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Delay-dependent Robust Resilient H_∞ Control for Uncertain Singular Time-delay Systems

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Abstract— **In this paper, the delay-dependent robust resilient H_∞ control is investigated for singular time-delay systems with norm-bounded parameter uncertainty. First, a new delay-dependent bounded real lemma (BRL) for the nominal singular time-delay system is established. Then the problem of robust resilient H_∞ control is solved via a resilient controller with respect to additive and multiplicative controller gain variations, which guarantees the closed-loop system to be regular, impulse free, internally stable and satisfies an H_∞ norm condition for all admissible uncertainties. The explicit expression for the desired robust resilient H_∞ controller is deigned by using the LMIs and the cone complementarity linearization iterative algorithm.**

Index Terms — **uncertain singular time-delay systems, resilient H_∞ control, delay-dependent criteria, linear matrix inequality(LMI).**

I. INTRODUCTION

Robust H_∞ control is an important method of disturbance rejection in the field of system control. Many results have been reported about the robust H_∞ control for time-delay systems. Recently, robust H_∞ control for singular time-delay systems has also been considered, see [1]-[2]. However, all the results obtained are delay-independent, so they are likely to be conservative.

On the other hand, it is worth noting that in practice, controllers do have a certain degree of errors due to finite word length in any digital systems, the imprecision inherent in analog systems and need for additional tuning of parameters in the final controller implementation. Such controllers are often termed “fragile”. Hence, it is considered beneficial that the designed controllers should be capable of tolerating some level of controller gain variations. Recently, some efforts have been developed to tackle the non-fragile controller design problem [3]-[5]. To the best of our knowledge, there are no previous results on delay-dependent robust resilient H_∞ control for singular time-delay systems in the literature, which has motivated our research.

In this paper, the problem of delay-dependent robust resilient H_∞ control is discussed for uncertain singular time-delay systems. The considered time-delay is constant but unknown. First, a new delay-dependent bounded real lemma (BRL) for the nominal singular time-delay system is established. Then by using the idea of generalized quadratic stabilization, the sufficient conditions are proposed for the existence of the delay-dependent robust resilient H_∞ controllers respect to additive and multiplicative controller gain variations. And cone complementarity linearization iterative algorithms are given to design the resilient H_∞ controllers such that the resultant closed-loop systems are regular, impulse free, robust internally stable, and satisfy an H_∞ norm condition with a prescribed level.

The notation $L_2[0,\infty)$ represents the space of square-integrable vector functions over $[0,\infty)$, the norm of $f(t)\in L_2[0,\infty)$ is defined as $\| f \|_2 = (\int_0^\infty f^T(t)f(t)dt)^{\frac{1}{2}}$. For a given stable continuous-time transfer function matrix $G(s)$, its H_∞ norm is defined as $\| G(s) \|_\infty = \sup\limits_{\omega\in[0,\infty)} \sigma_{max}(G(j\omega))$. The symbol $*$ will be used in some matrix expressions to induce a symmetric structure, for example, $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following class of uncertain singular time-delay system:

$$\left\{\begin{array}{rcl} E\dot{x}(t) & = & (A+\triangle A)x(t)+(A_\tau+\triangle A_\tau)x(t-\tau) \\ & & +B_1u(t)+(B_2+\triangle B_2)w(t) \\ z(t) & = & (G+\triangle G)x(t)+Du(t) \\ x(t) & = & \phi(t), \quad t\in[-\tau,0] \end{array}\right. \tag{1}$$

where $x(t)\in R^n$ is the state, $u(t)\in R^q$ is the control input, $w(t)\in R^r$ is the disturbance input that belongs to $L_2[0,\infty), w(0)=0, z(t)\in R^s$ is the controlled output. E,A,A_τ,B_1,B_2,G and D are known real constant matrices with appropriate dimensions, $0<\text{rank}E=p<n$. τ is an unknown but constant delay and satisfies $0<\tau\le\tau_m$, $\phi(t)\in C_{n,\tau}$ is a compatible vector valued initial function. $\triangle A,\triangle A_\tau,\triangle B_2,\triangle G$ are unknown time-invariant matrices representing norm-bounded uncertainties which are assumed to be of the following forms:

$$\begin{bmatrix} \triangle A & \triangle A_\tau & \triangle B_2 \\ \triangle G & \star & \star \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} F \begin{bmatrix} E_1 & E_\tau & E_2 \end{bmatrix} \tag{2a}$$

$$F^TF\le I, \;\; F\in R^{i\times j} \tag{2b}$$

where $D_1\in R^{n\times i}$, $D_2\in R^{s\times i}$, $E_1\in R^{j\times n}$, $E_\tau\in R^{j\times n}$, $E_2\in R^{j\times r}$ are known real constant matrices and F is an uncertain real constant matrix.

