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Sampled-data observer design: A dynamical approach

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***Abstract*— The problem of sampled-data state reconstruction in linear time invariant systems is considered. A new full order observer structure that can generate intersample state estimation is introduced. The observer synthesis is carried out using the H_∞ framework and is shown to have some important advantages over the classical lifting technique.**

I. Introduction

The observer design problem is a very important problem that has various applications such as output feedback control, system monitoring, process identification and fault detection. The classical approach to the solution of this problem is by using the *Luenberger* observer structure [6] in the deterministic case, or the well known *Kalman* filter [4] in the special case of stochastic noise and disturbances. The concept of *unknown input observer* (UIO) has also been introduced by Wang [12] to decouple the effect of an unknown input from the observer error. This approach was later extended in a series of papers [1] to the cases of modeling errors, plant disturbances as well as system faults. Optimization techniques have also been widely used in fault detection observers to minimize the disturbance effect and maximize the fault effect when complete decoupling is not possible [3], [1]. All of these works, however, consider the continuous-time (or the discrete-time) problem in which a continuous (discrete)-time observer is designed to observe the state of a continuous-time (discrete-time) plant. In this paper, our interest is the sampled-data observer (SDO) design problem. It is the problem of reconstructing the states of a continuous-time plant using a discrete-time observer, which can operate with a rate higher than the sample and hold devices connected to the plant. An important advantage of this sampled-data framework is the possibility to provide intersample estimation and, therefore, better piecewise approximate reconstruction of the continuous-time states of interest. Such information is very useful for observer applications but is, however, hard to obtain given the fact that the output information is only available at the slow rate of the sample device.

A classical approach used for the SDO design problem mainly in control applications is the *inferential control* approach where primary measurements of inputs and outputs are used to estimate the effect of secondary measurements (these may include unmeasured states, disturbances, etc.) and then a standard control system is used to adjust the control effort [13]. The most important part of this technique is the design of an estimator that minimizes the estimation error of inferred measurements at fast sampling points where an actual measurement is unavailable [10]. In most cases, however, inferential control methods are restricted to specific types of control schemes or processes. Besides, the issues of practical importance (such as model uncertainty, system dynamics, restrictions on the controller structure) are not incorporated [5].

This encouraged much research to be done in the area of sampled-data control and one of the successful approaches that has been introduced is the *lifting technique* [2], [8]. The main idea of the lifting technique is to generate slow rate control inputs that depend on fast rate information of the reference input, controlling an augmented output which represents the fast rate error signal. Design of the controller can then be done within the multirate digital control framework. This technique can also be used for the dual SDO design problem. However, the lifting technique has important drawbacks such as the increased dimensional complexity and the time lag problem [10]. This can have negative performance implications in real time observer applications.

In this paper, we study the sampled-data observer design problem using a novel approach. An observer design based on the fast rate plant model is introduced. In order to achieve intersample state estimation using this observer, we proceed as follows: two signals are fed to the observer; namely, the plant input and the plant output. The plant input is constant during the intersample, owing to the hold device and is therefore fed to the observer at the fast rate. The output is only available at the sampling instants and is therefore fed at the slow rate of the sample device. To obtain a robust estimate with respect to this unknown intersample output information, we formulate the problem as an H_∞ optimal control problem, making use of the dynamic observer structure introduced in [7]. We then show that the proposed H_∞ problem is equivalent to a modified H_∞ problem which satisfies the standard assumptions. A design procedure solvable using commercially available software is then presented. The proposed H_∞ approach is also compared to the classical lifting approach through simulations and is shown to have some important advantages over the lifting technique when applied to a fast rate fault detection problem.

II. Preliminaries and notation

Our attention is focused on the sampled-data observer design (SDO) design problem for sampled-data systems shown in Fig. 1 where the sample and hold devices are operating with the speed $1/h_s$ (h_s being the sampling time). The objective of the SDO design is to provide state estimates at a faster rate $1/h_f$ using the available input-output information.

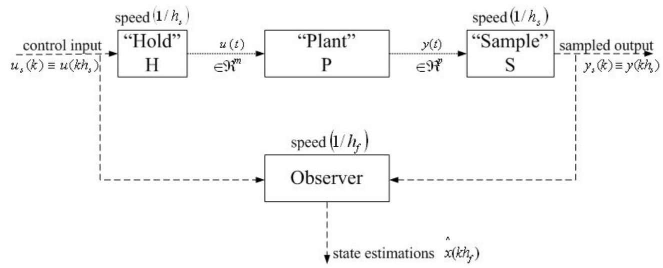

Fig. 1. Sampled-data observer (SDO) design problem.

