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Control of Mechanical Systems with Cyclic Coordinates using Higher Order Averaging

Amit K. Sanyal,
Department of Mechanical and Aerospace Engineering,
Arizona State University,
Tempe, AZ 85287
sanyal@asu.edu

Anthony M. Bloch[2], N. Harris McClamroch[3]
[2]Department of Mathematics,
[3]Department of Aerospace Engineering,
University of Michigan,
Ann Arbor, MI 48109
{abloch@math.lsa, nhm@engin}.umich.edu

***Abstract*— The control and dynamics of complex mechanical systems with unactuated cyclic coordinates, using only internal controls, is treated here. The goal is to achieve full controllability of the reduced dynamics obtained by eliminating the cyclic coordinates using standard Routh reduction. The reduced system is also underactuated. We use high frequency, high amplitude periodic inputs and the framework of chronological calculus and averaging theory, for this purpose. A feedback scheme based on this approach is applied to the example of a dumbbell body in planar motion with an attitude control input in a central gravitational field. From our earlier work on this model, based on linearization, we know that the system is controllable at its relative equilibria. This work supplements earlier research on the possible use of internal controls for orbital maneuvers of underactuated spacecraft.**

I. INTRODUCTION

This paper deals with averaging-based control of underactuated mechanical systems with cyclic coordinates. Although the general theory presented here can be applied to all underactuated mechanical systems with drift, we focus on applications to systems that have unactuated cyclic coordinate(s). For such systems, we deal with the reduced dynamics obtained by eliminating the cyclic coordinates.

In Section II, we obtain the reduced equations of motion by Routh reduction, which have the general structure of a *1-homogeneous* system with drift, as described in [1]. In Section III, we give some general theory on averaging-based control of underactuated mechanical systems (see [1], [2], [3], [4] and references therein). We also make use of nonlinear Floquet theory, as given in [5], [1]. Although the full dynamics of such systems are simple mechanical systems as described in [6], [7], the reduced dynamics obtained from Routh reduction are described as 1-homogeneous systems, which have terms linear in velocity. An alternative way of looking at the "reduced dynamics" and control of mechanical systems with a cyclic variable is given in [9].

A specific example of a dumbbell-shaped body in central gravity, is considered for an application of the general theory in Section IV. The dumbbell body in planar motion is modeled as two identical mass particles that are connected by a rigid link. This model has been treated in our prior work [10], [11]. An alternate scheme for stabilization using potential shaping is applied to the three dimensional model of the dumbbell body in [12]. Since the only external force on the dumbbell body is central gravity, the orbital angular momentum is conserved. The reduced equations of motion for the orbit and attitude dynamics have been previously obtained in [11]. These reduced equations incorporate the attitude control inputs, and they form the basis for application of the control techniques presented here.

A control scheme to stabilize the unstable relative equilibria of this model is obtained in Section V. To verify the performance of the control law obtained for a maneuver, we carry out numerical simulations based on a variational (symplectic-momentum) integration algorithm given in [13]. Such integrators can yield accurate simulations of complex mechanical systems over long time periods. This is necessary for applications where the control scheme needs to act over long time periods. An excellent overview of variational integrators, with an emphasis on integrators that preserve geometric structures, is given in [14].

II. UNDERACTUATED MECHANICAL SYSTEMS

In this section, we describe the structure of underactuated mechanical systems with cyclic coordinates on which the Lagrangian is not dependent.

