t. the output-FCC holds for Σ and for its dual system.
- (s&d) P has a stabilizable and detectable realization.
- (j.s&d) P has a jointly stabilizable and detectable realization.

Condition (IL) is equivalent to the following:

- (ILreal) P has a realization that is stabilizable by a controller (system) with internal loop.

The analogous variant of (CC) is also equivalent.

If $\dim U, \dim Y < \infty$, then "with internal loop" can be removed from (ILreal), and any of the above conditions is equivalent to the following *Corona condition*:

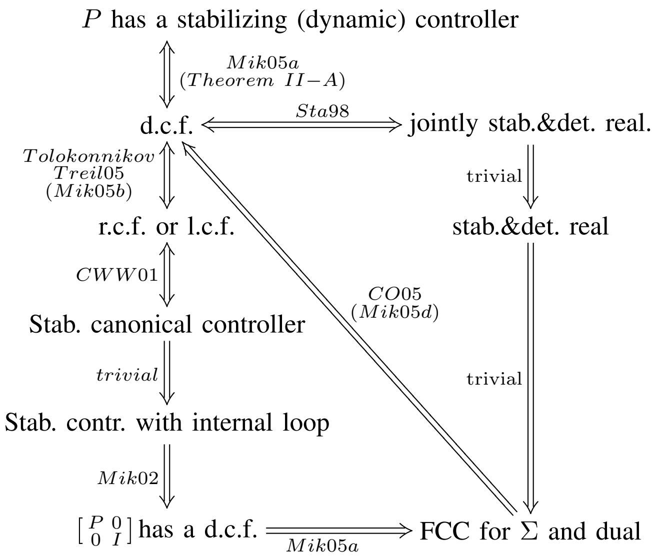
(C) $P = FG^{-1}$ with $F, G \in \mathrm{H}^\infty$, $F^*F + G^*G \geq \epsilon I$ on $\mathbb{C}_0^+$ for some $\epsilon > 0$, and $\det G \not\equiv 0$.

Given a d.c.f. $\begin{bmatrix} g & F \\ f & G \end{bmatrix} = \begin{bmatrix} \tilde{G} & -\tilde{F} \\ -\tilde{f} & \tilde{g} \end{bmatrix}^{-1} \in \mathrm{H}^\infty(U \times Y)$ of P, all stabilizing controllers with internal loop for P are obtained from the standard Youla parameterization

$$(F + gJ)(G + fJ)^{-1} \quad (J \in \mathrm{H}^\infty(Y, U)). \qquad (13)$$

(The controller is proper iff $(G + fJ)^{-1} \in \mathrm{H}^\infty_\infty(U)$.)

We list below the previously known and the newly established implications between the conditions (this is also a sketch of the proof):



The implication from a r.c.f. to a d.c.f. is from [Tol81] in the matrix-valued case and [Tre04] in the (separable) infinite-dimensional case (the nonseparable case from [Mik05b]). Implication (FCC)⇒d.c.f. is from [CO05] (in the separable case under the assumption that $\sigma(A)$ does not contain the imaginary axis; the general case from [Mik05d]). Carleson's Corona condition (C) was extended to the matrix-valued case in [Fuh68]. The claim on (ILreal) is from [Mik02], and the Youla parameterization is well known. On the other implications the above chart is self-explanatory.

Moreover, a stabilizing controller with internal loop is a stabilizing controller (without internal loop) for $\left[\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right]$, which thus has a d.c.f., by Theorem II-A. Using [Sta98], we obtain a jointly stabilizable realization of $\left[\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right]$, and the top-left part of this realization satisfies (FCC). The result in [CO05] is based on showing that the canonical normalized factorization of P has a Hankel norm strictly less than one, which implies left invertibility, i.e., r.c. (The standard d.c.f. formula method cannot be used, as explained below "joint stabilizability and detectability" below.)

From Theorem V-A it easily follows that if the function $\left[\begin{smallmatrix} P & 0 \\ 0 & I_Z \end{smallmatrix}\right]$ has a d.c.f. for some Z, then it actually has a d.c.f. for any Z.