In the sequel, we consider the model of the plant $\mathbf{P}$ to be a known LTI system $\mathbf{\Sigma}$ of the form (1) with (A, C) observable.

$$\mathbf{\Sigma}: \begin{cases} \dot{x}(t) = A\ x(t) + B\ u(t),\ A \in \mathcal{R}^{n\times n}, B \in \mathcal{R}^{n\times m} \\ y(t) = C\ x(t) + D\ u(t),\ C \in \mathcal{R}^{p\times n}, D \in \mathcal{R}^{p\times m} \end{cases} \tag{1}$$

Knowing h_s and h_f, it follows that the two systems $\mathbf{\Sigma_s}$ and $\mathbf{\Sigma_f}$ (seen as the exact discretizations of $\mathbf{\Sigma}$ at the sampling rates $1/h_s$ and $1/h_f$ respectively) are known and given by the following *step-invariant transformations* [2]:

$$\mathbf{\Sigma_s}: \begin{cases} x_s(k+1) = \ A_s\ x_s(k) + B_s\ u_s(k) \\ y_s(k) = \ C\ x_s(k) + D\ u_s(k) \end{cases} \tag{2}$$

$$\mathbf{\Sigma_f}: \begin{cases} x_f(k+1) = \ A_f\ x_f(k) + B_f\ u_f(k) \\ y_f(k) = \ C\ x_f(k) + D\ u_f(k) \end{cases} \tag{3}$$

where for $i = (s, f)$: $A_i = e^{h_i A},\ B_i = \int_0^{h_i} e^{\tau A} d\tau B$ and where $x_i(k) \triangleq x(kh_i)$, $u_i(k) \triangleq u(kh_i)$ and $y_i(k) \triangleq y(kh_i)$. The sampling times h_s and h_f are assumed to satisfy:

(i) h_f is strictly less than h_s and the ratio between them is an integer number, i.e:

$$r = \frac{h_s}{h_f}, \qquad \text{where } r \in \mathcal{Z}^+ \text{ and } r > 1 \tag{4}$$

(ii) The sampling time h_f is *non-pathological*, i.e, no two eigenvalues of A differ by $(j.k\frac{2\pi}{h_f}),\ \ k \in \mathcal{Z},\ k \neq 0$.

Assumption (i) is a technical assumption that guarantees that the slow rate data is a proper subset of the fast rate data. Assumption (ii) implies that the observability assumption is preserved for the pair (A_f, C) [2].

Luenberger observers for $\mathbf{\Sigma_s}$ and $\mathbf{\Sigma_f}$ will be denoted by $\mathbf{\Psi_s}$ and $\mathbf{\Psi_f}$ respectively and have the following structure [6]:

$$\mathbf{\Psi_i}: \begin{cases} \hat{x}_i(k+1) = A_i\hat{x}_i(k) + B_i u_i(k) + L_i\left[y_i(k) - \hat{y}_i(k)\right] \\ \hat{y}_i(k) = C\hat{x}_i(k) + Du_i(k) \end{cases} \tag{5}$$

where $i = (s, f)$ and where L_i (the observer gain) is a static n by p matrix designed to ensure that all of the eigenvalues of the matrix $(A_i - L_i C)$ lie in the open left half complex plane. Throughout the paper, we will also make use of the following definitions and notations:

Definition 1: ($\mathcal{L}_2$ Space) The space $\mathcal{L}_2$ consists of all $\mathcal{L}$ebesque measurable functions $u : \mathcal{Z}^+ \rightarrow \mathcal{R}^q$, having a finite $\mathcal{L}_2$ norm $\|u\|_{\mathcal{L}_2}$, where $\|u\|_{\mathcal{L}_2} \triangleq \sqrt{\sum_{k=0}^{\infty}\|u(k)\|^2}$, with $\|u(k)\|$ as the *Euclidean* norm of the vector $u(k)$.

For a discrete-time system $\mathbf{G} : \mathcal{L}_2 \rightarrow \mathcal{L}_2$, we will represent by $\gamma(\mathbf{G})$ the $\mathcal{L}_2$ gain of that system given by $\gamma(\mathbf{G}) = \sup_u \frac{\|\mathbf{G}u\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}$. In the case of a linear time-invariant system $\mathbf{G} : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ with a stable transfer matrix $\hat{G}(z)$, $\gamma(\mathbf{G})$ is equivalent to the H_∞ norm of $\hat{G}(z)$ defined as:

$$\gamma(\mathbf{G}) \triangleq \parallel \hat{G}(z) \parallel_\infty = \max_{\theta\in[0,2\pi]} \sigma_{\max}(\hat{G}(e^{-j\theta}))$$

where $\sigma_{\max}(.)$ represents the maximum singular value of $\hat{G}(e^{-j\theta})$. We will use small letters to represent scalar variables and vectors, capital letters for matrices, and capital bold letters for operators and systems. $\mathbf{S}$ and $\mathbf{H}$ will be used to represent the sample and hold operators with slow sampling time h_s, while $\mathbf{S_f}$ and $\mathbf{H_f}$ will be used for the same operators with fast sampling time h_f. We will use the matrices I_n, 0_n and 0_{nm} to represent the identity matrix of order n, the zero square matrix of order n and the zero n by m matrix respectively. The symbol $\hat{T}_{yu}(z)$ represents the *z-transform* transfer matrix from the input u to the output y. The partitioned matrix $\mathbf{K} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ (when used as an operator from u to y, i.e, $y = \mathbf{K}u$) represents the state space representation $(\xi(k+1) = \ A\ \xi(k) + B\ u(k);\ y(k) = \ C\ \xi(k) + D\ u(k))$, and in that case the transfer matrix is $\hat{K}(z) = C(zI - A)^{-1}B + D$. And finally, in all the block diagrams used in this paper, solid and dashed lines represent continuous-time and discrete-time signals respectively.