A. Underactuated Mechanical Systems with Cyclic Variables

Let $\mathbf{Q}$ be the configuration manifold of the system, and let there be c cyclic variables for the system. Then the configuration manifold can be represented trivially as $\mathbf{Q} = \mathbf{S} \times \mathbb{T}^c$, with the local coordinates (x^i, ν^α), $i = 1, \dots, m$ and $\alpha = 1, \dots, c$. The Lagrangian has the form

$$\mathcal{L}(x, \dot{x}, \dot{\nu}) = \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j + g_{(m+\alpha)i}(x)\dot{x}^i\dot{\nu}^\alpha + \frac{1}{2} g_{(m+\alpha)(m+\beta)}\dot{\nu}^\alpha\dot{\nu}^\beta - V(x),$$

where $g_{ab}(x)$ are the components of the metric at the point x, and $V(x)$ is the potential energy. The conjugate momenta p_α corresponding to the ν^α are conserved. The classical Routhian is defined by setting the p_α constant and performing a partial Legendre transformation in the ν^α: $\mathcal{R}(x, \dot{x}) = \mathcal{L}(x, \dot{x}, \dot{\nu}) - p_\alpha \dot{\nu}^\alpha$, where $\dot{\nu}^\alpha$ is a function of x and $\dot{x}$ as follows:

$$\dot{\nu}^\alpha(x) = g^{(m+\alpha)(m+\beta)}(x)(p_\beta - g_{(m+\beta)i}(x)\dot{x}^i).$$

The metric on the reduced space is defined by $h_{ij} = g_{ij} - g_{i(m+\alpha)} A^\alpha_j$, where $A^\alpha_j = g^{(m+\alpha)(m+\beta)} g_{(m+\beta)j}$ are

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the connection coefficients. The modified potential for the reduced system is

$$V_p(x) = V(x) + \frac{1}{2} g^{(m+\alpha)(m+\beta)}(x) p_\alpha p_\beta,$$

and the Routhian in terms of these quantities is

$$\mathcal{R}(x, \dot{x}) = \frac{1}{2} h_{ij}(x)\dot{x}^i \dot{x}^j + A_j^\alpha(x) p_\alpha \dot{x}^j - V_p(x).$$

Introducing the modified Routhian [2], [18]

$$\widetilde{\mathcal{R}}(x, \dot{x}) = \frac{1}{2} h_{ij}(x)\dot{x}^i \dot{x}^j - V_p(x), \tag{1}$$

the equations for the reduced dynamics take the form

$$\frac{d}{dt}\frac{\partial \widetilde{\mathcal{R}}}{\partial \dot{x}^i} - \frac{\partial \widetilde{\mathcal{R}}}{\partial x^i} = -B_{ij}^\alpha p \dot{x}^j, \tag{2}$$

and the B_{ij}^α are curvature coefficients of the connection A,

$$B_{ij}^\alpha = \frac{\partial A_i^\alpha}{\partial x^j} - \frac{\partial A_j^\alpha}{\partial x^i}.$$

If there are m actuated degrees of freedom, then applying d'Alembert's principle, the equations of motion obtained from (2) are

$$\ddot{x}^i = -\Gamma_{jk}^i \dot{x}^j \dot{x}^k - C_k^i \dot{x}^k - h^{il}\frac{\partial V_p}{\partial x^l} + h^{ia} u_a, \tag{3}$$

where

$$\Gamma_{jk}^i = \frac{1}{2} h^{il}\left(\frac{\partial h_{lj}}{\partial x^k} + \frac{\partial h_{lk}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^l}\right),$$

are the Christoffel symbols for the metric h [18], $C_k^i = h^{il} B_{lk}^\alpha p_\alpha$, and u^a, $a \in \{1, 2, \dots, m\}$, are the control inputs.

B. Homogeneous Structure of Such Systems

The fibers of the tangent bundle $\mathrm{T}\mathbf{S}$ are given by the kernel of $\mathrm{T}\pi$, where $\pi : \mathrm{T}\mathbf{S} \to \mathbf{S}$ is the (tangent bundle) projection. Let V denote the fiber space, then $V = \mathrm{Ker}(\mathrm{T}\pi)$. The notion of homogeneity with respect to the fiber is determined by the dilation operator, δ_t, which dilates the fiber as $\delta_t : \mathrm{T}\mathbf{S} \to \mathrm{T}\mathbf{S}, \quad (x, v) \mapsto (x, e^t v)$, so that $(\delta_t)^p = \delta_{pt}$. In local coordinates, the infinitesimal generator of the dilation operator is the Liouville vector field on $\mathrm{T}\mathbf{S}$ (see [15]) $\Delta = v^i \frac{\partial}{\partial v^i}$.