Our aim is to develop a resilient controller:

$$u(t)=(K+\triangle K)x(t) \tag{3}$$

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such that the closed-loop system constructed by (1)-(3):

$$\begin{cases} E\dot{x}(t) = (A_k + \triangle A_k)x(t) + (A_\tau + \triangle A_\tau)x(t-\tau) \\ \qquad\quad + (B_2 + \triangle B_2)w(t) \\ z(t) = (G_k + \triangle G_k)x(t) \\ x(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{4}$$

for any constant time-delay τ satisfying $0 < \tau \le \tau_m$:

1) is regular, impulse free and robustly internally stable (i. e. the closed-loop system is robustly asymptotically stable when $w(t) \equiv 0$);

2) $\| G_{zw}(s) \|_\infty \le \gamma$, where $G_{zw}(s)$ is the transfer function from the disturbance input $w(t)$ to the controlled output $z(t)$, $\gamma > 0$ is a prescribed constant (that is, under zero initial condition, $\| z \|_2 \le \gamma \| w \|_2$ for all nonzero $w \in L_2[0,\infty)$); where $A_k = A + B_1 K, \triangle A_k = \triangle A + B_1 \triangle K, G_k = G + DK, \triangle G_k = \triangle G + D \triangle K$.

In this case, the controller (3) is called a robust resilient H_∞ controller for the system (1).

Remark 1: About the definition that the singular time-delay system is regular, impulse free and asymptotically stable, it can be referred to [6].

About the controller gain variations, two forms will be considered:

a) Additive controller gain variations:

$$\triangle K = D_3 F_3 E_3, \quad F_3^T F_3 \le I; \tag{5a}$$

b) Multiplicative controller gain variations:

$$\triangle K = D_4 F_4 E_4 K, \quad F_4^T F_4 \le I \tag{5b}$$

where $D_i, i = 3,4$, are known real constant matrices and $F_i, i = 3,4$, are uncertain real constant matrices.

$\triangle A, \triangle A_\tau, \triangle B_2, \triangle G$ and $\triangle K$ are said to be admissible if (2) and (5) are satisfied.

To get the main results of this paper, the following lemmas are needed.

Lemma 1 [6]: The singular time-delay system

$$\begin{cases} E\dot{x}(t) &= Ax(t) + A_\tau x(t-\tau), \\ x(t) &= \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{6}$$

is regular, impulse free and asymptotically stable for any constant delay τ satisfying $0 < \tau \le \tau_m$, if there exist matrices $Q > 0, X \ge 0, Z > 0$ and P, Y satisfying

$$PE = E^T P^T \ge 0,$$

$$\begin{bmatrix} \Gamma & PA_\tau - Y + \tau_m A^T Z A_\tau \\ * & -Q + \tau_m A_\tau^T Z A_\tau \end{bmatrix} < 0,$$

$$\begin{bmatrix} X & Y \\ * & E^T Z E \end{bmatrix} \ge 0,$$

where $\Gamma = A^T P^T + PA + Q + \tau_m X + Y + Y^T + \tau_m A^T Z A$.

Lemma 2 [7]: For any matrices D, E, F with appropriate dimensions and a scalar $\varepsilon > 0$, where F satisfies: $F^T F \le I$, the following inequality holds:

$$DFE + E^T F^T D^T \le \varepsilon^{-1} D D^T + \varepsilon E^T E.$$

III. ROBUST RESILIENT H_∞ CONTROL

First of all, consider the nominal unforced system of (1):

$$\begin{cases} E\dot{x}(t) &= Ax(t) + A_\tau x(t-\tau) + B_2 w(t) \\ z(t) &= Gx(t) \\ x(t) &= \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{7}$$

For delay-free normal systems, the necessary and sufficient condition guaranteeing the H_∞ cost is stated in the well-known bounded real lemma (BRL). In the following lemma, we will present a new delay-dependent BRL for singular time-delay system (7). However, unlike the BRL for delay-free normal systems, the proposed BRL gives only a sufficient condition.

Lemma 3: For some prescribed value $\gamma > 0$, if there exist matrices $Q > 0, X \ge 0, Z > 0$ and P, Y satisfying

$$PE = E^T P^T \ge 0, \tag{8a}$$

$$\begin{bmatrix} A^T P^T + PA + Q + \tau_m X + Y + Y^T + G^T G & PA_\tau - Y & \tau_m A^T Z & PB_2 \\ * & -Q & \tau_m A_\tau^T Z & 0 \\ * & * & -\tau_m Z & \tau_m Z B_2 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{8b}$$

$$\begin{bmatrix} X & Y \\ * & E^T Z E \end{bmatrix} \ge 0, \tag{8c}$$

then the system (7) is regular, impulse free, internally stable and satisfies $\| G_{zw} \|_\infty \le \gamma$ for any $\tau : 0 < \tau \le \tau_m$.

