

Filtering and Identification of Interest Rate Model with Stochastic Volatility

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Abstract— We consider the dynamics of forward rate process which is modeled by the parabolic type infinite-dimensional factor model with stochastic volatility. Before using this model to the portfolio construction problem, we need to estimate the parameters included in this parabolic model. Usually this identification is performed by using the yield curve as the observation data. In this paper, we propose the filtering and identification method for the parabolic type factor model by using the maximum likelihood technique.

I. INTRODUCTION

We consider the actual modeling of the real-world term structure dynamics which gives the statistical description of the movements of interest rate. One model of potential interest might include a 'curvature reducing' term in the drift of forward rates in [1], [2]. We consider an infinite-dimensional term-structure model for this purpose. A parabolic stochastic partial differential equation is proposed for the term structure. Although the instantaneous forward rate is unobservable, Libor, FRA's (forward rate agreements) etc are market instruments. Hence the yield curve process is constructed from these instruments and we can use this process to estimate the intracutaneous rate and parameters included in the model.

Let $f(t, x)$ be a forward rate of the bond process which is a solution of the following stochastic partial differential equation:

$$df(t, x) = \{\text{Drift term}\}dt + dw(t, x)$$

where x denotes the "time-to-maturity" and $w(t, x)$ is an infinite-dimensional Brownian motion process with $E\{w(t, x_1)w(t, x_2)\} = tq(x_1, x_2)$. At first we assume that the existence of the solution f in $L^2(\Omega; C(T_f; L^2(G)))$ where G denotes the spatial region for x .

From the property of Ito's integral, we have

$$\sum (f(t_{i+1}, x_1) - f(t_i, x_1))(f(t_{i+1}, x_2) - f(t_i, x_2)) \sim tq(x_1, x_2). \quad (1)$$

Hence we can estimate q as

$$q(x_1, x_2) \sim \frac{1}{t} \sum (f(t_{i+1}, x_1) - f(t_i, x_1)) \times (f(t_{i+1}, x_2) - f(t_i, x_2))$$

This identification may work well from the macroscopic view point [3]. Recently, the stochastic volatility model has been

introduced to the stock model, i.g., Heston' model [4]. For the factor model, in [5] the stochastic volatility has been already proposed. If we use this stochastic volatility modeling, we need to construct the estimation algorithm of the volatility process and identification of the included parameters.

II. AN EMPIRICAL EXAMPLE

First we consider the simple example of stochastic volatility case. The forward rate $f(t, x)$ is a solution of

$$df(t, x) = \{\text{Drift term}\}dt + \sqrt{\sigma(t)}dw(t, x), \quad (2)$$

where the stochastic volatility process $\sigma(t)$ is given by

$$d\sigma(t) = \alpha(m - \sigma(t))dt + k\sqrt{\sigma(t)}dv(t) \quad (3)$$

where $v(t)$ is a R^1 -valued Brownian motion process.

In the usual bond data case, through the yield curve data we calculate $f(t, x)$ process numerically. From (2), by using Ito's formula, the following observation data is given;

$$\begin{aligned} y(t) &= \int_G [f^2(t, x) - f^2(0, x) - 2 \int_0^t f(s, x)df(s, x)]dx \\ &= \int_0^t \sigma(s) \int_G q(x, x)dx ds. \end{aligned} \quad (4)$$

Hence differentiate the left hand side of (4) with respect to t , and we have

$$\frac{dy(t)}{dt} = \frac{d \int_0^t \sigma(s) Tr\{Q\} ds}{dt} = \sigma(t) Tr\{Q\}. \quad (5)$$

However the obtained results are not satisfactory. In practice, the above relation can not be realized, because we don't obtain the exact continuous data and always some errors are corrupted. Here we can show the following simulation results by using the US treasury bond data. In Fig. 1, the real data of the forward rate $f(t, x)$ derived from the real yield curve data is shown. The observation by using (4) is shown in Fig. 2. From (5), we obtain the volatility estimate shown in Fig. 3. From this figure, the obtained results is hardly to be the estimate of the volatility process, as mentioned in [6], [7].