Definition 1. *A vector field $X \in \chi(\mathrm{T}\mathbf{S})$ is said to be homogeneous of order $p \in \mathbb{Z}$ if $[\Delta, X] = pX$, for some $p > -2$.*

Here $[\cdot, \cdot]$ denotes the Jacobi-Lie bracket and $\chi(\mathrm{T}\mathbf{S})$ denotes the set of vector fields on $\mathrm{T}\mathbf{S}$. The only smooth vector field of order less than -1 is the zero vector field and the only vector field of order -1 are the vertical lifts. We state below a few other properties of homogeneous vector fields, arising in mechanical systems applications.

Proposition 1. *Given $X,\ Y \in \chi(\mathrm{T}\mathbf{S})$, homogeneous of order p and q respectively, $[X, Y]$ is homogeneous of order $p + q$.*

From this proposition, we get the following corollary.

Corollary 1. *If $X,\ Y \in \chi(\mathrm{T}\mathbf{S})$ are vertical lifts, then* $[X, Y] = 0$.

Hence, if X, Y are vertical vector fields, the Jacobi identity implies the symmetry of the Jacobi-Lie bracket $[X, [\Gamma, Y]] = [Y, [\Gamma, X]]$, for any $\Gamma \in \chi(\mathrm{T}\mathbf{S})$.

Definition 2. *The* symmetric product of vertical lifts with respect to the vector field $\Gamma \in \chi(\mathrm{T}\mathbf{S})$ *is defined as*

$$\langle X : Y \rangle^\Gamma \equiv [X, [\Gamma, Y]],$$

where $X,\ Y \in \chi(\mathrm{T}\mathbf{S})$ are vertical vector fields.

This is a generalization of the symmetric product of Lewis and Murray [7] and the definition of Crouch [8] in terms of gradient vector fields. In application to a mechanical system with a drift vector field Γ, we will simply write $\langle X : Y \rangle$ for this symmetric product on $\chi(\mathrm{T}\mathbf{S})$.

There is a $\mathbb{Z}$-gradation of homogeneous spaces created from homogeneous vector fields in the set $\chi(\mathrm{T}\mathbf{S})$. The subspace of vector fields of homogeneous order $k \in \mathbb{Z}$ is denoted $\mathcal{P}_k$. The following properties hold for these homogeneous spaces: $[\mathcal{P}_i, \mathcal{P}_j] \subset \mathcal{P}_{i+j}$, and $\mathcal{P}_k = \{0\}$, $\forall\ k < -1$. Accordingly, we may define the following direct sum of homogeneous spaces

$$\mathcal{M}_k = \oplus_{i=-1}^{i \le k} \mathcal{P}_i, \tag{4}$$

which inherit the properties of its homogeneous parts, $[\mathcal{M}_i, \mathcal{M}_j] \subset \mathcal{M}_{i+j}$, and $\mathcal{M}_k = \{0\}$, $\forall\ k < -1$. For mechanical systems of the form (3), the drift vector field is in $\mathcal{M}_1$ (we call such systems *1-homogeneous*), and the control vector fields are in $\mathcal{M}_{-1}$ since they are lifts.

III. Averaging for Underactuated Mechanical Systems with Drift

In this section, we use averaged feedback using the series expansion methods in [15] for mechanical systems to obtain control algorithms that are asymptotically stabilizing.