Proof: By (8) and Lemma 1, it is easy to see that the system (7) is regular, impulse free and internally stable. Next to prove $\| G_{zw} \|_\infty \le \gamma$, that is, under the zero initial condition $\phi(t) = 0, t \in [-\tau, 0]$, we have $\| z \|_2 \le \gamma \| w \|_2$, $\forall w \in L_2[0,\infty), w(0) = 0$. Noticing that (E, A) is regular, impulse free and $\phi(t) = 0, t \in [-\tau, 0]$, so there exist nonsingular matrices M, N such that the system (7) r. s. e. (restricted system equivalence) to

$$\begin{cases} \bar{E}\dot{y}(t) &= \bar{A}y(t) + \bar{A}_\tau y(t-\tau) + \bar{B}_2 w(t), \\ z(t) &= \bar{G}y(t), \\ y(t) &= 0, \quad t \in [-\tau, 0] \end{cases} \tag{9}$$

where

$$y(t) = N^{-1}x(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad \bar{E} = MEN = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \tag{10a}$$

$$\bar{A} = MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-p} \end{bmatrix}, \tag{10b}$$

$$\bar{B}_2 = MB_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \quad \bar{G} = GN = \begin{bmatrix} G_1 & G_2 \end{bmatrix}, \tag{10c}$$

$$y_1(t) \in R^p, \quad B_{21} \in R^{p \times q}, \quad G_1 \in R^{s \times p}. \tag{10d}$$

And define matrices $\bar{P}, \bar{Q}, \bar{X}, \bar{Y}, \bar{Z}$ as follows:

$$\bar{P} := N^T P M^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \bar{Q} := N^T Q N = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}, \tag{11a}$$

$$\bar{X} := N^T X N = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}, \bar{Y} := N^T Y N = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}, \quad (11b)$$

$$\bar{Z} := M^{-T} Z M^{-1} = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix}. \quad (11c)$$

It is obvious that the inequalities (8) hold if and only if

$$\bar{P}\bar{E} = \bar{E}^T \bar{P}^T \geq 0, \quad (12a)$$

$$\begin{bmatrix} \bar{A}^T\bar{P}^T + \bar{P}\bar{A} + \bar{Q} + \tau_m \bar{X} + \bar{Y} + \bar{Y}^T + \bar{G}^T\bar{G} & \bar{P}\bar{A}_\tau - \bar{Y} & \tau_m \bar{A}^T \bar{Z} & \bar{P}\bar{B}_2 \\ * & -\bar{Q} & \tau_m \bar{A}_\tau^T \bar{Z} & 0 \\ * & * & -\tau_m \bar{Z} & \tau_m \bar{Z}\bar{B}_2 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad (12b)$$

$$\begin{bmatrix} \bar{X} & \bar{Y} \\ * & \bar{E}^T\bar{Z}\bar{E} \end{bmatrix} \geq 0, \quad (12c)$$

It is easy to prove that the matrix $\bar{P}$ satisfying (12a) is of the form: $\bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}$. Furthermore, we can obtain form Schur complement argument and (12b) that $\bar{P}$ is nonsingular and then $P_{11} > 0$. It follows from (10), (11) and (12c) that

$$\begin{bmatrix} X_{11} & X_{12} & Y_{11} & Y_{12} \\ * & X_{22} & Y_{21} & Y_{22} \\ * & * & Z_{11} & 0 \\ * & * & * & 0 \end{bmatrix} \geq 0, \quad (13)$$

so $Y_{12} = 0, Y_{22} = 0$, i. e.,

$$\bar{Y} = \begin{bmatrix} Y_{11} & 0 \\ Y_{21} & 0 \end{bmatrix}. \quad (14)$$

Define Lyapunov-Krasovskii functional:

$$V(y_t) = y^T(t)\bar{P}\bar{E}y(t) + \int_{t-\tau}^{t} y^T(s)\bar{Q}y(s)ds + \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{y}^T(\alpha)\bar{E}^T\bar{Z}\bar{E}\dot{y}(\alpha)d\alpha d\beta. \quad (15)$$

Using $y_1(t) - y_1(t-\tau) = \int_{t-\tau}^{t} \dot{y}_1(\alpha)d\alpha$, calculating the time-derivative of $V(y_t)$ along with the solution of the system (9) yields:

$$\begin{aligned} &\dot{V}(y_t)\mid_{(9)} + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\ &= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix} \xi(t) \\ &+ \xi^T(t) \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}^T \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \\ &- \begin{bmatrix} y_1(t-\tau) \\ y_2(t-\tau) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \begin{bmatrix} y_1(t-\tau) \\ y_2(t-\tau) \end{bmatrix} \\ &+ \tau \dot{y}^T(t)\bar{E}^T\bar{Z}\bar{E}\dot{y}(t) - \int_{t-\tau}^{t} \dot{y}_1^T(\alpha)Z_{11}\dot{y}_1(\alpha)d\alpha \\ &+ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}^T \begin{bmatrix} G_1^T G_1 & G_1^T G_2 \\ * & G_2^T G_2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \\ &- \gamma^2 w^T(t)w(t) \end{aligned} \quad (16)$$

where

$$\begin{aligned} \xi(t) = &\begin{bmatrix} A_1 + A_{\tau 11} \\ A_{\tau 21} \end{bmatrix} y_1(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} y_2(t) \\ &- \begin{bmatrix} A_{\tau 11} \\ A_{\tau 21} \end{bmatrix} \int_{t-\tau}^{t} \dot{y}_1(\alpha)d\alpha \\ &+ \begin{bmatrix} A_{\tau 12} \\ A_{\tau 22} \end{bmatrix} y_2(t-\tau) + \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} w(t). \end{aligned}$$

