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Exact Smoothers for Discrete-Time Hybrid Stochastic Systems

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Abstract— **In this article we compute the exact smoothing algorithm for discrete-time Gauss-Markov models whose parameter-sets switch according to a known Markov law. The smoothing algorithm we present is general, but can be readily configured into any of the three main classes of smoothers of interest to the practitioner, that is, fixed point, fixed lag and fixed interval smoothers.**

All smoothers are functions of their corresponding filter. The filter we use to develop our smoother is the exact information-state filter for hybrid Gauss Markov models due to Elliott, Dufour and Sworder, [14]. Our approach is in contrast to some other smoothing schemes in the literature, which are often based upon ad-hoc schemes.

It is well known that the fundamental impediment in all estimation for jump Markov systems, is the management of an exponentially growing number of hypotheses. In our scheme, we propose a method to maintain a fixed number of candidate paths in a history, each identified as optimal by a probabilistic criterion. The outcome of this approach is a new and general smoothing scheme, based upon the exact filter dynamics, and whose memory requirements remain fixed in time.

I. INTRODUCTION

In this article the reference probability method is used to compute smoothed state estimates for a discrete-time hybrid dynamical system. This particular problem has received relatively little attention in the literature, that is, compared to the corresponding filtering problem for hybrid stochastic systems. This is perhaps due to the inherent complexity arising from stochastic hybrid systems and the basic complexity one generally expects in smoothing algorithms. In the articles [5] and [6], a smoothing scheme based upon the Interacting Multiple Model (IMM) algorithm is proposed. The originator of the IMM, namely Henk Blom, also studied the smoothing problem by using a time-reversal method in [4]. Further, a two-filter form of a smoother, similar in spirit to the ideas of the so-called Fraser-Potter smoother, was presented in [17].

Traditionally smoothing has been considered largely an offline processing scheme and therefore has received relatively scant attention from communities focussed on real-time estimation, such as the tracking community. However, with the advent of increasingly powerful computing and the potential benefits of smoothing schemes, it is indeed timely to revisit this particular smoothing problem. Some potential applications of smoothing in tracking are, fixed lag smoothers, fixed interval smoothing for track reconstruction and parameter estimation using the EM algorithm.

Using a general result, we propose a new suboptimal smoothing algorithm which provides an exact hypothesis management scheme, circumventing growth in algorithmic complexity.

II. STOCHASTIC DYNAMICS

Much of the detail in this section is now standard and can be found in texts such as [2], [9], [10]. Further, a companion paper to this article which considers the associate filtering problem, appears in this publication, see [20].

All processes are defined, initially, on a fixed probability space $\big(\Omega, \mathcal{F}, P\big)$.

A. Markov Chain Dynamics

We consider a time-homogeneous discrete-time m-state Markov chain Z. We represent the state space of Z on a canonical basis of indicator function $\boldsymbol{e}_i$, where the vector $\boldsymbol{e}_i = (0, 0, \dots, 1, 0, \dots, 0)'$ has unity in the i-th position. Our Markov chain has statistics (Π, p_0), where $\Pi = \big[\pi_{(j,i)}\big]_{\substack{1\le j\le m\\ 1\le i\le m}}$ is the transition matrix of Z, with elements

$$\pi_{(j,i)} \triangleq P(Z_k = \boldsymbol{e}_j \mid Z_{k-1} = \boldsymbol{e}_i), \quad \forall k \in \mathbb{N} \tag{1}$$

and $E\big[Z_0\big] = p_0$.

B. State Process Dynamics

We suppose the indirectly observed state vector $x \in \mathbb{R}^{n\times 1}$, has dynamics

$$x_k = \sum_{j=1}^{m} \big\langle Z_k, \boldsymbol{e}_j \big\rangle A_j x_{k-1} + \sum_{j=1}^{m} \big\langle Z_k, \boldsymbol{e}_j \big\rangle B_j w_k. \tag{2}$$

Here w is a vector-valued Gaussian process with $w \sim N(0, \boldsymbol{I}_n)$. A_j and B_j are $n \times n$ matrices and for each $j \in \{1, 2, \dots, m\}$, are nonsingular.