A. Nonlinear Floquet Theory

We consider a non-autonomous nonlinear system

$$\dot{y} = f(y, t/\epsilon), \tag{5}$$

where $y \in M$, M is a smooth manifold, and $f(y, \tau)$ is a time-varying vector field that is periodic with period T. Introducing a time scaling $\tau = t/\epsilon$, we get

$$y' = \frac{dy}{d\tau} = \epsilon f(y, \tau). \tag{6}$$

Let $\Phi_{0,\tau}^{\epsilon f}$ be the flow generated by the vector field ϵf in the time interval from 0 to τ. Hence, $\Phi_{0,\tau}^{\epsilon f}$ is a family of diffeomorphisms (on M) satisfying the differential equation

$$d\Phi_{0,\tau}^{\epsilon f}/d\tau = \Phi_{0,\tau}^{\epsilon f} \circ \epsilon f(\cdot, \tau), \quad \Phi_{0,0}^{\epsilon f} = \mathrm{Id}, \tag{7}$$

where Id denotes the identity map. an integral series form (see [5]), which is denoted compactly by $\Phi_{0,\tau}^{\epsilon f} = \overrightarrow{\exp} \int_0^\tau \epsilon f(\cdot, \sigma) d\sigma$, and called the *right chronological exponential*, following [16]. The diffeomorphism $M_\epsilon = \Phi_{0,T}^{\epsilon f}$, which gives the flow generated by (6) over one period, is called the *monodromy map*; obviously, $\Phi_{0,\tau+T}^{\epsilon f} = M_\epsilon \circ \Phi_{0,\tau}^{\epsilon f}$. We now present an important result stated in [5].

Proposition 2. *The origin is aymptotically stable for the system (6) if it is an asymptotically stable fixed point for the monodromy map M_ϵ.*

It is more convenient to deal with the logarithm of M_ϵ, which (if it exists) is a linear object. Assuming that this logarithm (vector field) exists, we denote it as Λ_ϵ, so that $M_\epsilon = \exp \Lambda_\epsilon$. We now state the analog of Floquet theorem for nonlinear time-periodic systems.

Theorem 1. *(Nonlinear Floquet Theorem) Let $\Phi_{0,\tau}^{\epsilon f}$ denote the flow of the time-periodic nonlinear system (6), where $f(y, \tau+T) = f(y,\tau)$. If the vector field Λ_ϵ is the logarithm of the diffeomorphism $M_\epsilon = \Phi_{0,T}^{\epsilon f}$, then the flow $\Phi_{0,\tau}^{\epsilon f}$ can be represented as a composition $\Phi_{0,\tau}^{\epsilon f} = P(\tau) \circ \exp(\Lambda_\epsilon \tau / T)$ where $P(\tau) = P(\tau+T)$ is a periodic flow.*

B. Averaging Control of Underactuated Mechanical Systems

An underactuated mechanical system with high frequency, high amplitude, periodic control inputs can be expressed as

$$\dot{y} = X(y) + \frac{1}{\epsilon} Y_a^{\mathrm{lift}}(y) u_0^a(y, t/\epsilon), \tag{8}$$

where $y = (x, \dot{x}) \in \mathrm{T}\mathbf{S}$, X is the drift vector field, $0 < \epsilon \ll 1$ is a small parameter, Y_a^{lift} and $u_0^a(\cdot,\cdot)$, $a = 1, \ldots, n < m$ are control vector fields and control inputs. For systems like (3), $X \in \mathcal{M}_1$, and we have an underactuated *1-homogeneous* system. We assume that the directly actuated states can be stabilized by state feedback, and the control inputs have additional time-periodic vibrational terms;

$$u_0^a(y,t) = \epsilon f^a(y) + u^a(t/\epsilon), \tag{9}$$

with $u^a(\cdot)$ T-periodic. We scale time such that $t/\epsilon \mapsto \tau$, to obtain

$$y' = \epsilon X_S(y) + Y_a^{\mathrm{lift}}(y) u^a(\tau), \tag{10}$$

where $X_S(y) = X(y) + Y_a^{\mathrm{lift}}(y) f^a(y) \in \mathcal{M}_1$. The flow of equation (10) can be obtained from the *variation of constants* formula [16]. We define the vector field