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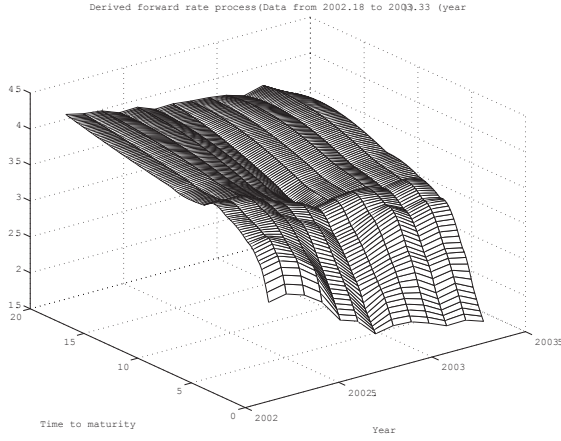


Fig. 1. Forward rate process(US-bonds)

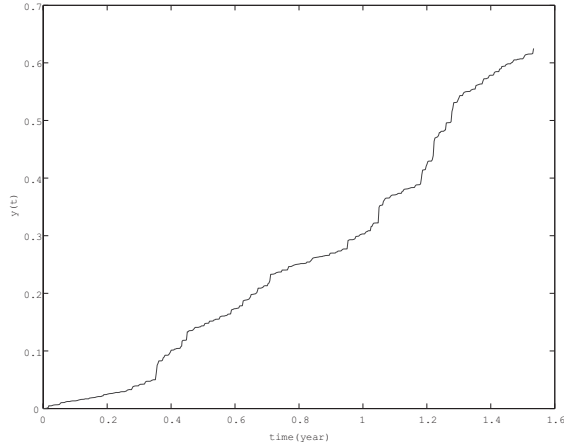


Fig. 2. Derived process $y(t)$ from US-Bonds

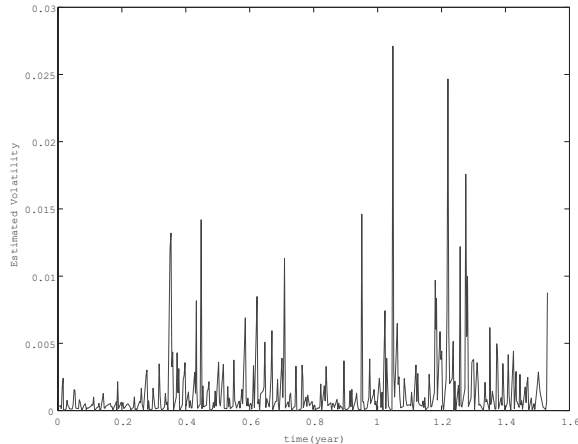


Fig. 3. Estimated volatility from (5)

III. VOLATILITY ESTIMATION AND IDENTIFICATION

From the previous example, we find that the estimation algorithm (5) does not work well. We need to use the technique proposed in [6]. To estimate the volatility process, we can construct the observation data like (4) from the factor process $f(t, x)$ numerically. Hence the obtained data is always corrupted with observation noise.

Here we consider the simple situation: the observation data $y(t)$ is given by

$$y(t) = \int_0^t \sigma(s)Tr\{Q\}ds + \epsilon(t) \quad (6)$$

where $Tr\{Q\} = \int_G q(x, x)dx$ and $\epsilon(t)$ is a R^1 -valued Brownian motion process independent of v and this noise denotes the error to calculate (4) from the yield curve numerically.

Now we can reformulate our volatility estimation problem as the filtering and identification problem as follows:
(System dynamics)

$$\begin{cases} d\sigma(t) = \alpha(m - \sigma(t))dt + k\sqrt{\sigma(t)}dv(t) \\ dy(t) = Tr\{Q\}\sigma(t)dt + d\epsilon(t) \end{cases} \quad (7)$$

where unknowns are m, k, σ_0 , and $Tr\{Q\}$.

If we derive the usual Zakai equation to this system, the second order partial differential operator becomes degenerate and this causes the numerical instability to solve the Zakai equation with the aid of the finite difference method.