C. Observation Process Dynamics

Consider a vector-valued observation process with values in $\mathbb{R}^{d\times 1}$ and dynamics

$$y_k = \sum_{j=1}^{m} \big\langle Z_k, \boldsymbol{e}_j \big\rangle C_j x_k + \sum_{j=1}^{m} \big\langle Z_k, \boldsymbol{e}_j \big\rangle D_j v_k. \tag{3}$$

Here v is a vector-valued Gaussian process with $v \sim N(0, \boldsymbol{I}_d)$. We suppose the matrices $D_j \in \mathbb{R}^{d\times d}$, for each $j \in \{1, 2, \dots, m\}$, are nonsingular. Our filtrations are as follows:

$$\mathcal{F}_k = \sigma\big\{x_\ell, 0 \le \ell \le k\big\}, \tag{4}$$

$$\mathcal{Z}_k = \sigma\big\{Z_\ell, 0 \le \ell \le k\big\}, \tag{5}$$

$$\mathcal{Y}_k = \sigma\big\{y_\ell, 0 \le \ell \le k\big\}, \tag{6}$$

$$\mathcal{G}_k = \sigma\big\{Z_\ell, x_\ell, y_\ell, 0 \le \ell \le k\big\}. \tag{7}$$

D. Reference Probability

The dynamics given at (2) and (3), are each defined on a measurable space $(\Omega, \mathcal{F})$, under a measure P. However, consider a new measure $P^\dagger$, under which the dynamics for the processes Z, x and y, are, respectively

$$P^\dagger \quad \begin{cases} Z_k & = \Pi Z_{k-1} + L_k, \\ x_k & \text{are iid and } N(0, \boldsymbol{I}_n), \\ y_k & \text{are iid and } N(0, \boldsymbol{I}_d). \end{cases} \tag{8}$$

<u>Notation</u>:
The symbol $\Phi(\cdot)$ will be used to denote the zero mean normal density on $\mathbb{R}^d$:

$$\Phi(\xi) = (2\pi)^{-d/2} \exp\big(-\tfrac{1}{2}\xi'\xi\big). \tag{9}$$

Similarly we shall also use the symbol $\Psi(\cdot)$ to denote a standardised Gaussian density. The space dimension on which these densities is defined will be clear by context. To avoid cumbersome notation with matrices, we sometimes denote the inverse of a matrix A by $\operatorname{inv}(A)$ and the space of $m \times n$ matrices in our context is written as $\mathbb{M}^{m\times n}$.

We now define the measure P, by setting the restriction of a Radon-Nikodym derivative to $\mathcal{G}_k$ to

$$\Lambda_{0,k} \triangleq \frac{dP}{dP^\dagger}\Big|_{\mathcal{G}_k} = \prod_{\ell=0}^{k} \lambda_\ell,$$

where

$$\lambda_\ell = \sum_{j=1}^{m} \big\langle Z_\ell, \boldsymbol{e}_j \big\rangle \frac{\Phi\big(D_j^{-1}(y_\ell - C_j x_\ell)\big)}{|D_j|\Phi(y_\ell)} \times \frac{\Psi\big(B_j^{-1}(x_\ell - A_j x_{\ell-1})\big)}{|B_j|\Psi(x_\ell)}. \tag{10}$$

Theorem 1 *Suppose $\gamma = \{\gamma_\ell, 0 \le \ell \le k\}$ is an integrable $\mathcal{G}$-adapted process. then*

$$E\big[\gamma_k \mid \mathcal{Y}_k\big] = \frac{E^\dagger\big[\Lambda_{0,k}\gamma_k \mid \mathcal{Y}_k\big]}{E^\dagger\big[\Lambda_{0,k} \mid \mathcal{Y}_k\big]}. \tag{11}$$