$$Y(y,\tau) = \left(\Phi_{0,\tau}^{Y_a^{\mathrm{lift}} u^a}\right)^* (\epsilon X_S)(y), \tag{11}$$

where $(\Phi^G)^*$ denotes the pull-back of the flow map Φ^G, along the vector field G. The variation of constants formula gives the perturbed flow as the composition of flows

$$\Phi_{0,\tau}^{\epsilon X_S + Y_a^{\mathrm{lift}} u^a} = \Phi_{0,\tau}^{Y_a^{\mathrm{lift}} u^a} \circ \Phi_{0,\tau}^{Y}. \tag{12}$$

If X_1 and X_2 are time-varying, then as given in [16], [17],

$$\left(\Phi_{0,\tau}^{X_2}\right)^* X_1(y,\tau) = X_1(y,\tau) + \sum_{k=1}^{\infty} \int_0^\tau \cdots \int_0^{\sigma_{k-1}} \left(\mathrm{ad}_{X_2(y,\sigma_k)} \cdots \mathrm{ad}_{X_2(y,\sigma_1)} X_1(y,\tau)\right) d\sigma_k \cdots d\sigma_1. \tag{13}$$

This series does not converge in general, but for 1-homogeneous systems, it is always convergent since only the first two terms of the summation are nonvanishing. Hence, (11) takes the form

$$Y = \epsilon X_S + \epsilon U_{(1)}^{(a)}(\tau)[Y_a^{\mathrm{lift}}, X_S] - \frac{1}{2}\epsilon U_{(1,1)}^{(a,b)}(\tau) \langle Y_a^{\mathrm{lift}} : Y_b^{\mathrm{lift}} \rangle, \tag{14}$$

where the $U_{(\cdot)}^{(\cdot)}(\tau)$ terms are the *averaging coefficients*, given, for example, by

$$U_{(1,1)}^{(a,b)}(\tau) = U_{(1)}^{(a)} U_{(1)}^{(b)} = \Big(\int_0^\tau u^a(\sigma_1) d\sigma_1 \Big) \Big(\int_0^\tau u^b(\sigma_1) d\sigma_1 \Big),$$

$$U_{(n)}^{(a)}(\tau) = \int_0^t \int_0^{\sigma_{n-1}} \cdots \int_0^{\sigma_2} u^a(\sigma_1) d\sigma_1 \cdots d\sigma_n.$$

When time-averaged, these coefficients become the *averaged coefficients* for the averaged system. Since the flow $\Phi_{0,\tau}^{Y_a^{\mathrm{lift}}(y) u^a(\tau)}$ is also T-periodic, the averaged system corresponding to (10) is the averaged flow along along $Y(y,\tau)$.

C. Averaged System

Nonlinear Floquet theory decomposes the flow of the vector field Y in (12) as $\Phi_{0,\tau}^{Y} = P(\tau) \circ \exp(Z\tau)$, where $P(\tau)$ is a T-periodic mapping and Z is the autonomous, averaged vector field. Since $\Phi_{0,\tau}^{Y_a^{\mathrm{lift}}(y) u^a(\tau)}$ is also T-periodic, the flow of the system (10) is given by

$$\Phi_{0,\tau}^{\epsilon X_S + Y_a^{\mathrm{lift}} u^a} = \Phi_{0,\tau}^{Y_a^{\mathrm{lift}} u^a} \circ P(\tau) \circ \exp(Z\tau). \tag{15}$$