From (13) we have

$$\begin{aligned} &-2\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} A_{\tau 11} \\ A_{\tau 21} \end{bmatrix} \int_{t-\tau}^{t} \dot{y}_1(\alpha)d\alpha \\ &= \int_{t-\tau}^{t} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(\alpha) \end{bmatrix}^T \begin{bmatrix} 0 & 0 & \Theta_1 \\ * & 0 & \Theta_2 \\ * & * & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(\alpha) \end{bmatrix} d\alpha \\ &\leq \int_{t-\tau}^{t} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(\alpha) \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} & \Pi_1 \\ * & X_{22} & \Pi_2 \\ * & * & Z_{11} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(\alpha) \end{bmatrix} d\alpha \\ &= \tau \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \\ &\quad + 2\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}^T \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} (y_1(t) - y_1(t-\tau)) \\ &\quad + \int_{t-\tau}^{t} \dot{y}_1^T(\alpha)Z_{11}\dot{y}_1(\alpha)d\alpha, \end{aligned} \quad (17)$$

where

$$\Theta_1 = -P_{11}A_{\tau 11} - P_{12}A_{\tau 21}, \quad \Theta_2 = -P_{22}A_{\tau 21},$$
$$\Pi_1 = Y_{11} + \Theta_1, \quad \Pi_2 = Y_{21} + \Theta_2.$$

So

$$\begin{aligned} &\dot{V}(y_t)\mid_{(9)} + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\ &\leq \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_1(t-\tau) \\ y_2(t-\tau) \\ w(t) \end{bmatrix}^T \Xi \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_1(t-\tau) \\ y_2(t-\tau) \\ w(t) \end{bmatrix} \\ &\quad + \tau_m \dot{y}^T(t)\bar{E}^T\bar{Z}\bar{E}\dot{y}(t) \\ &= \begin{bmatrix} y(t) \\ y(t-\tau) \\ w(t) \end{bmatrix}^T \Pi \begin{bmatrix} y(t) \\ y(t-\tau) \\ w(t) \end{bmatrix} \end{aligned} \quad (18)$$

with

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} \\ * & \Xi_{22} & P_{22}A_{\tau 21} - Y_{21} & P_{22}A_{\tau 22} & P_{22}B_{22} \\ * & * & -Q_{11} & -Q_{12} & 0 \\ * & * & * & -Q_{22} & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}$$

$$
\begin{aligned}
\Xi_{11} &= P_{11}A_1 + A_1^T P_{11} + Q_{11} + \tau_m X_{11} \\
&\quad + Y_{11} + Y_{11}^T + G_1^T G_1, \\
\Xi_{12} &= Y_{21}^T + P_{12} + \tau_m X_{12} + Q_{12} + G_1^T G_2, \\
\Xi_{13} &= P_{11}A_{\tau 11} + P_{12}A_{\tau 21} - Y_{11}, \\
\Xi_{14} &= P_{11}A_{\tau 12} + P_{12}A_{\tau 22}, \\
\Xi_{15} &= P_{11}B_{21} + P_{12}B_{22}, \\
\Xi_{22} &= P_{22} + P_{22}^T + Q_{22} + \tau_m X_{22} + G_2^T G_2,
\end{aligned}
$$

$$
\Pi = \begin{bmatrix} \begin{matrix} \bar{P}\bar{A} + \bar{A}^T\bar{P}^T + \bar{Q} + \tau_m\bar{X} \\ +\bar{Y} + \bar{Y}^T + \bar{G}^T\bar{G} \end{matrix} & \bar{P}\bar{A}_\tau - \bar{Y} & \bar{P}\bar{B}_2 \\ * & -\bar{Q} & 0 \\ * & * & -\gamma^2 I \end{bmatrix}
$$

$$
+ \tau_m \begin{bmatrix} \bar{A}^T \\ \bar{A}_\tau^T \\ \bar{B}_2^T \end{bmatrix} \bar{Z} \begin{bmatrix} \bar{A}^T \\ \bar{A}_\tau^T \\ \bar{B}_2^T \end{bmatrix}^T .
$$

From (12b) and Schur complement argument, it gets $\Pi < 0$. Then integrating from 0 to ∞ on both sides of (18) yields:

$$
\begin{aligned}
&\int_0^\infty \dot{V}(y_t)\mid_{(9)} dt + \int_0^\infty z^T(t)z(t)dt \\
&\quad -\gamma^2 \int_0^\infty w^T(t)w(t)dt \le 0, \qquad (19)
\end{aligned}
$$

that is,

$$
\| z \|_2^2 \le \gamma^2 \| w \|_2^2 + V(y_t)\mid_{t=0} - V(y_t)\mid_{t=\infty} . \qquad (20)
$$

Noticing $V(y_t)\mid_{t=0} = 0, V(y_t)\mid_{t=\infty} \ge 0$, we lead to

$$
\| z \|_2 \le \gamma \| w \|_2, \ \ \forall w \in L_2[0,\infty), \ \ w \neq 0, \qquad (21)
$$

which implies that $\| G_{zw} \|_\infty \le \gamma$ and the proof is completed. ■

Based on Lemma 3, we are now in the position to give the design method for the robust resilient H_∞ controller (3) in the light of two forms of controller gain variations.