III. Review Of Exact Hybrid filter Dynamics

The exact state estimation filter given in [14] is written in unnormalised form, that is, dynamics satisfied by an unnormalised probability density. These dynamics are computed using reference probability techniques, see [9], [10] and [2]. Briefly, we are interesting in computing conditional probabilities for joint events of the form $P(x \in dx, Z_k = \boldsymbol{e}_j \mid \mathcal{Y}_k)$. Omitting the details, we assume the existence unnormalised probability densities corresponding to our events of interest, where, for example,

$$P(x \in dx, Z_k = \boldsymbol{e}_j \mid \mathcal{Y}_k) = \frac{q_k^j(x)dx}{\int_{\mathbb{R}^n} q_k^j(\xi)d\xi}. \tag{12}$$

What we would like to compute, is recursive dynamics whose solutions are the densities $q^j(x) : \mathbb{R}^{n\times 1} \to \mathbb{R}_+$, where $j \in \{1, 2, \dots, m\}$. To this end, we recall the fundamental contribution of [14] in the next Theorem.

Theorem 2 (Elliott, Dufour, Sworder, 1996) *The unnormalised probability density $q_k^j(x)$, is computed by the equation,*

$$q_k^j(x) = \frac{\Phi\big(D_j^{-1}(y_k - C_j x)\big)}{\Phi(y_k)|D_j||B_j|} \times \sum_{r=1}^{m} \pi_{(j,r)} \int_{\mathbb{R}^n} \Psi\big(B_j^{-1}(x - A_j\xi)\big) q_{k-1}^r(\xi)d\xi. \tag{13}$$

IV. Exact Hybrid Smoother Dynamics

In this section we present our main results. We compute a a general smoother by applying similar techniques to those developed in [19] and [12].

We first note that

$$E\big[\langle Z_k, \boldsymbol{e}_j\rangle f(x_k) \mid \mathcal{Y}_{0,T}\big] = \frac{E^\dagger\big[\Lambda_{0,T}\langle Z_k, \boldsymbol{e}_j\rangle f(x_k) \mid \mathcal{Y}_{0,T}\big]}{E^\dagger\big[\Lambda_{0,T} \mid \mathcal{Y}_T\big]}. \tag{14}$$

Write

$$\widetilde{\mathcal{F}}_k \triangleq \sigma\big\{x_\ell, Z_\ell, 0 \le \ell \le k\big\}. \tag{15}$$

Using repeated conditioning, the numerator in the quotient of equation (14) may be written as

$$\begin{aligned} & E^\dagger\big[\Lambda_{0,T}\langle Z_k, \boldsymbol{e}_j\rangle f(x_k) \mid \mathcal{Y}_{0,T}\big] \\ &= E^\dagger\big[\Lambda_{0,k}\Lambda_{k+1,T}\langle Z_k, \boldsymbol{e}_j\rangle f(x_k) \mid \mathcal{Y}_{0,T}\big] \\ &= E^\dagger\Big[E^\dagger\big[\Lambda_{0,k}\Lambda_{k+1,T}\langle Z_k, \boldsymbol{e}_j\rangle f(x_k) \mid \widetilde{\mathcal{F}}_k \vee \mathcal{Y}_{0,T}\big] \mid \mathcal{Y}_{0,T}\Big] \\ &= E^\dagger\Big[\Lambda_{0,k}\langle Z_k, \boldsymbol{e}_j\rangle f(x_k) E^\dagger\big[\Lambda_{k+1,T} \mid \widetilde{\mathcal{F}}_k \vee \mathcal{Y}_{0,T}\big] \mid \mathcal{Y}_{0,T}\Big]. \end{aligned} \tag{16}$$

Further, the Markov property of our processes ensures the following equality

$$E^\dagger\big[\Lambda_{k+1|T} \mid \widetilde{\mathcal{F}}_k \vee \mathcal{Y}_{0,T}\big] = E^\dagger\big[\Lambda_{k+1|T} \mid Z_k, x_k, \mathcal{Y}_{0,T}\big]. \tag{17}$$

Write

$$v_k^j(x) \triangleq E^\dagger\big[\Lambda_{k+1,T} \mid Z_k = \boldsymbol{e}_j, x_k = x, \mathcal{Y}_{0,T}\big]. \tag{18}$$

Theorem 3 (Pardoux [24]) *The exact and general, unnormalised smoother density for the stochastic hybrid system whose dynamics include equations (2), (3) and the Markov chain Z, is given by the expectation*

$$\begin{aligned} & E^\dagger\big[\Lambda_{0,k}\langle Z_k, \boldsymbol{e}_j\rangle f(x_k) v_k^j(x_k) \mid \mathcal{Y}_{0,T}\big] = \\ & \int_{\mathbb{R}^n} \Big\{ q_k^j(\xi) v_k^j(\xi) \Big\} f(\xi) d\xi. \end{aligned} \tag{19}$$

We now compute a backwards recursion for the function $v_k^j(x)$, similar in form to the forward recursion for the function $q_k^j(x)$, given in equation (13).