We define the time-averaged coefficients

$$\overline{U_{(N)}^{(A)}} = \frac{1}{T} \int_0^T U_{(N)}^{(A)}(\tau) d\tau,$$

where the multi-index notation $(A) = (a_1, a_2, \ldots, a_{|A|})$ and $(N) = (n_1, n_2, \ldots, n_{|N|})$ of [1] is used. A second order (in ϵ) approximation to Z is given by

$$\begin{aligned} Z = {} & \epsilon X_S + \epsilon \overline{U_{(1)}^{(a)}} [Y_a^{\mathrm{lift}}, X_S] - \frac{1}{2}\epsilon \overline{U_{(1,1)}^{(a,b)}} \langle Y_a^{\mathrm{lift}} : Y_b^{\mathrm{lift}} \rangle \\ & + \epsilon^2 C_1 [[Y_a^{\mathrm{lift}}, X_S], X_S] - \frac{1}{2}\epsilon^2 C_2 [\langle Y_a^{\mathrm{lift}} : Y_b^{\mathrm{lift}} \rangle, X_S] \\ & + \frac{1}{2}\epsilon^2 \overline{U_{(2,1)}^{(a,b)}} [[Y_a^{\mathrm{lift}}, X_S], [Y_b^{\mathrm{lift}}, X_S]] + \frac{1}{2}\epsilon^2 \\ & \left(\overline{U_{(2,1,1)}^{(a,b,c)}} - \frac{1}{2} T\ \overline{U_{(1)}^{(a)}}\ \overline{U_{(1,1)}^{(b,c)}} \right) \langle Y_a^{\mathrm{lift}} : \langle Y_b^{\mathrm{lift}} : Y_c^{\mathrm{lift}} \rangle\rangle, \end{aligned} \tag{16}$$

where

$$C_1 = \left(\overline{U_{(2)}^{(a)}} - \frac{1}{2} T\ \overline{U_{(1)}^{(a)}} \right), \quad C_2 = \left(\overline{U_{\widehat{(1,1)}}^{\widehat{(a,b)}}} - \frac{1}{2} T\ \overline{U_{(1,1)}^{(a,b)}} \right).$$

The $\widehat{\ }$ symbol denotes integrals within the product structure, for example,

$$U_{(\widehat{(0,0)},1)}^{(\widehat{(a,b)},c)}(\tau) = \Big(\int_0^\tau U_{(0,0)}^{(a,b)}(\sigma) d\sigma \Big) \Big(U_{(1)}^{(c)}(\tau) \Big).$$

Changing back to the original time coordinate, the flow of the (second order) averaged autonomous system is given by

$$\dot{z} = \frac{1}{\epsilon} Z(z). \tag{17}$$

D. Feedback Stabilization with Sinusoidal Inputs

By modulating the values of the averaged coefficients $\overline{U_{(\cdot)}^{(\cdot)}}$, we can control the averaged system (17). We use sinusoidal input signals parametrized by their amplitudes:

$$u^a(t) = \alpha^a \omega f(\sin(\omega t)), \tag{18}$$

where $T = 2\pi/\omega$ is the period and f is a polynomial. Let the ordered brackets in the second order expansion of the averaged vector field be denoted by $\bar{Y}_j$, and their corresponding averaged coefficients be denoted $T^j(\alpha)$, $j = 1, \ldots, N$. The averaged equations can then be put into the form

$$\dot{z} = X_S(z) + \frac{1}{\epsilon}T^j(\alpha)\bar{Y}_j(z) = X_S(z) + \frac{1}{\epsilon}B(z)H(\alpha), \quad (19)$$

where the matrices B and H are given by $B(z) = [\bar{Y}_1 \ldots \bar{Y}_N](z)$ and $H(\alpha) = [T^1 \ldots T^N]^\top(\alpha)$.

Asymptotic stability of the averaged system (17) will imply asymptotic stability of the actual system by Proposition 3. We now present a continuous feedback law that stabilizes an equilibrium of the reduced system in $\mathrm{T}\mathbf{S}$.