To avoid this difficulty, noting that the original system (7) has a strong solution, we transform the system to the state independent case; set $z(t) = \sqrt{\sigma(t)}$. By using the Ito formula, we have

$$\begin{cases} dz(t) = \frac{1}{2}\{(\alpha m - \frac{k^2}{4})\frac{1}{z(t)} - \alpha z(t)\}dt + \frac{k}{2}dv(t) \\ dy(t) = Tr\{Q\}z^2(t)dt + d\epsilon(t) \end{cases} \quad (8)$$

At least to keep $z(t) > 0$ a.s., we set the following condition:

$$\alpha m - \frac{k^2}{4} > 0. \quad (9)$$

For the transformed system, the system noise is independent of the state, and however drift terms become non-Lipschitz functions. In this paper, we assume that near singular point $z = 0$ these nonlinear functions of the drift terms are adjusted as the Lipschitz functions in the neighborhood of the singular point $z = 0$. However we omits to describe such a mathematical adjustment in details to avoid the mathematically complicated descriptions.

A. Zakai equation

First we assume that all unknown parameters and the initial distribution of σ_0 are known. Hence it is possible to formulate the nonlinear filtering problem. To solve the Zakai equation numerically, the numerical procedure can be easily performed for the case that system noise term is independent of the system state.

So for the state $z(t)$, we have

$$E\{g(\sqrt{z(t)})|\mathcal{F}_t^y\} = \frac{p(t)(g)}{p(t)(1)} \text{ for } g \in C_b^2 \quad (10)$$

where $p(t)$ is a solution of the Zakai equation:

$$dp(t) + \mathcal{L}^*p(t)dt = \mathcal{M}p(t)dy(t), \quad (11)$$

and where

$$\begin{aligned} \mathcal{L}^*\phi &= -\frac{k^2}{8} \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{2} \left\{ (\alpha m - \frac{k^2}{4}) \frac{1}{z} - \alpha z \right\} \frac{\partial \phi}{\partial z} \\ &\quad - \frac{1}{2} \left\{ (\alpha m - \frac{k^2}{4}) \frac{1}{z^2} + \alpha \right\} \phi \\ \mathcal{M}\phi &= \text{Tr}\{Q\}z^2\phi. \end{aligned} \quad (12)$$

$$\mathcal{M}\phi = \text{Tr}\{Q\}z^2\phi. \quad (13)$$

Noting that \mathcal{M} is a bounded operator, the robust form of the Zakai equation can be derived. Defining the transformation:

$$q(t, z) = p(t, z) \exp[-\text{Tr}\{Q\}z^2y(t)] \quad (14)$$

we have

$$\frac{\partial q}{\partial t} + \mathcal{L}^*q - \frac{k^2}{2} \text{Tr}\{Q\}zy(t) \frac{\partial q}{\partial z} + h(t)q = 0, \quad (15)$$

$$q(0, z) = p_o(z) \quad (16)$$

where

$$\begin{aligned} h(t) &= (\alpha m - \frac{k^2}{4} - \alpha z^2)y(t) \text{Tr}\{Q\} \\ &\quad - \frac{k^2}{2} y^2(t) z^2 (\text{Tr}\{Q\})^2 + \frac{z^4}{2} (\text{Tr}\{Q\})^2. \end{aligned} \quad (17)$$

B. A simulation example

Before using the real data, we present a simulation example to estimate the stochastic volatility.

We set

$$m = 0.4, k = 1, \alpha = 5, \sigma_o = 0.4, \text{Tr}\{Q\} = 100$$

The simulated observation $y(t)$ is given in the following figure:

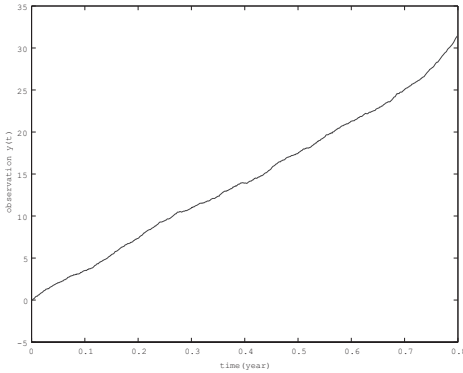


Fig. 4. Observation data $y(t)$

The transformed conditional probability density $p(t, x|\mathcal{Y}_t)_{x=z^2}$ for the original process σ is shown in

Fig. 5. The true and estimated volatilities are demonstrated in Fig. 6.