III. A Lifting formulation for sampled-data observer design

In this section, we formulate the sampled-data observer (SDO) design problem using the classical Lifting technique. We first give a brief introduction to this technique, then we present how it can be used to solve the problem.

A. The Lifting technique

The Lifting technique is one of the classical approaches used in multirate digital control. It relies on the use of a linear, time-varying, non-causal operator $\mathbf{L}_\alpha$, which operating on a discrete signal $v(k) \equiv \{v(0), v(1), \ldots\}$ gives another discrete signal referred to as the lifted signal $\underline{v}(k)$ where:

$$\underline{v}(k) \triangleq \mathbf{L}_\alpha v(k) \equiv \left\{ \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(\alpha-1) \end{bmatrix}, \begin{bmatrix} v(\alpha) \\ v(\alpha+1) \\ \vdots \\ v(2\alpha-1) \end{bmatrix}, \ldots \right\} \tag{6}$$

Here $\alpha \in \mathcal{Z}^+$ is referred to as the Lifting order. The Lifting operator transforms a fast rate signal into a slow rate signal that contains the same information. This is clear by noting that if $v(k)$ is a discrete-time signal of vectors of order "q" sampled every "h" seconds, $\underline{v}(k)$ can be considered as a signal of vectors of order "αq" (sampled every "αh" seconds) storing the same information in $v(k)$. Hence, the two signals $x_s(k) \in \mathcal{R}^n$ and $x_f(k) \in \mathcal{R}^n$ (defined in (2) and (3)) are two discrete signals of different sampling times (h_s and h_f respectively), while $x_s(k) \in \mathcal{R}^n$ and $\underline{x_f}(k) \in \mathcal{R}^{rn}$

($\triangleq \mathbf{L_r} x_f(k)$) have the same sampling time and are related by: $x_s(k) = \begin{bmatrix} I_n & 0_n & \dots & 0_n \end{bmatrix}_{1\times r} \underline{x_f}(k)$. The inverse of $\mathbf{L}_\alpha$ is denoted by $\mathbf{L}_\alpha{}^{-1}$ and is also a linear, time varying (but causal) operator. Both $\mathbf{L}_\alpha$ and $\mathbf{L}_\alpha{}^{-1}$ preserve the $\mathcal{L}_2$ norms [2]. In addition to lifting signals, the Lifting operators are also used to lift systems as follows: consider $\mathbf{G_d}$; a discrete-time, LTI, single rate system (inputs and outputs are discrete signals sampled every "h" seconds) that has n states, m inputs and p outputs, and that is represented as $\mathbf{G_d} \triangleq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where $A,\ B,\ C$ and D are matrices of appropriate dimensions. The lifted system $\underline{\mathbf{G_d}}$ is defined as the system $\underline{\mathbf{G_d}} \triangleq \mathbf{L}_\alpha\ \mathbf{G_d}\ \mathbf{L}_\alpha{}^{-1}$. It can be shown that the representation of $\underline{\mathbf{G_d}}$ is as follows [2]:

$$\underline{\mathbf{G_d}} \equiv \left[\begin{array}{c|ccccc} A^\alpha & A^{\alpha-1}B & A^{\alpha-2}B & \dots & B \\ \hline C & D & 0_{pm} & \dots & 0_{pm} \\ CA & CB & D & \dots & 0_{pm} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{\alpha-1} & CA^{\alpha-2}B & CA^{\alpha-3}B & \dots & D \end{array}\right] \tag{7}$$

The MIMO system $\underline{\mathbf{G_d}}$ is considered as a slow rate representation of $\mathbf{G_d}$. Throughout the paper, we will assume the Lifting order to be the constant r in (4), and we will use $\mathbf{L}$ to refer to $\mathbf{L_r}$. The following Lifting relations will also be used (for proof refer to [2]):

$$\mathbf{S_f}\ \mathbf{H} \equiv \mathbf{L}^{-1}\ \mathbf{Q} \tag{8}$$

where $\mathbf{Q}$ is the static matrix $\begin{bmatrix} I_m & I_m & \dots & I_m \end{bmatrix}^T_{1\times r}$. And,

$$\mathbf{S}\ \mathbf{P}\ \mathbf{H_f} \equiv\ \mathbf{R}\ \mathbf{L}\ \mathbf{S_f}\ \mathbf{P}\ \mathbf{H_f} \tag{9}$$

where $\mathbf{R}$ is the static matrix $\begin{bmatrix} I_p & 0_p & \dots & 0_p \end{bmatrix}_{1\times r}$.