We call a function R a *stabilizing controller with internal loop* for P if $R \in \mathrm{H}^\infty_\infty(Y \times \Xi, U \times \Xi)$ for some Hilbert space Ξ and $(I - S)^{-1} \in \mathrm{H}^\infty$, where $S = \left[\begin{smallmatrix} 0 & R_{11} & R_{12} \\ P & 0 & 0 \\ 0 & R_{21} & R_{22} \end{smallmatrix}\right]$. Note from

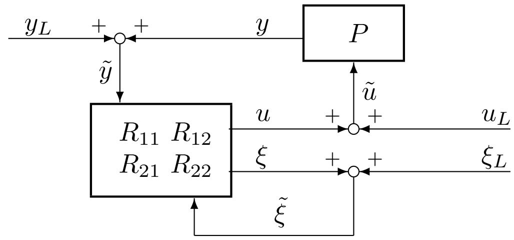


Fig. 5. Controller R with internal loop for $P \in \mathrm{H}^\infty_\infty(U, Y)$

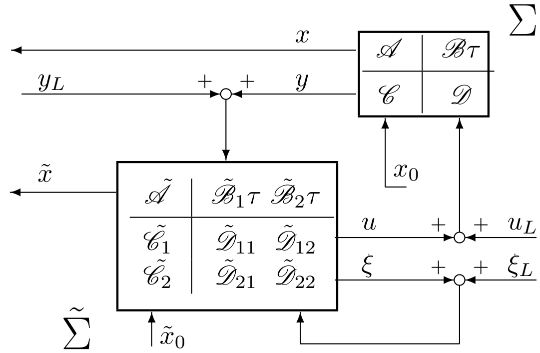


Fig. 6. Controller $\tilde{\Sigma}$ with internal loop for the WPLS Σ

Figure 5 that $(I - S)^{-1} - I$ maps $\left[\begin{smallmatrix} u_L \\ y_L \\ \xi_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \\ \xi \end{smallmatrix}\right]$. Thus, R is stabilizing iff the maps $\left[\begin{smallmatrix} u_L \\ y_L \\ \xi_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \\ \xi \end{smallmatrix}\right]$ are well-posed and stable. [WC97] [CWW01] An equivalent condition is that $\left[\begin{smallmatrix} I & -P_0 \\ -R & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty(Y \times \Xi \times U \times \Xi)$, where $P_0 := \left[\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right]$. [Mik02]

The equivalent condition (ILreal) is formally stronger: it means (the existence of a realization Σ for this fixed P and of another system $\tilde{\Sigma}$ such) that all 25 maps from initial states and external inputs to states and outputs in Figure 6 are stable. (It obviously follows that the transfer function of $\tilde{\Sigma}$ is a stabilizing controller for P with internal loop.) See Section 7.2 of [Mik02] for further details.

If $F \in \mathrm{H}^\infty(Y, U)$ and $G \in \mathrm{H}^\infty(U)$ are r.c., then $R := \left[\begin{smallmatrix} 0 & F \\ I & I-G \end{smallmatrix}\right]$ is called a (right) *canonical controller* (see [CWW01] or [Mik02]). Sometimes we denote it by FG^{-1}, as in the Youla parameterization above.

Two controllers with internal loop for P are considered *equivalent* if the closed-loop signals $\left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ are the same (but those from ξ_L or to ξ may differ).

The Youla parameterization covers all stabilizing controllers in the sense that the others are equivalent to exactly one of those given by the parameterization.

A controller R with internal loop is equivalent to a controller (without internal loop, i.e., to a $\mathcal{B}(Y, U)$-valued function) iff $(I - R_{22})^{-1}$ exists somewhere. If so, then it is equivalent to $Q = R_{11} + R_{12}(I - R_{22})^{-1}R_{21} \in \mathrm{H}^\infty_\infty(Y, U)$. Similarly, R is equivalent to a proper controller iff $(I - R_{22})^{-1}$ is proper. Even in the scalar case, some stabilizing controllers with internal loop do not correspond to any function ($I - R_{22}$ is nowhere invertible), and such controllers have practical engineering applications ("short circuit stabilization") [CWW01]. However, one can always choose the J in (13) so that the controller becomes a function (or $G + fJ$ becomes invertible near a predetermined point of $\mathbb{C}_0^+$ where also P is defined). It is not known whether the controller can always be made proper (in the matrix-valued case the answer is positive [Qua04], as mentioned above).