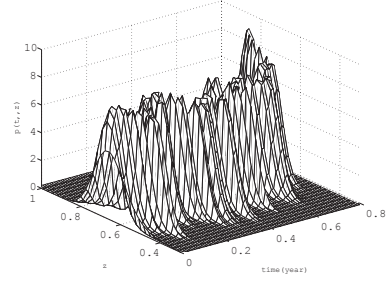


Fig. 5. Normalized conditional probability density

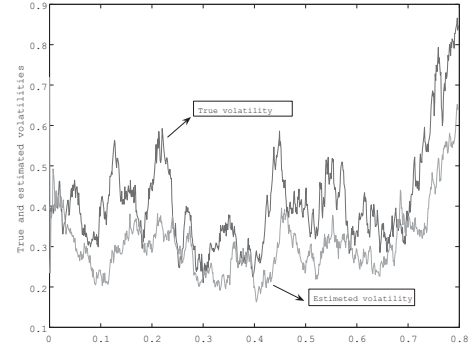


Fig. 6. True and estimated volatilities

IV. PARAMETER IDENTIFICATION

The log likelihood functional is given by

$$\begin{aligned} LLF(t_f) &= -\frac{1}{2} \int_0^{t_f} |\text{Tr}\{Q\}\hat{\sigma}(t)|^2 dt \\ &\quad + \int_0^{t_f} \text{Tr}\{Q\}\hat{\sigma}(t)dy(t). \end{aligned} \quad (18)$$

We represent the unknown parameters $m, k, \text{Tr}\{Q\}, \sigma_o$ as

$$\theta = [m, k, \text{Tr}\{Q\}, \sigma_o] \in \Theta = \text{Bounded subset of } R^4$$

The unknown parameter θ is estimated as

$$\hat{\theta}^o = \arg \max_{\theta \in \Theta} LLF(t_f).$$

In practice, this optimization procedure is performed by using the genetic algorithm in [8]. It is possible to use the characteristic function method in [9]. Here we already have a filter equation and the maximum likelihood estimates seem to be easily realized.

A. Simulation Studies

Corresponding to the previous example, we identify the parameters m, k, α, \dots in the sense of the maximum likelihood by using the generic algorithm by using MATLAB

toolbox [8].

The obtained M.L.E. for $m, k, \alpha, \sigma_o, Tr\{Q\}$ are :

$$\hat{m} = 1.5, \hat{k} = 0.799, \hat{\sigma}_o = 0.5, \\ Tr\{\hat{Q}\} = 80.273$$

These estimates are different from true values given in the previous example. However by using these values for solving the Zakai equation again, the obtained estimate $\hat{\sigma}(t)$ is shown in Fig. 7 and is closely to the true value than the estimate with true values of parameters. This may be caused from the choice of the initial probability density function $p_o(z)$.

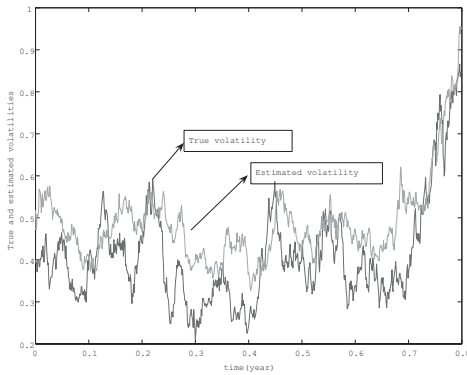


Fig. 7. True and estimated volatilities (M.L.E.)

We also present the estimated observation:

$$\hat{y}(t) = \int_0^t Tr\{\hat{Q}\}\hat{\sigma}(s)ds,$$

in Fig. 8 .

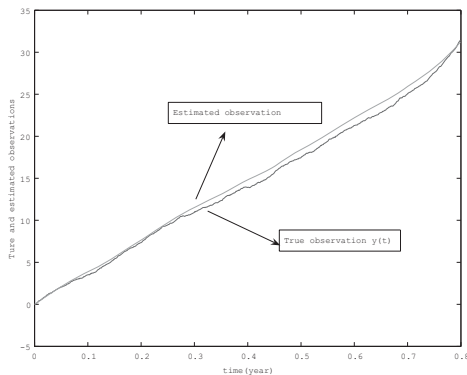


Fig. 8. True and estimated observations

B. Simulation for Real Data

Here we treat a real data. From the derived data $f(t, x)$ as shown in Fig. 1, we can obtain the observation data $y(t)$ from (4) with the observation noise shown in Fig. 2.