Theorem 4 *The un-normalised function $v_k^j(x)$, satisfies the backward recursion:*

$$v_k^j(x) = \sum_{r=1}^{m} \frac{\pi_{(r,j)}}{|D_r||B_r|\Phi(y_{k+1})} \times \int_{\mathbb{R}^n} \Phi\big(D_r^{-1}(y_{k+1} - C_r\xi)\big)\Psi(B_r^{-1}(\xi - A_r x))v_{k+1}^r(\xi)d\xi. \tag{20}$$

Proof: From definition (18), we again use repeated conditioning to write

$$\begin{aligned}
v_k^j(x) &= E^\dagger\big[\Lambda_{k+1,T} \mid Z_k = \boldsymbol{e}_j, x_k = x, \mathcal{Y}_{0,T}\big] \\
&= E^\dagger\big[\lambda_{k+1}\Lambda_{k+2,T} \mid Z_k = \boldsymbol{e}_j, x_k = x, \mathcal{Y}_{0,T}\big] \\
&\qquad Z_k = \boldsymbol{e}_j, x_k = x, \mathcal{Y}_{0,T}\Big] \\
&= E^\dagger\Big[\lambda_{k+1}E^\dagger\big[\Lambda_{k+2,T} \mid Z_{k+1}, x_{k+1}, Z_k = \boldsymbol{e}_j, \\
&\quad x_k = x, \mathcal{Y}_{0,T}\big] \mid Z_k = \boldsymbol{e}_j, x_k = x, \mathcal{Y}_{0,T}\Big] \\
&= \sum_{r=1}^{m} E^\dagger\Big[\lambda_{k+1}\langle Z_{k+1}, \boldsymbol{e}_r\rangle E^\dagger\big[\Lambda_{k+2,T} \mid Z_{k+1} = \boldsymbol{e}_r, \\
&\quad x_{k+1}, Z_k = \boldsymbol{e}_j, x_k = x, \mathcal{Y}_{0,T}\big] \mid \\
&\qquad Z_k = \boldsymbol{e}_j, x_k = x, \mathcal{Y}_{0,T}\Big] \\
&= \sum_{r=1}^{m} E^\dagger\Big[\lambda_{k+1}\langle Z_{k+1}, \boldsymbol{e}_r\rangle \times \\
&\quad E^\dagger\big[\Lambda_{k+2,T} \mid Z_{k+1} = \boldsymbol{e}_r, \\
&\quad x_{k+1}, \mathcal{Y}_{0,T}\big] \mid \qquad Z_k = \boldsymbol{e}_j, x_k = x, \mathcal{Y}_{0,T}\Big].
\end{aligned} \tag{21}$$

Since Z is a Markov chain and under $P^\dagger$, we can continue this calculation now by noting that

$$\begin{aligned}
v_k^j(x) &= \sum_{r=1}^{m} E^\dagger\Big[\big\langle Z_{k+1}, \boldsymbol{e}_r\big\rangle \frac{\Phi\big(D_r^{-1}(y_{k+1} - C_r x_{k+1})\big)}{|D_r|\Phi(y_{k+1})} \times \\
&\quad \frac{\Psi\big(B_r^{-1}(x_{k+1} - A_r x_k)\big)}{|B_r|\Psi(x_{k+1})} v_{k+1}^r(x_{k+1}) \mid \\
&\qquad Z_k = \boldsymbol{e}_j, x_k = x, \mathcal{Y}_{0,T}\Big] \\
&= \sum_{r=1}^{m} \frac{\pi_{(r,j)}}{|D_r||B_r|\Phi(y_{k+1})} \times \\
&\quad \int_{\mathbb{R}^n} \Phi\big(D_r^{-1}(y_{k+1} - C_r\xi)\big) \times \\
&\qquad \Psi(B_r^{-1}(\xi - A_r x))v_{k+1}^r(\xi)d\xi
\end{aligned} \tag{22}$$