In Section IV we explained that if the state-space is finite-dimensional, then stabilizing state feedback means an operator $K \in \mathcal{B}(X, U)$ such that $\sigma(A + BK) \subset \mathbb{C}^-$. Similarly, then detectability means the existence of $T \in \mathcal{B}(Y, X)$ such that $\sigma(A+TC) \subset \mathbb{C}^-$. An equivalent condition is that when we connect the second output to the first input or the first input to the second output in the augmented system $\Sigma_{\text{Joint}} := \left(\begin{array}{c|cc} A & B & T \\ \hline C & D & 0 \\ K & 0 & 0 \end{array}\right)$, the resulting closed-loop system becomes stable. We call this *joint stabilizability and detectability*, and the I/O maps of the resulting two closed-loop systems (plus identities) are the inverses of each other; this produces a d.c.f. [Sta98] [WC97].

When $\dim X = \infty$ and K and T are stabilizing and detecting, respectively, the situation is otherwise the same but the "mixed terms" (e.g., $T(\cdot - A - BK)^{-1}K$) in the I/O maps mentioned above need not be stable (not even be proper if both T and K are unbounded). Thus, joint stabilizability and detectability is strictly stronger than stabilizability and detectability.

The factorizations (and controllers) presented above can be found by solving corresponding algebraic or integral Riccati equations [Mik05d] [Mik05c]; see [CO05] and [Mik02, Theorem 7.3.12(c)] for simpler constructive formulas in certain special cases.

VI. Further results and notes

All positive results mentioned above and below are true also in their obvious "discrete-time" forms.

Sometimes one wants to stabilize a system dynamically through *partial feedback* (measurement-feedback), where the controller can measure only to a part of the input and affect only a part of the output. The standard results, as presented in, e.g., [Fra87] for finite-dimensional systems (or rational transfer functions), can be extended to the infinite-dimensional setting using the above results.

Naturally, all above results have analogies for exponential stabilization (and exponential coprime factorizations: the maps belong to H^∞_ω for some $\omega < 0$). Moreover, the results lead to many other implications between different forms of stabilizability, detectability and dynamic stabilizability (including external and exponential stabilizability) for a fixed system.

Weak coprimeness ("quasi-coprimeness") was presented in [Mik02], motivated by the fact that w.r.c. output-stabilization is the weakest form of stabilization that allows the reduction of optimal control problems to the output-stable case. By the comments below Theorem IV-A, this approach applies to any typical solvable control problem.

As shown in [Mik05d] and [Mik05c], weak coprimeness is in many ways a more natural (direct) extension of coprimeness to irrational functions than the (Bézout) one used in this article. E.g., the following are equivalent for any proper function P:

(i') P has a weakly right coprime factorization.
(ii') $P = fg^{-1}$ with $f, g \in \mathrm{H}^\infty$ and g^{-1} proper.
(iii') P has a stabilizable realization.
(iv') P has an output-stabilizable realization.
(v') A certain integral LQR Riccati equation for some realization of P has a nonnegative solution.
(vi') A certain algebraic LQR Riccati equation for some realization of P has a nonnegative solution.

(From (iii') and (iv) it follows that also the condition (v) in the introduction is equivalent to (i)–(iv).)

However, not every P satisfying (i')–(vi') has an r.c.f. For those that do, every w.r.c.f. is an r.c.f. Naturally, not all proper functions satisfy even (i')–(vi') (e.g., $P(s) = (s-1)^{-1/2}$).

Weak coprimeness also plays a key role in the proofs of Theorem II-A, which in turn was needed for Theorem V-A.

The definitions are mainly from [Sta98] (or [Sta05]), but equivalent definitions are wide-spread, most of them originating in [Sal87], [Sal89], [Wei94b], [Wei94a] etc. Otherwise the sources of previously known results have been explained above. See, e.g., [Mik05c], [WC97] and [Mik02] for further historical notes.

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