B. Application to the SDO design problem

To solve the SDO design problem introduced in section II and represented by Fig. 1, it is necessary to find a model that captures the fast rate states to be estimated (i.e, $x_f(k)$) and which is also function of an available set of input/output information. The multirate system $\mathbf{SPH_f}$ (mapping the fast rate input $u_f(k)$ into the slow rate output $y_s(k)$ as in (10)) is therefore a possible candidate to solve the problem.

$$y_s(k) = \mathbf{SPH_f}\ u_f(k) \tag{10}$$

To find a model for that system, it is easy to see (by using the notation in (8)) that $u_f(k)$ is related to $u_s(k)$ as:

$$\underline{u_f}(k) = \mathbf{Q}\ u_s(k) \tag{11}$$

To reflect the response of $x_f(k)$, it is important to remark that the model of P in (1) can also be represented as:

$$\mathbf{P} = \mathbf{M_1}\mathbf{P}' + \mathbf{M_2} \tag{12}$$

where $\mathbf{P}' = \left[\begin{array}{c|c} A & B \\ \hline I_n & 0_{nm} \end{array}\right]$, $\mathbf{M_1} = \left[\begin{array}{c|c} 0_n & 0_n \\ \hline 0_{pn} & C \end{array}\right]$ and $\mathbf{M_2} = \left[\begin{array}{c|c} 0_n & 0_{nm} \\ \hline 0_{pn} & D \end{array}\right]$ Therefore, using the Lifting properties introduced in section III-A, along with equations (11) and (12), the multirate system $\mathbf{SPH_f}$ can be represented by the following state space representation (see [9] for more details about the derivation of this model):

$$\xi(k+1) = A_f^r \xi(k) + \begin{bmatrix} A_f^{r-1}B_f & A_f^{r-2}B_f & \dots & B_f \end{bmatrix} \underline{u_f}(k)$$

$$\underline{x_f}(k) = \underbrace{\begin{bmatrix} I_n \\ A_f \\ \vdots \\ A_f^{r-1} \end{bmatrix}}_{\bar{C}} \xi(k) + \underbrace{\begin{bmatrix} 0_{nm} & \dots & 0_{nm} \\ B_f & \ddots & 0_{nm} \\ \vdots & \ddots & \vdots \\ A_f^{r-2}B_f & \dots & 0_{nm} \end{bmatrix}}_{\bar{D}} \underline{u_f}(k) \tag{13}$$

$$y_s(k) = [C\ 0_{pn}\ \dots\ 0_{pn}]\,\underline{x_f}(k) + [D\ 0_{pm}\ \dots\ 0_{pm}]\,\underline{u_f}(k)$$

The following theorem presents an observer for this model and provides conditions for its convergence:

Theorem 1: The system given by the following equations:

$$\begin{aligned} \hat{\xi}(k+1) = A_f^r\ \hat{\xi}(k) + \begin{bmatrix} A_f^{r-1}B_f & A_f^{r-2}B_f & \dots & B_f \end{bmatrix} \underline{u_f}(k) \\ + L_\ell\,(y_s(k) - \hat{y}_s(k)) \end{aligned}$$

$$\underline{\hat{x}_f}(k) = \bar{C}\ \hat{\xi}(k) + \bar{D}\ \underline{u_f}(k) \tag{14}$$

$$\hat{y}_s(k) = [\,C\ \ 0_{pn}\ \dots\ 0_{pn}\,]\,\underline{x_f}(k) + [D\ 0_{pm}\ \dots\ 0_{pm}]\,\underline{u_f}(k)$$

is a SDO for the system in Fig. 1 if and only if the observer gain L_ℓ is chosen such that $(A_f^r - L_\ell C)$ is Hurwitz.

Proof: By defining the error variables as $e_x = \underline{x_f}(k) - \underline{\hat{x}_f}(k)$ and $e_\xi = \xi(k) - \hat{\xi}(k)$, we have:

$$e_x(k) = \bar{C}\ e_\xi(k)$$

But using (13) and (14) we have:

$$\begin{aligned} e_\xi(k+1) &= A_f^r\ \xi(k) - \left(A_f^r\ \hat{\xi}(k) + L_\ell\,(y_s(k) - \hat{y}_s(k))\right) \\ &= A_f^r\ e_\xi(k) - L_\ell\left(\begin{bmatrix} C & 0_{pn} & \dots & 0_{pn} \end{bmatrix}\ e_x(k)\right) \\ &= \left(A_f^r - L_\ell C\right) e_\xi(k) \end{aligned}$$

Therefore, $(A_f^r - L_\ell C)$ Hurwitz is necessary and sufficient for the error $e_\xi(k)$ to converge to zero. It follows that (14) is a SDO for the system in Fig. 1 with $\underline{\hat{x}_f}(k)$ as the required fast rate state estimation. △

Remarks:

1) A necessary and sufficient condition for arbitrary pole placement of the SDO in (14) is the observability of the pair (A_f^r, C). This is not guaranteed by the "non-pathological" assumption on h_f in section II.
2) The observer has a time delay of h_s. This is clear by noting that $\underline{\hat{x}_f}(0)$ is based on the initial guess for $\hat{\xi}(0)$. The correction term $(y_s(k) - \hat{y}_s(k))$ has effect on $\underline{\hat{x}_f}(k)$ only starting from $k = 1$.
3) The observer developed in this section is equivalent to two observers performing in parallel: Ψ_s in (5) as a slow rate closed loop observer having $L_s \equiv L_\ell$, and Ψ_f (also given in (5) but with $L_f = 0_{np}$) as an open loop observer updating its initial conditions every "r" steps with the new state of Ψ_s. This is similar to the filter structure in [11].