■

V. Gaussian Mixture Approximations

The dynamics for the v process given in Theorem 4 are exact, however, to implement such dynamics, the immediate question is how one might deal with the integral over $\mathbb{R}^n$ in equation (20). The process v is a nonnegative process. We suppose that the function $v_{k+1}^r(\xi)$, in equation (20), can be approximated arbitrarily closely by a weighted Gaussian mixture. This assumption is supported by a fundamental approximation result in function space, which is detailed in [30] and [18].

Theorem 5 *Suppose the function $v_{k+1}^r(\xi)$, (in equation (20)), is approximated as a weighted Gaussian mixture with $M^v \in \mathbb{N}$ components. That is, suppose*

$$v_{k+1}^r(\xi) = \sum_{s=1}^{M^v} \rho_{k+1}^{r,s} \frac{1}{(2\pi)^{n/2}|\Sigma_{k+1|T}^{r,s}|^{\frac{1}{2}}} \times \exp\Big\{-\tfrac{1}{2}(\xi - \alpha_{k+1|T}^{r,s})' \operatorname{inv}(\Sigma_{k+1|T}^{r,s})(\xi - \alpha_{k+1|T}^{r,s})\Big\}. \tag{23}$$

Here $\Sigma_{k+1|T}^{r,s} \in \mathbb{M}^{n\times n}$, and $\alpha_{k+1|T}^{r,s} \in \mathbb{R}^n$, are both $\mathcal{Y}_{k+1,T}$-measurable functions for all pairs $(r,s) \in \{1,2,\ldots,m\} \times \{1,2,\ldots,M^v\}$. Using the Gaussian mixture (23), the recursion for $v_k^j(x)$, at times $k \in \{1,2,\ldots,T-1\}$, has the form

$$v_k^j(x) \triangleq \frac{1}{(2\pi)^{(n+d)/2}\,\Phi(y_{k+1})} \sum_{r=1}^{m}\sum_{s=1}^{M^v} K_{k+1,T}^v(j,r,s) \times \exp\Big\{-\tfrac{1}{2}\Big(x - \operatorname{inv}(S_{k+1|T}^{r,s})\tau_{k+1|T}^{r,s}\Big)' S_{k+1,T}^{r,s}\Big(x - \operatorname{inv}(S_{k+1|T}^{r,s})\tau_{k+1|T}^{r,s}\Big)\Big\}. \tag{24}$$