For the additive controller gain variations, the uncertain matrix $\triangle A_k$ and $\triangle G_k$ in the system (4) can be written as

$$
\triangle A_k = D_{k1}F_kE_k, \ \ \triangle G_k = D_{k2}F_kE_k, \qquad (22a)
$$

where

$$
D_{k1} = \begin{bmatrix} D_1 & B_1D_3 \end{bmatrix}, \ D_{k2} = \begin{bmatrix} D_2 & DD_3 \end{bmatrix},
$$

$$
E_k = \begin{bmatrix} E_1 \\ E_3 \end{bmatrix}, \ F_k = \begin{bmatrix} F & 0 \\ 0 & F_3 \end{bmatrix}. \qquad (22b)
$$

By Lemma 3 and using a Schur complement argument, it can be obtained that for some prescribed given $\gamma > 0$, if there exist matrices $\bar{Q} > 0, \bar{X} \ge 0, \bar{Z} > 0, \bar{P}, \bar{Y}$ satisfying

$$
\bar{P}E = E^T\bar{P}^T \ge 0, \qquad (23a)
$$

$$
\Omega_1 := \begin{bmatrix} \Upsilon_1 & \bar{P}(A_\tau + \triangle A_\tau) - \bar{Y} & \tau_m(A_k + \triangle A_k)^T\bar{Z} & \bar{P}(B_2 + \triangle B_2) & (G_k + \triangle G_k)^T \\ * & -\bar{Q} & \tau_m(A_\tau + \triangle A_\tau)^T\bar{Z} & 0 & 0 \\ * & * & -\tau_m\bar{Z} & \tau_m\bar{Z}(B_2 + \triangle B_2) & 0 \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \qquad (23b)
$$

$$
\begin{bmatrix} \bar{X} & \bar{Y} \\ * & E^T\bar{Z}E \end{bmatrix} \ge 0, \qquad (23c)
$$

with $\Upsilon_1 = (A_k + \triangle A_k)^T\bar{P}^T + \bar{P}(A_k + \triangle A_k) + \bar{Q} + \tau_m\bar{X} + \bar{Y} + \bar{Y}^T$, then for any $\tau : 0 < \tau \le \tau_m$, the closed-loop system (4) is regular, impulse free, robustly internally stable and satisfies $\| G_{zw} \|_\infty \le \gamma$.

Define

$$
\Omega_2 := \begin{bmatrix} \Upsilon_2 & \bar{P}A_\tau - \bar{Y} & \tau_m A_k^T\bar{Z} & \bar{P}B_2 & G_k^T \\ * & -\bar{Q} & \tau_m A_\tau^T\bar{Z} & 0 & 0 \\ * & * & -\tau_m\bar{Z} & \tau_m\bar{Z}B_2 & 0 \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & -I \end{bmatrix}
$$

with$\Upsilon_2 = A_k^T\bar{P}^T + \bar{P}A_k + \bar{Q} + \tau_m\bar{X} + \bar{Y} + \bar{Y}^T$, then by Lemma 2, we have

$$
\begin{aligned}
&\Omega_1 = \Omega_2 \\
&+ \begin{bmatrix} \bar{P}D_{k1} & \bar{P}D_1 \\ 0 & 0 \\ \tau_m\bar{Z}D_{k1} & \tau_m\bar{Z}D_1 \\ 0 & 0 \\ D_{k2} & 0 \end{bmatrix} \begin{bmatrix} F_k & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} E_k^T & 0 \\ 0 & E_\tau^T \\ 0 & 0 \\ 0 & E_2^T \\ 0 & 0 \end{bmatrix}^T \\
&+ (\begin{bmatrix} \bar{P}D_{k1} & \bar{P}D_1 \\ 0 & 0 \\ \tau_m\bar{Z}D_{k1} & \tau_m\bar{Z}D_1 \\ 0 & 0 \\ D_{k2} & 0 \end{bmatrix} \begin{bmatrix} F_k & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} E_k^T & 0 \\ 0 & E_\tau^T \\ 0 & 0 \\ 0 & E_2^T \\ 0 & 0 \end{bmatrix}^T)^T \\
&\le \Omega_2 + \\
&\varepsilon \begin{bmatrix} \bar{P}D_{k1} & \bar{P}D_1 \\ 0 & 0 \\ \tau_m\bar{Z}D_{k1} & \tau_m\bar{Z}D_1 \\ 0 & 0 \\ D_{k2} & 0 \end{bmatrix} \begin{bmatrix} \bar{P}D_{k1} & \bar{P}D_1 \\ 0 & 0 \\ \tau_m\bar{Z}D_{k1} & \tau_m\bar{Z}D_1 \\ 0 & 0 \\ D_{k2} & 0 \end{bmatrix}^T \\
&+\varepsilon^{-1} \begin{bmatrix} E_k^T & 0 \\ 0 & E_\tau^T \\ 0 & 0 \\ 0 & E_2^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_k^T & 0 \\ 0 & E_\tau^T \\ 0 & 0 \\ 0 & E_2^T \\ 0 & 0 \end{bmatrix}^T
\end{aligned} \qquad (24)
$$