With the aid of generic algorithm, we can obtain the maximum likelihood estimate of unknown parameters:

$$\hat{m} = 1.5723, \hat{k} = 0.6840, \hat{\alpha} = 0.2031, \\ Tr\{\hat{Q}\} = 19.1449, \hat{\sigma}_o = 0.01$$

So by using these estimate, we can estimate the stochastic volatility process:

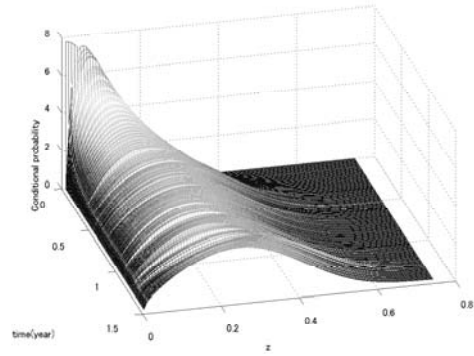


Fig. 9. Conditional probability density $p(t, z|Y_t)$

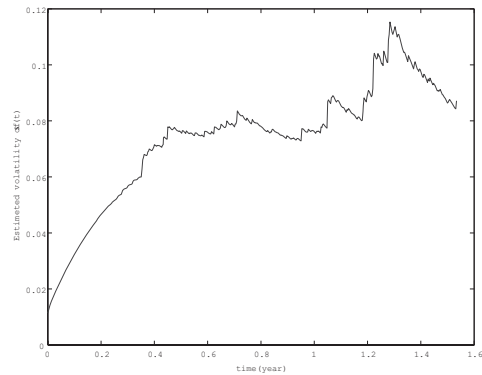


Fig. 10. Estimated stochastic volatility $\hat{\sigma}(t)$

V. TERM STRUCTURE MODEL AND FILTERING

Now we set the mathematical mode of term structure for US-bonds. The evaluation of the term structure of interest rates is parameterized by the instantaneous forward rate curve denoted by $f(t, x)$ where t denotes time and x the time to maturity in some bounded region, $]0, \hat{T}[$.

As mentioned in the previous section, if the mathematical model of the forward rate $f(t, x)$ is formulated, we can construct the observation (6) by using the formula (4). This implies that the stochastic volatility estimation established in Sec.IV can be applied to any types of factor model.

The infinite-dimensional term structure model with stochastic volatility is given by

$$df(t, x) = \frac{\partial f(t, x)}{\partial x}dt + \mu(t, x)dt + \sqrt{\sigma(t)}dw(t, x), \quad (19)$$

where $\mu(t, x)$ denotes the market price of volatility risk whose form is given by

$$\mu(t, x) = \lambda\sigma(t)\sqrt{q(x, x)} + \sigma(t) \int_0^x q(x, y)dy. \quad (20)$$

Recently from empirical observations, Bouchaud et.al. [2] proposed the parabolic type systems to support the smoothness property with respect to maturity. The natural extension of (19) to the parabolic system with stochastic volatility is

$$df(t, x) = \frac{a}{2} \frac{\partial^2 f(t, x)}{\partial x^2} dt + \frac{\partial f(t, x)}{\partial x} dt + \mu(t, x) dt + \sqrt{\sigma(t)} dw(t, x), \quad (21)$$

$$d\sigma(t) = \alpha(m - \sigma(t))dt + k\sqrt{\sigma(t)}dv(t) \quad (22)$$

where $v(t)$ is a R^1 -Brownian motion process. By constructing the boundary conditions, we set the short rate $r(t)$ and the long rate $\ell(t)$ as the boundary value $f(t, 0)$ and $f(t, 1)$.

Noting that the short and long rate are jointly Markovian [10], set for simplicity the linear boundary condition as

$$f(0, x) = f_o(x), \quad x \in G \quad (23)$$

$$\frac{a}{2} \frac{\partial f(t, 0)}{\partial x} = \mu_0(m_r - f(t, 0)), \quad t \in]0, t_f[\quad (24)$$

$$-\frac{a}{2} \frac{\partial f(t, 1)}{\partial x} = \mu_1(m_\ell - f(t, 0)), \quad t \in]0, t_f[\quad (25)$$