At the final time T, $\forall\, j \in \{1,2,\ldots,m\}$,

$$v_T^j(x) \triangleq 1. \tag{25}$$

Here

$$\begin{aligned}
&K_{k+1,T}^v(j,r,s) \triangleq \\
&\frac{\pi_{(r,j)}\rho_{k+1}^{r,s}}{|D_r||B_r||\Sigma_{k+1|T}^{r,s}|^{\frac{1}{2}}\Big|\gamma^r + \operatorname{inv}\big(\Sigma_{k+1|T}^{r,s}\big)\Big|^{\frac{1}{2}}} \times \\
&\exp\Big\{-\tfrac{1}{2}\big(\alpha_{k+1|T}^{r,s}\big)' \operatorname{inv}(\Sigma_{k+1|T}^{r,s}) \times \\
&\Big[\boldsymbol{I} - \widetilde{\Sigma}_{k+1|T}^{r,s}\operatorname{inv}\big(\Sigma_{k+1|T}^{r,s}\big)\Big]\alpha_{k+1|T}^{r,s}\Big\} \times \\
&\exp\Big\{-\tfrac{1}{2}y_{k+1}'\operatorname{inv}(D_r D_r')y_{k+1}\Big\} \times \\
&\exp\Big\{-\tfrac{1}{2}\Big(\widetilde{\Sigma}_{k+1|T}^{r,s}\operatorname{inv}\big(\Sigma_{k+1|T}^{r,s}\big)\alpha_{k+1|T}^{r,s}\Big)' \times \\
&\operatorname{inv}\big(\widetilde{\Sigma}_{k+1|T}^{r,s}\big)\Big(\widetilde{\Sigma}_{k+1|T}^{r,s}\operatorname{inv}\big(\Sigma_{k+1|T}^{r,s}\big)\alpha_{k+1|T}^{r,s}\Big)\Big\} \times \\
&\exp\Big\{\tfrac{1}{2}\Big(C_r\operatorname{inv}(D_r D_r')y_{k+1} + \\
&\operatorname{inv}\big(\Sigma_{k+1|T}^{r,s}\big)\alpha_{k+1|T}^{r,s}\Big)' \times \\
&\operatorname{inv}\big(\widetilde{\Sigma}_{k+1|T}^{r,s}\big)\Big(C_r\operatorname{inv}(D_r D_r')y_{k+1} + \\
&\operatorname{inv}\big(\Sigma_{k+1|T}^{r,s}\big)\alpha_{k+1|T}^{r,s}\Big)\Big\} \times \\
&\exp\Big\{\tfrac{1}{2}(\tau_{k+1|T}^{r,s})'\operatorname{inv}(S_{k+1|T}^{r,s})\tau_{k+1|T}^{r,s}\Big\} \in \mathbb{R}.
\end{aligned} \tag{26}$$

$$S^{r,s}_{k+1|T} \triangleq A'_r \operatorname{inv}(B_r B'_r) A_r - A'_r \operatorname{inv}(B_r B'_r) \widetilde{\Sigma}^{r,s}_{k+1|T} \times \operatorname{inv}(B_r B'_r) A_r \in \mathbb{R}^{n\times n} \quad (27)$$

$$\tau^{r,s}_{k+1|T} \triangleq A'_r \operatorname{inv}(B_r B'_r) \widetilde{\Sigma}^{r,s}_{k+1|T} \times \Big(C'_r \operatorname{inv}(D_r D'_r) y_{k+1} + \operatorname{inv}\big(\Sigma^{r,s}_{k+1|T}\big) \alpha^{r,s}_{k+1|T} \Big) \in \mathbb{R}^n, \quad (28)$$

$$\gamma_r \triangleq \operatorname{inv}(B_r B'_r) + C'_r \operatorname{inv}(D_r D'_r) C_r \in \mathbb{R}^{n\times n}, \quad (29)$$

$$\mu^r_{k+1} \triangleq C'_r \operatorname{inv}(D_r D'_r) y_{k+1} + \operatorname{inv}(B_r B'_r) A_r x \in \mathbb{R}^n, \quad (30)$$

$$\widetilde{\Sigma}^{r,s}_{k+1|T} \triangleq \operatorname{inv}\Big(\gamma^r + \Sigma^{r,s}_{k+1|T}\Big) \in \mathbb{R}^{n\times n}. \quad (31)$$

Corollary 1 *Write*

$$v^j_k \triangleq \int_{\mathbb{R}^n} v^j_k(\xi) d\xi. \quad (32)$$

The scalar-valued quantity v^j_k, is computed by the double sum

$$v^j_k = \frac{1}{(2\pi)^{d/2}\Phi(y_{k+1})} \sum_{r=1}^{m} \sum_{s=1}^{M^v} \vartheta^v_{k+1,T}(j,r,s). \quad (33)$$

Here

$$\vartheta^v_{k+1,T}(j,r,s) \triangleq K^v_{k+1,T}(j,r,s) |S^{r,s}_{k+1|T}|^{-\frac{1}{2}} \quad (34)$$

A. Sub-Optimal Smoother Dynamics

1) Hypothesis Management: Write

$$\Gamma^v \triangleq \{1,2,\dots,m\} \times \{1,2,\dots,M^v\}, \quad (35)$$

$$\widetilde{S}^v_{k+1,T}(j,r,s) \triangleq \{\vartheta^v_{k+1,T}(j,r,s)\}_{(r,s)\,\in\,\Gamma^v}. \quad (36)$$