IV. H_∞ SAMPLED-DATA OBSERVER DESIGN

To avoid the drawbacks of the Lifting technique, a direct use of the fast rate model in (3) to design a SDO is necessary. However, any Luenberger observer for this model (such as $\mathbf{\Psi_f}$ in (5)) is not a feasible solution due to the unavailable output information $y_f(k)$. And if $y_f(k)$ is replaced by $\tilde{y}_f(k)$:

$$\tilde{y}_f(k) = y_f(k) + d(k) \tag{15}$$

(which is an arbitrary approximation to $y_f(k)$ with an error vector $d(k)$), then the observer $\mathbf{\Psi_f}$ has an estimation error $e = x_f - \hat{x}_f$ with dynamics given from:

$$e(k+1) = (A_f - L_f C)\ e(k) - L_f\ d(k) \tag{16}$$

which is affected by $d(k)$ causing divergence of the observer. In this section, we solve the SDO design problem by using a dynamic structure (instead of (5)). The idea is to replace the static observer gain L_f by a dynamic controller, and we here show how this dynamic controller can be used to minimize the effect of $d(k)$ on e. Towards that goal, the proposed dynamical observer for the fast rate model (3) is:

$$\hat{x}_f(k+1) = A_f\ \hat{x}_f(k) + B_f\ u_f(k) + \eta(k) \tag{17}$$

$$\hat{y}_f(k) = C\ \hat{x}_f(k) + D\ u_f(k) \tag{18}$$

where $\eta(k)$ is obtained as follows:

$$z(k+1) = A_L\ z(k) + B_L\ (\tilde{y}_f(k) - \hat{y}_f(k)) \tag{19}$$

$$\eta(k) = C_L\ z(k) + D_L\ (\tilde{y}_f(k) - \hat{y}_f(k)) \tag{20}$$

with A_L, B_L, C_L, D_L having appropriate dimensions, and where $\tilde{y}_f(k)$ is an approximation to $y_f(k)$ with an error vector $d(k)$ as given in (15). We will also write

$$\mathbf{K} = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right] \tag{21}$$

to represent the compensator in (19)-(20). It is easy to see that the error dynamics in (16) is now given by

$$e(k+1) = A_f\ e(k) - \eta(k) \tag{22}$$

$$\eta(k) = \mathbf{K}\ \ (Ce(k) + d(k)) \tag{23}$$

which can also be represented by the standard setup in Fig. 2

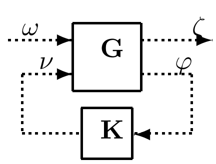

Fig. 2. Standard setup.

having the variables in (24), with controller $\mathbf{K}$ in (21), and plant $\mathbf{G}$ as the standard state space representation in (25).

$$\omega = d(k),\ \zeta = \ e(k) = x_f(k) - \hat{x}_f(k)$$

$$\nu = \eta(k),\ \varphi = \ Ce(k) + d(k) \tag{24}$$

$$\mathbf{G} \triangleq \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|cc} A_f & 0_{np} & -I_n \\ \hline I_n & 0_{np} & 0_n \\ C & I_p & 0_{pn} \end{array}\right] \tag{25}$$

Therefore, the SDO design problem reduces to the *input/output* stability problem of the setup in Fig. 2 which has as input $d(k)$ and as output $e(k)$. With an arbitrary choice for $\tilde{y}_f(k)$ in (15), one can ensure that $d(k)$ is a bounded signal and the problem in Fig. 2 can then be solved as an $\mathcal{L}_1$ optimization problem. However, we here focus on the use of H_∞ optimization assuming $d(k)$ to be of finite energy (i.e, $d(k) \in \mathcal{L}_2$). Unfortunately, the SDO design problem cannot be carried out directly using the standard H_∞ solution since the standard form in (25) does not satisfy all of the regularity assumptions in the H_∞ framework.

A. Problem regularization

By adding a "weighted" disturbance term in the state equation of the fast rate model (3), now we tackle the problem of designing an observer for the following system:

$$x_f(k+1) = A_f\ x_f(k) + B_f\ u_f(k) + \epsilon\ \phi(k),\ \ \epsilon > 0 \tag{26}$$

$$y_f(k) = C\ x_f(k) + D\ u_f(k) \tag{27}$$

where the vector $\phi(k)$ is a disturbance term. Using the same observer defined by (17)-(21), the observer error dynamics can still be represented by the setup in Fig. 2 with the same variables in (24), except for replacing ω by $\bar{\omega}$ defined as:

$$\bar{\omega} \triangleq \begin{bmatrix}\phi(k) & d(k)\end{bmatrix}^T \tag{28}$$

and redefining the plant $\mathbf{G}$ as:

$$\mathbf{G} \triangleq \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|cc} A_f & \begin{bmatrix}\epsilon I_n & 0_{np}\end{bmatrix} & -I_n \\ \hline I_n & \begin{bmatrix}0_n & 0_{np}\end{bmatrix} & 0_n \\ C & \begin{bmatrix}0_{pn} & I_p\end{bmatrix} & 0_{pn} \end{array}\right] \tag{29}$$