We use the variational framework [11] to study the stochastic partial differential equation. Let $G =]0, \hat{T}[$. We work in the following Hilbert spaces:

$$V = H^1(G) \subset H = L^2(G) \subset V' = \text{dual of } V.$$

Define

$$\langle A\phi_1, \phi_2 \rangle = \int_G \left\{ \frac{a}{2} \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_1}{\partial x} \phi_2 \right\} dx - \{ \mu_0 \phi_0(0) \phi_1(0) + \mu_1 \phi_1(1) \phi_2(1) \}, \quad \forall \phi_1, \phi_2 \in V.$$

The weak form of the proposed system is

$$\begin{aligned} & (f(t), \phi) + \int_0^t \{ \langle Af(s), \phi \rangle + (g, \phi)_\Gamma \} ds \\ & = (f_o, \phi) - \int_0^t (\mu(s), \phi) ds + (w(t), \phi) \quad \forall \phi \in V \end{aligned} \quad (26)$$

where $\Gamma = L^2(\partial G)$ and

$$\begin{aligned} (\phi_1, \phi_2) &= \int_G \phi_1(x) \phi_2(x) dx \\ (g, \phi)_\Gamma &= \mu_0 m_r \phi(0) + \mu_1 m_\ell \phi(1) \end{aligned}$$

and w, w_1 and w_0 are mutually independent Brownian motion processes; $\forall \phi_1, \phi_2 \in H$

$$E\{(w(t), \phi_1)(w(t), \phi_2)\} = t(\phi_1, Q\phi_2)$$

with

$$Tr\{Q\} < \infty. \quad (27)$$

Remark 5.1: It is possible to represent $w(t, x)$ as

$$w(t, x) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i(x) \beta_i(t) \quad (28)$$

where $\{e_i(x)\}_{i=1}^{\infty}$ are orthonormal basis in H , and $\{\beta_i(t)\}_{i=1}^{\infty}$ are mutually independent Brownian motion processes and $Tr\{Q\} = \sum_{i=1}^{\infty} \lambda_i$.

Theorem 5.1: Under (27),

$$a > 0$$

$$f_o \in L^2(\Omega; H) \quad g \in L^2(\Omega \times]0, t_f[; H)$$

and

$$\mu \in L^2(\Omega \times]0, t_f[; V'),$$

(26) has a unique solution in

$$L^2(\Omega; C([0, t_f]; H) \cap L^2(]0, t_f[; V)).$$

Proof: The parabolic type stochastic evolution equation with boundary noise has been studied by many authors. For example the used method can be found in the book by Rozovskii [12] and Paradoux [13].

Proposition 5.1: Under

$$f_o \in L^2(\Omega; V), \quad \mu \in L^2(\Omega \times]0, t_f[; H) \quad (29)$$

and

$$Tr\left\{ \frac{\partial}{\partial x} \left(\frac{\partial Q}{\partial x} \right)^* \right\} < \infty, \quad (30)$$

we have

$$f \in L^2(\Omega; C([0, t_f]; V) \cap L^2(]0, t_f[; H^2)) \quad (31)$$

and the spot rate $r(t) = f(t, 0)$ and the long rate $\ell(t) = f(t, 1)$ respectively satisfy

$$r, \ell \in L^2(\Omega; C([0, t_f]; R^1)). \quad (32)$$

Proof: By using the technique proposed by Bardos [14], it is easy to derive the above regularity property.

A. Observation process constructed from yield curve

Noting that the relation between the yield curve and the forward rate process is theoretically given by

$$Yield(t, T) = \frac{1}{T-t} \int_0^{T-t} f(t, x) dx, \quad (33)$$

Hence, we have

$$f(t, x) = \frac{\partial Yield(t, T)}{\partial(T-t)} \Big|_{T-t=x}. \quad (34)$$

It follows from (4) and (34) that

$$y(t) = \int_0^t \sigma(s) Tr\{Q\} ds + \epsilon(t)$$

where we add the observation noise ϵ because the calculations of (34) is performed numerically and always some errors exhibit. Now it is possible to estimate the stochastic volatility process from the algorithm shown in Sec.III.

Next problem is to estimate the factor process $f(t, x)$ and system parameters contained in(26). We set the observation mechanism:

$$dY(t, T_k) = \frac{1}{T_k - t} \int_0^{T_k-t} f(t, x) dx dt + d\epsilon_{2k}(t) \quad (35)$$

where $\epsilon_{2k}(t)$ denotes the Brownian motion process in R^1 which is independent of v, w, ϵ and this measurement noise is for example due to bid-ask spread.