We propose, at each time k, to identify the M^v-best candidate functions, (components in the Gaussian mixture), for each function $v^j_k(x)$. This maximisation procedure is as follows:

$$\vartheta^v_{k+1,T}(j,r^*_1,s^*_1) \triangleq \max_{(r,s)\,\in\,\Gamma^v} \widetilde{S}^v_{k+1,T}(j,r,s), \quad (37)$$

$$\vdots \qquad \vdots$$

$$\vartheta^v_{k+1,T}(j,r^*_{M^v},s^*_{M^v}) \triangleq \max_{(r,s)\,\in\,\Gamma^v \setminus \{(r^*_1,s^*_1),\dots\}} \widetilde{S}^v_{k+1,T}(j,r,s). \quad (38)$$

The optimal index set, for function $v^j_k(x)$, is:

$$\mathcal{I}_k(j) \triangleq \{(r^*_{k,1},s^*_{k,1}),(r^*_{k,2},s^*_{k,2}),\dots,(r^*_{k,M^v},s^*_{k,M^v})\}. \quad (39)$$

Using these indexes, the order M^v equation for $v^j_k(x)$, whose memory requirements are fixed in time, has the form;

$$v^j_k(x) \triangleq \frac{1}{(2\pi)^{(d+n)/2}\Phi(y_{k+1})} \times \sum_{\ell=1}^{M^v} K^v_{k+1,T}(j,r^*_{k+1,\ell},s^*_{k+1,\ell}) \times \exp\Big\{-\tfrac{1}{2}\Big(x - \operatorname{inv}(S^{r^*_{k+1,\ell},s^*_{k+1,\ell}}_{k+1|T})\tau^{r^*_{k+1,\ell},s^*_{k+1,\ell}}_{k+1|T}\Big)' \times S^{r^*_{k+1,\ell},s^*_{k+1,\ell}}_{k+1,T} \times \Big(x - \operatorname{inv}(S^{r^*_{k+1,\ell},s^*_{k+1,\ell}}_{k+1|T})\tau^{r^*_{k+1,\ell},s^*_{k+1,\ell}}_{k+1|T}\Big)\Big\}. \quad (40)$$

Remark V.1 In the companion paper to this article, see [20], a corresponding suboptimal scheme is developed to compute the q process. Combining the suboptimal schemes for the q and the v, according the Theorem of Pardoux given by equation (19), one may compute smoothed state estimates with a scheme whose memory requirements remain fixed in time.

References

[1] G. Ackerson and K. Fu, "On state estimation in switching environments", IEEE Transactions on Automatic Control 15(1), pp. 10-17, 1970.

[2] L. Aggoun and R. J. Elliott, *Measure Theory and Filtering*, Cambridge University Press, 2004.

[3] H. Blom, An effi cient fi lter for abruptly changing systems, 23rd IEEE Conference on Decision and Control, Las Vegas USA, November 1984.

[4] H. Blom and Y. Bar Shalom, Time-reversion of a hybrid state stochastic difference system with a jump-linear smoothing application, IEEE Transactions on Information Theory, volume 36, number 4, pp. 836-847, 1990.

[5] B. Chen and J. K. Tugnait, An interacting multiple model fi xed-lag smooth ing algorithm for Markovian switching systems, 37th IEEE Conference on Decision and Control, Tampa Florida US A, December 1986.

[6] B. Chen and J. K. Tugnait, Interacting multiple model fi xed-lag smoothing algorithm for Markovian switching systems, IEEE Transactions on Aerospace and Electronic Systems, volume 36, Number 1, January 2000.

[7] J. Cloutier, J. Evers and J. Freely, "Assessment of air-to-air missile guidance and control technology", IEEE Control systems magazine, 9, pp. 27-34, 1989

[8] R. J. Elliott and W. P. Malcolm, Reproducing Gaussian Densities and Linear Gaussian Detection, Systems and Control Letters, 40 (2000), pp. 133-138.

[9] R. J. Elliott, *Stochastic Calculus and its Applications*, Springer Verlag 1982.