This standard form, however, still does not satisfy the regularity assumptions in the H_∞ problem. Fortunately, regularization can be done by extending the external output ζ to include the "weighted" vector $\beta\eta(k)$, $\beta > 0$. This adds another change in Fig. 2 by replacing ζ by $\bar{\zeta}$ defined as:

$$\bar{\zeta} = \begin{bmatrix}e(k) & \beta\eta(k)\end{bmatrix}^T \tag{30}$$

The plant $\mathbf{G}$ is then given by

$$\mathbf{G} \triangleq \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|cc} A_f & \begin{bmatrix}\epsilon I_n & 0_{np}\end{bmatrix} & -I_n \\ \hline \begin{bmatrix}I_n \\ 0_n\end{bmatrix} & \begin{bmatrix}0_n & 0_{np} \\ 0_n & 0_{np}\end{bmatrix} & \begin{bmatrix}0_n \\ \beta I_n\end{bmatrix} \\ C & \begin{bmatrix}0_{pn} & I_p\end{bmatrix} & 0_{pn} \end{array}\right] \tag{31}$$

It follows that all of the regularity assumptions summarized below [2], [14] are now satisfied:

1) (A,B_2) stabilizable: satisfied for any matrix A.
   (C_2,A) detectable: satisfied iff (A_f, C) is detectable.
2) $D_{21}D_{21}^T = I_p$, which is nonsingular.
   $D_{12}^T D_{12} = \beta^2 I_n$, which is nonsingular.
3) rank $\begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix} = 2n =$ full column rank $\forall\lambda$.
   rank $\begin{bmatrix} A - \lambda I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p =$ full row rank $\forall\lambda$.
4) $D_{22} = 0$.

Therefore, all the regularity assumptions are satisfied iff the pair (A_f, C) is detectable.

B. Proof of equivalence

Let T_1 be the setup in Fig. 2 associated with (25), T_2 the one associated with (29) and T_3 the one associated with (31) where the three share the same controller K in (21). And let $\hat{T}_1(z)$, $\hat{T}_2(z)$ and $\hat{T}_3(z)$ be their corresponding transfer matrices. The following two lemmas demonstrate a certain equivalence relationships among these setups (the proof is omitted and is included in [9]).

Lemma 1: Consider a stabilizing controller $\mathbf{K}$ for the setups T_1 and T_2, then $\| \hat{T}_1(z) \|_\infty < \gamma$ if and only if $\exists\, \epsilon > 0$ such that $\| \hat{T}_2(z) \|_\infty < \gamma$.

Lemma 2: Given $\epsilon > 0$ and a stabilizing controller $\mathbf{K}$ for the setups T_2 and T_3, then $\| \hat{T}_2(z) \|_\infty < \gamma$ if and only if $\exists\, \beta > 0$ such that $\| \hat{T}_3(z) \|_\infty < \gamma$.

We are now ready to present our main result in the form of a theorem showing that the observer gain $\mathbf{K}$ needed to minimize the energy ($\mathcal{L}_2$ norm) of the estimation error for the SDO in (17)-(21) must solve a regular H_∞ optimal control problem. To this end, we define the regular discrete H_∞ problem "Problem 1" as follows:

Problem 1: *Given $\epsilon > 0$ and $\beta > 0$, find $\mathcal{S}$, the set of admissible controllers $\mathbf{K}$ satisfying $\| \hat{T}_{\zeta\omega}(z) \|_\infty < \gamma$ for the setup in Fig. 2 having the plant $\mathbf{G}$ in (31).*

The main result is summarized in the following theorem:

Theorem 2: Consider the SDO design problem in Fig. 1 with the plant $\mathbf{P}$ in (1) and the fast rate model in (3). Then the following two statements are equivalent:

(i) The observer (17)-(21) with the dynamic gain $\mathbf{K}$ has a minimum estimation error energy.

(ii) $\exists\, \epsilon^* > 0,\ \beta^* > 0$ s.t $\mathbf{K} \in \mathcal{S}^*$ (the set of controllers solving "Problem 1" with the minimum possible γ).

Proof: Since the observer's error dynamics is represented by T_1 (the setup in Fig. 2 associated with (25)), then the estimation error's energy satisfies: $\|e\|_{\mathcal{L}_2} \leq \|\hat{T}_1(z)\|_\infty\ \|d\|_{\mathcal{L}_2}$. Then, $\|e\|_{\mathcal{L}_2}$ is minimized, for a certain disturbance signal $d(k)$, if and only if the controller $\mathbf{K}$ minimizes $\|\hat{T}_1(z)\|_\infty$. The equivalence of the two statements then follows as a direct result of Lemma 1 and Lemma 2. $\triangle$

C. A new H_∞ design procedure

The following iterative "binary search" procedure is then proposed to evaluate the observer gain $\mathbf{K}$:

Design procedure:

Step 1 Set γ_{low} to an arbitrary small positive value and γ_{high} to an arbitrary large positive value.

Step 2 Set $\epsilon > 0$ and $\beta > 0$ and set $\gamma \leftarrow \frac{\gamma_{low}+\gamma_{high}}{2}$.

Step 3 Test solvability of "Problem 1". If test fails then go to Step 5 ; otherwise solve the problem, select any $\mathbf{K} \in \mathcal{S}$ as a candidate observer gain and set $\gamma_{high} \leftarrow \gamma$.

Step 4 If $|\gamma_{high} - \gamma_{low}| <$ a threshold value then *stop the algorithm*, otherwise go back to Step 2.