Denote $\tilde{D}\tilde{D}^T := D_1D_1^T + D_{k1}D_{k1}^T$, by Schur complement argument, if there exist matrices $\bar{Q} > 0, \bar{X} \ge 0, \bar{Z} > 0, \bar{P}, \bar{Y}$ and $\varepsilon > 0$ satisfying

$$
\begin{bmatrix} \Upsilon_3 & \bar{P}A_\tau - \bar{Y} & \tau_m A_k^T\bar{Z} + \varepsilon\tau_m\bar{P}\tilde{D}\tilde{D}^T\bar{Z} \\ * & -\bar{Q} & \tau_m A_\tau^T\bar{Z} \\ * & * & -\tau_m\bar{Z} + \varepsilon\tau_m^2\bar{Z}\tilde{D}\tilde{D}^T\bar{Z} \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}
$$

$$
\left.\begin{matrix}
\bar{P}B_2 & G_k^T+\varepsilon\bar{P}D_{k1}D_{k2}^T & E_k^T & 0 \\
0 & 0 & 0 & E_\tau^T \\
\tau_m\bar{Z}B_2 & \varepsilon\tau_m\bar{Z}D_{k1}D_{k2}^T & 0 & 0 \\
-\gamma^2 I & 0 & 0 & E_2^T \\
* & -I+\varepsilon D_{k2}D_{k2}^T & 0 & 0 \\
* & * & -\varepsilon I & 0 \\
* & * & * & -\varepsilon I
\end{matrix}\right] < 0, \tag{25}
$$

then the above $\bar{Q},\bar{X},\bar{Z},\bar{P},\bar{Y}$ are sure to satisfy (23b), where $\Upsilon_3 = \Upsilon_2 + \varepsilon\bar{P}\tilde{D}\tilde{D}^T\bar{P}^T$. Pre-multiplying by $\mathrm{diag}\{\bar{P}^{-1},\bar{P}^{-1},\bar{Z}^{-1},I,I,I,I\}$ and post-multiplying by $\mathrm{diag}\{\bar{P}^{-T},\bar{P}^{-T},\bar{Z}^{-1},I,I,I,I\}$ on both sides of (25), and letting

$$
\begin{aligned}
&P:=\bar{P}^{-1}, Q:=\bar{P}^{-1}\bar{Q}\bar{P}^{-T}, X:=\bar{P}^{-1}\bar{X}\bar{P}^{-T}, \\
&Y:=\bar{P}^{-1}\bar{Y}\bar{P}^{-T}, Z:=\bar{Z}^{-1}, W:=KP^T,
\end{aligned} \tag{26}
$$

then that the inequality

$$
\left[\begin{matrix}
\Upsilon_4 & A_\tau P^T-Y & \Upsilon_5 & B_2 \\
* & -Q & \tau_m PA_\tau^T & 0 \\
* & * & -\tau_m Z+\varepsilon\tau_m^2\tilde{D}\tilde{D}^T & \tau_m B_2 \\
* & * & * & -\gamma^2 I \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
PG^T+W^TD^T+\varepsilon D_{k1}D_{k2}^T & PE_k^T & 0 \\
0 & 0 & PE_\tau^T \\
\varepsilon\tau_m D_{k1}D_{k2}^T & 0 & 0 \\
0 & 0 & E_2^T \\
-I+\varepsilon D_{k2}D_{k2}^T & 0 & 0 \\
* & -\varepsilon I & 0 \\
* & * & -\varepsilon I
\end{matrix}\right] < 0. \tag{27}
$$

with

$$
\begin{aligned}
\Upsilon_4 &= PA^T+AP^T+B_1W+W^TB_1^T+Q+\tau_m X \\
&\quad +Y+Y^T+\varepsilon\tilde{D}\tilde{D}^T, \\
\Upsilon_5 &= \tau_m(PA^T+W^TB_1^T)+\varepsilon\tau_m\tilde{D}\tilde{D}^T
\end{aligned}
$$

has solutions $Q>0, X\geq 0, Z>0, P, Y, W$ and $\varepsilon>0$ is equivalent to that (25) has solutions $\bar{Q}>0, \bar{X}\geq 0, \bar{Z}>0, \bar{P}, \bar{Y}$ and $\varepsilon>0$, where $K=WP^{-T}$.

Analogously, combining with (26), we obtain that (23a) and (23c) have solutions $\bar{X}\geq 0, \bar{Z}>0, \bar{P}, \bar{Y}$ is equivalent to

$$
EP^T = PE^T \geq 0 \tag{28a}
$$

and

$$
\begin{bmatrix} X & Y \\ * & EP^TZ^{-1}PE^T \end{bmatrix} \geq 0 \tag{28b}
$$

have solutions $X\geq 0, Z>0, P, Y$. Thus the following theorem is obtained as:

Theorem 1: Consider the uncertain singular time-delay system (1), if there exist matrices $Q>0, X\geq 0, Z>0, P, Y, W$ and a scalar $\varepsilon>0$ with P nonsingular, satisfying inequalities (27) and (28), then the controller with respect to the additive controller gain variations:

$$
u(t) = (WP^{-T}+D_3F_3E_3)x(t) \tag{29}
$$

is a robust resilient H_∞ controller for the system (1).