[10] R. J. Elliott, L. Aggoun and J. B. Moore, *Hidden Markov Models Estimation and Control*, Springer Verlag Applications of Mathematics Series 29, 1995.

[11] R. J. Elliott and V. Krishnamurthy, New fi nite-dimensional fi lters for parameter estimation of discrete-time linear Gaussian models, IEEE Transactions on Automatic Control, Volume 44, Number 5, May, 1999.

[12] R. J. Elliott and W. P. Malcolm, Robust smoother dynamics for Poisson processes driven by an Itô diffusion, IEEE Conference on Decision and Control, December 2001, Orlando Florida, USA.

[13] R. J. Elliott, A general recursive discrete time fi lter, Journal of Applied Probability, 30, pp. 575-588, 1993.

[14] R. J. Elliott, F. Dufour, and D. Sworder, Exact hybrid fi lters in discrete time, IEEE Transactions on Automatic Control, 41, 1996, pp. 1807-1810.

[15] R. J. Elliott, F. Dufour and W. P. Malcolm, State and Mode Estimation for Discrete-Time Jump Markov Systems, SIAM Journal on Control and Optimization, *to appear in 2005*.

[16] R. J. Elliott and W. P. Malcolm, General smoothing formulas for Markov modulated Poisson observations, IEEE Transactions on Automatic Control, Volume 50, Number 8, 2005.
[17] R. E. Helmick and W. Dale Blair, Fixed-Interval Smoothing for Markovian Switching Systems, IEEE Transactions on Information Theory, Volume 41, Number 6, November 1995, pp. 1845-1855.
[18] J. Korevaar, Mathematical Methods, Academic Press, New York, 1968.
[19] W. P. Malcolm and R. J. Elliott, A general smoothing equation for Poisson observations, IEEE Conference on Decision and Control, December 1999, Phoenix USA.
[20] W. P. Malcolm, R. J. Elliott, F. Dufour and M. S. Arulampalam, An Algorithmic Estimation scheme for Hybrid Stochastic Systems, IEEE Conference on Decision and Control, December 2005, Seville Spain.
[21] M. Mariton, *Jump linear systems in automatic control*, Marcel Dekker, New York, 1990.
[22] P. Maybeck and R. Stevens, "Reconfi gurable fight control via multiple model adaptive control methods",IEEE Transactions on Aerospace an Electronic Systems 27, pp. 470-480, 1991
[23] S. Nardone and M. Graham, A closed form solution to bearings-only target motion analysis, IEEE Journal of Oceanic Engineering, 22(1) 1995.
[24] E. Pardoux, Equations du fi ltrage nonlineare de la predictions et du lissage, Stochastics, number 6, pp. 193-231, 1982.
[25] X. Rong Li and V. Jikov, A survey of maneuvering target tracking: Dynamic Models, In Signal and Data Processing of small Targets, SPIE2000, USA.
[26] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1966.
[27] A. N. Shiryaev, *Probability*, Second Edition, Springer Verlag 1996.
[28] D. Sworder, R. Vojak and R. Hutchins, Gain adaptive Tracking, Journal of Guidance, Control and Dynamics, 16(5), pp. 865-873, 1993.
[29] A. J. Viterbi, Error Bounds for the White Gaussian and Other Very Noisy Memoryless Channels with Generalized Decision Regions, IEEE Transactions on Information Theory, Volume IT, Number 2, March 1969.
[30] H. W. Sorenson and Alspach, D. L.,Recursive Bayesian Estimation Using Gaussian Sums, Automatica, Volume 7, Number 4, July 1971, pp. 465-479.
[31] D. Sworder, R. Vojak and R. Hutchins, Gain adaptive Tracking, Journal of Guidance, Control and Dynamics, 16(5), pp. 865-873, 1993.
[32] J. K. Tugnait, Adaptive Estimation and Identifi cation for Discrete Systems with Markov Jump Parameters, IEEE Transactions on Automatic Control, Volume AC-27, Number 5, October 1982.
[33] A. J. Viterbi, Error Bounds for the White Gaussian and Other Very Noisy Memoryless Channels with Generalised Decision Regions, IEEE Transactions on Information Theory, Volume IT, Number 2, March 1969.