Step 5 Set $\epsilon \leftarrow \frac{\epsilon}{2}$ and $\beta \leftarrow \frac{\beta}{2}$. If ϵ or $\beta <$ a threshold value then $\gamma_{low} \leftarrow \gamma$ and go to Step 4, otherwise go to Step 3.

Remarks

- The H_∞ design is guaranteed to converge if the pair (A_f, C) is detectable [14]. This condition is guaranteed by the "non-pathological assumption" on h_f.
- The H_∞ SDO does not introduce a time delay.
- The assumption of finite energy is easily satisfied in step tracking applications if $\tilde{y}_f(k)$ is selected as $\tilde{y}_f(k) = y_s(r\ (k\ mod\ r))$ (i.e, approximating the fast rate output as a constant signal between samples).

V. Simulation results

We here consider an illustrative example using the rotary inverted pendulum (*ROTPEN*) shown schematically in Fig. 3. The angle that the perfectly rigid link of length l_1 and inertia J_1 makes with the x-axis of an inertial frame is denoted θ_1 (degrees). Also, the angle of the pendulum (of length l_2 and mass m_2) from the z-axis of the inertial frame is denoted θ_2 (degrees). The *ROTPEN* has a state space model of the form

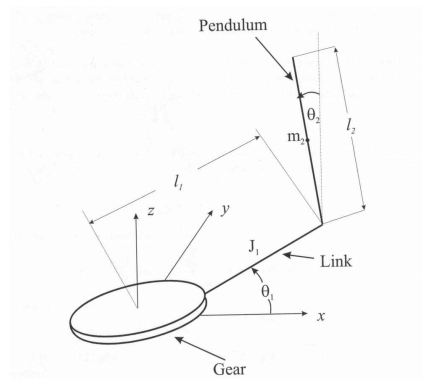

Fig. 3. The Rotary Inverted Pendulum (ROTPEN).

$\dot{x} = f(x)+g(x)u$ where $x = [\theta_1\ \theta_2\ \dot{\theta}_1\ \dot{\theta}_2]^T$ is the state vector and u is the scalar servomotor voltage input (Volt). The output is assumed to be θ_1 (the motor angle), i.e $y = x_1$. The system parameters are: $l_1 = 0.215$ m, $l_2 = 0.335$ m, $m_2 = 0.1246$ Kg and $J_1 = 0.0064$ Kg.m^2. Linearization about the link pendant configuration, i.e the equilibrium point ($\theta_1 =$ constant, $\theta_2 = 180$ degrees *and* $u = 0$) gives:

$$\dot{x}= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -23.9774 & -14.5255 & -0.2777 \\ 0 & -67.0081 & -13.9836 & -0.7760 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1439.3 \\ 1385.6 \end{bmatrix} u \tag{32}$$

This model is open-loop unstable. In our simulation, we first stabilize the system, furthermore the closed-loop poles and the feedforward gain are chosen to make the output track a step input of 10 degrees. The simulation time is taken as 50 sec. The observer design in case of H_∞ was done with the help of the Matlab command *hinfsyn* and using the *Bilinear* transformation approach [2] to get a discrete H_∞ controller. In case of lifting, the command *place* was used to place the *discrete* poles of the observer (14) at $\{0.0183,\ 0.0025, -0.0563 \pm 0.1231j\}$. The value of h_f is fixed to 0.1 sec and h_s is changed to take the values $\{0.2, 0.5, 0.8$ and $1\}$ sec. This represents a study for different values of r in (4). The approximated output $\tilde{y}_f(k)$ is chosen as the held output between samples (i.e, $\tilde{y}_f(k) = y_s(r\ (k\ mod\ r)))$ as shown in Fig. 4(a) for the case h_s=1. The approximation error $d(k)$ in (15) in this case is shown in Fig. 4(b). It can be seen that the disturbance term $d(k)$ is a decaying signal having a finite $\mathcal{L}_2$ norm (note that all $\mathcal{L}_2$ norms are computed for the interval $t = [0,\ 50\ sec]$).

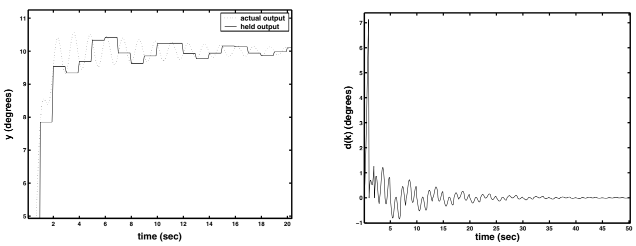

Fig. 4. (a) Actual vs approximated outputs. (b) Disturbance term $d(k)$.

Case study 1: The simulation is conducted to compare the Lifting technique with the H_∞ technique. The observer initial conditions (for both techniques) are taken as $\begin{bmatrix}0.2 & 0 & 0 & 0\end{bmatrix}^T$. Fig. 5 shows the output estimation error for the two cases when $h_s = 1$. Table I shows the trend of state estimation error's $\mathcal{L}_2$ norm with the change of h_s. With the increase of r, the number of inputs and outputs for the system in (13) increases making the lifting technique more complex. It is important to note that for very large values of h_s, the system in (13) could become unobservable making the use of the lifting technique impossible (as is the case for values of $h_s >> 1$). Two factors are important in choosing r: the computer speed to implement the observer in Fig. 1; and the required bound on the disturbance term since the norm of $d(k)$ in (15) increases with the increase of r.