For the multiplicative controller gain variations, the uncertain matrices $\triangle A_k$ and $\triangle G_k$ in the system (4) can be written as

$$
\triangle A_k = D_{c1}F_cE_c,\ \ \triangle G_k = D_{c2}F_cE_c, \tag{30a}
$$

where

$$
\begin{aligned}
&D_{c1} = \begin{bmatrix} D_1 & B_1D_4 \end{bmatrix},\ D_{c2} = \begin{bmatrix} D_2 & DD_4 \end{bmatrix}, \\
&E_c = \begin{bmatrix} E_1 \\ E_4K \end{bmatrix},\ F_c = \begin{bmatrix} F & 0 \\ 0 & F_4 \end{bmatrix}.
\end{aligned} \tag{30b}
$$

Denoting $\bar{D}\bar{D}^T := D_1D_1^T + D_{c1}D_{c1}^T$, we have:

Theorem 2: Consider the uncertain singular time-delay system (1), if there exist matrices $Q>0, X\geq 0, Z>0, P, Y, W$ and a scalar $\varepsilon>0$ with P nonsingular satisfying (28) and

$$
\left[\begin{matrix}
\Upsilon_6 & A_\tau P^T-Y & \Upsilon_7 & B_2 \\
* & -Q & \tau_m PA_\tau^T & 0 \\
* & * & -\tau_m Z+\varepsilon\tau_m^2\bar{D}\bar{D}^T & \tau_m B_2 \\
* & * & * & -\gamma^2 I \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
\Upsilon_8 & PE_1^T & W^TE_4^T & 0 \\
0 & 0 & 0 & PE_\tau^T \\
\varepsilon\tau_m D_{c1}D_{c2}^T & 0 & 0 & 0 \\
0 & 0 & 0 & E_2^T \\
-I+\varepsilon D_{c2}D_{c2}^T & 0 & 0 & 0 \\
* & -\varepsilon I & 0 & 0 \\
* & * & -\varepsilon I & 0 \\
* & * & * & -\varepsilon I
\end{matrix}\right] < 0, \tag{31}
$$

with

$$
\begin{aligned}
\Upsilon_6 &= PA^T+AP^T+B_1W+W^TB_1^T+Q+\tau_m X \\
&\quad +Y+Y^T+\varepsilon\bar{D}\bar{D}^T, \\
\Upsilon_7 &= \tau_m(PA^T+W^TB_1^T)+\varepsilon\tau_m\bar{D}\bar{D}^T, \\
\Upsilon_8 &= PG^T+W^TD^T+\varepsilon D_{c1}D_{c2}^T,
\end{aligned}
$$

then the controller with respect to the multiplicative controller gain variations:

$$
u(t) = (I+D_4F_4E_4)WP^{-T}x(t) \tag{32}
$$

is a robust resilient H_∞ controller for the system (1).

It is clear that the nonlinear term $EP^TZ^{-1}PE^T$ in (28b) makes that (28b) is not conformable to a LMI. However, by using the cone complementarity linearization iterative algorithm proposed in [8] by minor modification, we can convert (28b) to solving a sequence of convex optimization problems subject to LMIs.

Without loss of generality, it is assumed that $E = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$, then

$$
P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}, Y = \begin{bmatrix} Y_{11} & 0 \\ Y_{21} & 0 \end{bmatrix}, \tag{33}
$$

where $P_{11} \in R^{p\times p}, Y_{11} \in R^{p\times p}, P_{11} > 0$ and P is nonsingular. Introducing new variables $U > 0, T_{11} > 0, S_{11} > 0$ and $V_{11} > 0$, and denote

$$X = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}, U = \begin{bmatrix} U_{11} & U_{12} \\ * & U_{22} \end{bmatrix} \quad (34)$$

with $X_{11} \in R^{p\times p}, U_{11} \in R^{p\times p}$. Then it is easy to see that if there are solutions $P, X \geq 0, Y, Z > 0, U > 0$ and $V_{11} > 0, S_{11} > 0, T_{11} > 0$ to (33), (34) and

$$UZ = I, \quad (35)$$

$$\begin{bmatrix} X_{11} & X_{12} & Y_{11} \\ * & X_{22} & Y_{21} \\ * & * & T_{11} \end{bmatrix} \geq 0, \quad (36)$$

$$\begin{bmatrix} U_{11} & V_{11} \\ * & S_{11} \end{bmatrix} \geq 0 \quad (37)$$

$$P_{11}V_{11} = I, \quad T_{11}S_{11} = I, \quad (38)$$

then (28b) has solutions X, Y, Z and P. Then the problem of designing the robust resilient H_∞ controller with respect to the additive controller gain variations can be considered as a cone complementary problem subject to LMIs:

$$\text{Minimize } \{\text{tr}(UZ) + \text{tr}(P_{11}V_{11} + T_{11}S_{11})\} \quad (39)$$

subject to

$$\begin{cases} Q > 0, X \geq 0, Z > 0, U > 0, \varepsilon > 0, \\ P_{11} > 0, V_{11} > 0, T_{11} > 0, S_{11} > 0, \\ \begin{bmatrix} U & I \\ I & Z \end{bmatrix} \geq 0, \begin{bmatrix} P_{11} & I \\ I & V_{11} \end{bmatrix} \geq 0, \begin{bmatrix} T_{11} & I \\ I & S_{11} \end{bmatrix} \geq 0 \end{cases} \quad (40)$$

and (27), (33), (34), (36), (37).