Now at each time t we obtain the p -dimensional observation data

$$\vec{Y}(t) = [Y(t, T_1), Y(t, T_2), \dots, Y(t, T_p)]' \quad (36)$$

and (35) is represented by

$$d\vec{Y}(t) = H(t)f(t, \cdot)dt + d\vec{\epsilon}_2(t) \quad (37)$$

where

$$H(t)f(t, \cdot) = \left[\frac{1}{T_1 - t} \int_0^{T_1 - t} f(t, x)dx, \dots, \frac{1}{T_p - t} \int_0^{T_p - t} f(t, x)dx \right]' \quad (38)$$

and

$$\vec{\epsilon}_2 = [\epsilon_{21}, \epsilon_{22}, \dots, \epsilon_{2p}]'$$

B. Filtering for term structure

Noting that $v(t)$ and $w(t, x)$ are independent, we can formulate the filtering problem for $f(t, x)$ under $\vec{Y}(t)$ and $\sigma(t) = \hat{\sigma}(t)$ as the usual infinite-dimensional Kalman filter. Here we can separate the filtering of the factor process $f(t, x)$ and volatility process $\sigma(t)$.

Define for $\phi \in V$

$$(\hat{f}(t), \phi) = (E\{f(t)|\mathcal{F}_t^{\vec{Y}}, \sigma(t) = \hat{\sigma}(t)\}, \phi)$$

Hence

$$\begin{aligned} & d(\hat{f}(t), \phi) + \{ \langle A\hat{f}(t), \phi \rangle + (g, \phi)_\Gamma \} dt \\ & + E\{\sigma(t)|\mathcal{F}_t^y\} (\lambda \sqrt{q(x, x)} + \int_0^x q(x, s)ds) dt \\ & = (P_f(t)(d\vec{Y}(t) - H(t)\hat{f}(t, \cdot)dt), \phi) \text{ for } \phi \in V, \end{aligned}$$

where

$$\begin{aligned} & \left(\frac{dP_f(t)}{dt} \phi_1, \phi_2 \right) + \langle AP_f(t)\phi_1, \phi_2 \rangle + \langle A\phi_1, P_f(t)\phi_2 \rangle \\ & \quad + (P_f(t)\phi_1, H^*(t)H(t)P_f(t)\phi_2) \\ & \quad = E\{\sigma(t)|\mathcal{F}_t^y\} (Q\phi_1, \phi_2) \end{aligned}$$

for $\phi_1, \phi_2 \in V$ where

$$\begin{aligned} (P_f(0)\phi_1, \phi_2) &= E\{(f(0, x) - E\{f(0, x)\}, \phi_1) \\ & \quad \times (f(0, x) - E\{f(0, x)\}, \phi_2)\} \end{aligned}$$

and $E\{\sigma(t)|\mathcal{F}_t^y\}$ can be obtained through the Zakai equation.

C. Likelihood functional

In our setting, the likelihood functional is given by

$$\begin{aligned} L(\theta, t_f) &= -\frac{1}{2} \int_0^{t_f} |H(t)\hat{f}(t, \cdot)|_{R^p}^2 dt \\ & \quad + \int_0^{t_f} (H(t)\hat{f}(t, \cdot))' d\vec{Y}(t) \end{aligned} \quad (39)$$

where

$$\theta = (a, \mu_0, \mu_1, m_\ell, m_r, \lambda)'$$

The maximum likelihood estimate $\hat{\theta}^N$ of θ is given by

$$\hat{\theta}^{t_f} = \operatorname{argmax}_{\theta \in D(\text{bounded set of } R^6)} L(\theta, t_f)$$

By using the generic algorithm, we also have the M.L.E. in this model.

VI. CONCLUSIONS

The parabolic type factor model with stochastic volatility is proposed. The estimation scheme for stochastic volatility is constructed by using the nonlinear filtering theory. The identification procedure for the parameters included in the stochastic volatility dynamics and factor model is to use the maximum likelihood method with the aid of genetic algorithm. From the simulation studies the proposed algorithm works well. The parameter identification for these included in the factor model is now performing by using the US-bonds data.

In order to apply the model (26) to the option pricing we need to study the arbitrage free problem as used in the large financial markets in [15].

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