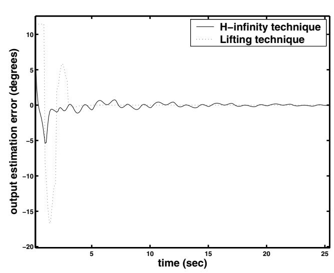

Fig. 5. Output estimation error for Lifting and H_∞ at $h_s = 1$ sec.

TABLE I

CASE STUDY 1 - EFFECT OF CHANGE OF h_s ON $\| e \|_{\mathcal{L}_2}$

Value of h_s (in sec.)	0.2	0.5	0.8	1
H_∞ technique	16.0328	16.9887	18.3506	19.5616
Lifting technique	22.9438	36.9739	46.2796	57.2925

Case study 2: In this case, the simulation shows the application of the Lifting and H_∞ SDOs in the fast rate fault detection problem. In this experiment, h_s is assumed to be 1 sec, and a sensor fault is assumed to start after 20 sec in the form of a small bias of magnitude 1.75 degrees. The residual signal $s(k)$ is taken as the summation of the output estimation error ($r(k)$) over a time window, and a simple decision scheme at step k is assumed as follows:

$$s(k) = \sum_{i=k-4}^{k} |r(k)| \ > \text{threshold} \ \Rightarrow \ \text{fault detected} \quad (33)$$

Applying both techniques (with the threshold as 2.0), the fault in H_∞ is detected at time = 20.1 sec, while in Lifting it is detected at time = 22 sec. The residual signals are shown in Fig. 6. This case study demonstrates that the proposed H_∞ observer scheme can provide updated residual signal in fast rate without introducing much time delay.

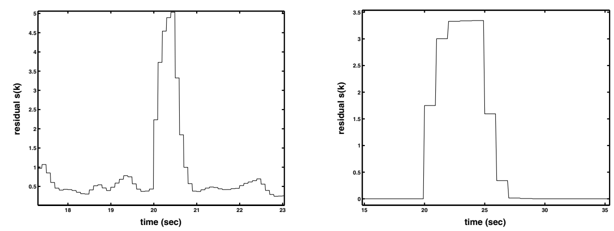

Fig. 6. (a) Residual for H_∞ technique. (b) Residual for Lifting technique.

VI. CONCLUSION

We considered the problem of sampled-data state reconstruction in LTI systems. An observer structure that generates intersample state estimations is introduced and the problem is shown to be equivalent to a standard H_∞ problem. A design algorithm to solve this problem is presented and can be carried out using commercially available software, such as MATLAB. The proposed H_∞ design has some important advantages over the classical lifting technique.

REFERENCES

[1] J. Chen, R. J. Patton, Robust model-based fault diagnosis for dynamic systems, Kluwer Academic Publishers, 1999.

[2] T. Chen, B. A. Francis, Optimal Sampled-data control systems, Springer, London, 1995.

[3] S. X. Ding, T. Jeinsch, P. M. Frank, E. L. Ding, "A unified approach to the optimization of fault detection systems," *Int. J. of Adaptive Contr. and Signal Process.* , 2000, 14, 725-745.

[4] A. H. Jazwinski, Stochastic processes and filtering theory, Academic press, New York, 1970.

[5] J. H. Lee, M. Morari, "Robust measurement selection," *Automatica* , vol. 27, 519-527, 1991.

[6] D. G. Luenberger, "An introduction to observers," *IEEE Trans. Automat. Contr.* , vol. 16, pp. 596-602, 1971.

[7] H. J. Marquez, "A frequency domain approach to state estimation," *J. of the Franklin Institute* , 340 (2003) 147-157.

[8] D. G. Meyer, "Cost translation and a lifting approach to the multirate LQG problem," *IEEE Trans. Automat. Contr.*, 37, 1411-1415, 1992.

[9] A. M. Pertew, H. J. Marquez and Q. Zhao, "H_∞ optimal sampled-data state observer design," *submitted to the IEE Proceedings Control Theory and Applications* , 2005.

[10] J. A. Rossiter, T. Chen, S. L. Shah, "A comparison of lifting and inferential control for multi-rate systems," *Proc. of the ACC*, Denver, Colorado, June 4-6, 2003.

[11] W. Sun, K. Nagpal, P. Khargonekar, "H_∞ control and filtering for sampled-data systems," *IEEE Trans. Automat. Contr.* , vol. 38, pp. 1162-1175, 1993.

[12] S. H. Wang, E. J. Davison, P. Dorato, "Observing the states of systems with unmeasurable disturbances," *IEEE Trans. Automat. Contr.* , vol. 20, pp. 716-717, 1975.

[13] R. Weber, C. Brosilow, "The use of secondary measurements to improve control," *AIChE J.* , vol. 18, 614-623, 1972.

[14] K. Zhou, J. C. Doyle, Essentials of robust control, Prentice-Hall, englewood Cliffs, NJ, 1998.