Notice that the above problem is a non-convex optimization problem. To solve the problem, we follow the iterative linearization method via minor modification proposed in [8] to solve this problem, and get the algorithm described as follows.

Algorithm 1:

(1) Make singular value decomposition to matrix E: $\bar{U}E\bar{V} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$, where $\bar{U}, \bar{V}$ are orthogonal matrices and $\Sigma \in R^{p\times p}$ is a diagonal matrix. Let $M = \begin{bmatrix} \Sigma^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \bar{U}$ and $N = \bar{V} \begin{bmatrix} \Sigma^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix}$, then $\bar{E} = MEN = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$. Take M, N as the transformation matrices, then system (1) is r. s. e. to:

$$\begin{cases} \bar{E}\dot{\bar{x}}(t) &= (\bar{A} + \triangle\bar{A})\bar{x}(t) + (\bar{A}_\tau + \triangle\bar{A}_\tau)\bar{x}(t-\tau) \\ & \quad + \bar{B}_1 u(t) + (\bar{B}_2 + \triangle\bar{B}_2)w(t), \\ z(t) &= (\bar{G} + \triangle\bar{G})\bar{x}(t) + Du(t), \\ \bar{x}(t) &= N^{-1}\phi(t), \quad t \in [-\tau, 0] \end{cases} \quad (41)$$

where, $\bar{A} = MAN$, $\bar{A}_\tau = MA_\tau N$, $\bar{B}_1 = MB_1$, $\bar{B}_2 = MB_2$, $\bar{G} = GN$, $\triangle\bar{A} = \bar{D}_1 F\bar{E}_1$, $\triangle\bar{A}_\tau = \bar{D}_1 F\bar{E}_\tau$, $\triangle\bar{B}_2 = \bar{D}_1 FE_2$, $\triangle\bar{G} = D_2 F\bar{E}_1$, $\bar{D}_1 = MD_1$, $\bar{E}_1 = E_1 N, \bar{E}_\tau = E_\tau N$. Correspondingly, let $\bar{E}_3 = E_3 N$. For convenience, we still denote $\bar{E}, \bar{A}, \bar{A}_\tau, \bar{B}_1, \bar{B}_2, \bar{G}, \bar{D}_1, \bar{E}_1, \bar{E}_\tau, \bar{E}_3$ as $E, A, A_\tau, B_1, B_2, G, D_1, E_1, E_\tau, E_3$.

(2) For given $\tau_m > 0$, find a feasible set $Q, X, Z, P, Y, W, U, P_{11}, V_{11}, S_{11}, T_{11}, \varepsilon$ satisfying (27), (33), (34), (36), (37) and (40). If there are none, exit. Otherwise set $U^{(0)} = U, Z^{(0)} = Z, P_{11}^{(0)} = P_{11}, V_{11}^{(0)} = V_{11}, T_{11}^{(0)} = T_{11}, S_{11}^{(0)} = S_{11}$, and verify the condition (28b). If (28b) is satisfied, then the robust resilient H_∞ controller is designed as

$$u(t) = (WP^{-T} + D_3F_3E_3)N^{-1}x(t). \quad (42)$$

If (28b) is not satisfied, set the index of the objective function in the next step as $k = 0$ and go to step (3).

(3) Solve the following convex optimization problem for the variables $Q, X, Z, P, Y, W, U, P_{11}, V_{11}, S_{11}, T_{11}$ and ε:

$$\text{Minimize } \{ \text{tr}(U^{(k)}Z + Z^{(k)}U) + \text{tr}(P_{11}^{(k)}V_{11} + V_{11}^{(k)}P_{11} + T_{11}^{(k)}S_{11} + S_{11}^{(k)}T_{11}) \}$$

subject to (27), (33), (34), (36), (37) and (40).

Set $U^{(k+1)} = U, Z^{(k+1)} = Z, P_{11}^{(k+1)} = P_{11}, V_{11}^{(k+1)} = V_{11}, T_{11}^{(k+1)} = T_{11}, S_{11}^{(k+1)} = S_{11}$.

(4) Verify the condition (28b). If condition (28b) is satisfied, then the robust resilient H_∞ controller is designed as (42). If condition (28b) is not satisfied within a specified number of steps of iterations, then exit. Otherwise, set the index k of the objective function in step (3) as $k+1$ and go to step (3).

Remark 2: The algorithm to design the the robust resilient H_∞ controller with respect to the multiplicative controller gain variations can be obtained by minor modification to Algorithm 1.

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