Theorem 2. *Consider the system (8) with controls given by (9) and an equilibrium, $y_e \in \mathrm{T}\mathbf{S}$. Let $\alpha \in \mathbb{R}^n$ denote the vector of amplitudes α^a of the T-periodic inputs in (18). Let $z(t)$ denote the averaged system response given by (19). With the directly controlled states linearly stabilized with state feedback as in (9), assume that*

$$G = \frac{\partial H}{\partial \alpha}\Big|_{\alpha=0} \in \mathbb{R}^{N\times n}$$

is of maximal rank. If there exists a $K \in \mathbb{R}^{n\times 2m}$ such that the matrix $A - \frac{1}{\epsilon}BGK$ has a stable spectrum, where

$$A = \frac{\partial X_S(z)}{\partial z}\Big|_{z=y_e} + T^j(0)\frac{\partial \bar{Y}_j(z)}{\partial z}\Big|_{z=y_e}, \; B = B(y_e),$$

then control inputs of the form (18) with

$$\alpha = -K\big(z(t) - y_e\big), \quad (20)$$

will stabilize the system about the equilibrium y_e.

Proof: The linearized dynamics of the averaged system (19) about $z = y_e$ and $\alpha = 0$ is given by

$$\dot{e}(t) = Ae(t) + \frac{1}{\epsilon}BG\alpha, \;\; e(t) = z(t) - y_e, \quad (21)$$

where α parametrizes the control inputs. Now we define

$$\bar{A} = A - \frac{1}{\epsilon}BGK,$$

such that the linearized closed-loop system with the control law (20) has the form $\dot{e}(t) = \bar{A}e(t)$. Hence the averaged system is stabilized if $\bar{A}$ has eigenvalues with negative real part. □

A difficulty with this control scheme is that it is based on feedback of the averaged state. From [1] we get the following series expansion for $P(\tau)$ in equation (15) upto first order in ϵ:

$$P(\tau) = \mathrm{Id} + \epsilon\int_0^\tau \widetilde{U}^{(a)}_{(1)}(\sigma)d\sigma[Y_a^{\text{lift}}, X_S] - \frac{1}{2}\epsilon\int_0^\tau \widetilde{U}^{(a,b)}_{(1,1)}(\sigma)d\sigma\langle Y_a^{\text{lift}} : Y_b^{\text{lift}}\rangle + O(\epsilon^2), \quad (22)$$

where $\widetilde{U}^{(A)}_{(N)}(\tau) = U^{(A)}_{(N)}(\tau) - \overline{U^{(A)}_{(N)}}$. The inverse of $P(\tau)$ is given upto $O(\epsilon)$ by

$$P^{-1}(\tau) = \mathrm{Id} - \epsilon\int_0^\tau \widetilde{U}^{(a)}_{(1)}(\sigma)d\sigma[Y_a^{\text{lift}}, X_S] + \frac{1}{2}\epsilon\int_0^\tau \widetilde{U}^{(a,b)}_{(1,1)}(\sigma)d\sigma\langle Y_a^{\text{lift}} : Y_b^{\text{lift}}\rangle + O(\epsilon^2). \quad (23)$$

Hence, one can obtain the averaged state from the actual state by inverting equation (15) as follows

$$z(t) = P^{-1}(t)\circ\Phi^{u^a Y_a^{\text{lift}}}_{t,0}(y(t)). \quad (24)$$

Since $u^a Y_a^{\text{lift}}$ is a T-periodic control vector field and the $Y_a^{\text{lift}} \in \mathcal{M}_{-1}$ commute, we can evaluate the flow $\Phi^{u^a Y_a^{\text{lift}}}_{0,t}$ and its inverse quite easily. In the following sections, we apply these general results to the example problem of a planar dumbbell body in central gravity.

IV. Reduced Dynamics of Dumbbell Body in Space

We apply the control scheme given in the last section to a rigid dumbbell body in central gravity, as depicted in Figure 1. The polar coordinates (r, ν), $\nu \in \mathbb{S}$, give the position vector of the center of the dumbbell body in an inertial frame fixed to the central body. The attitude θ is the angle between the longitudinal axis of the dumbbell body and its position vector. Let m and $2l$ be the mass of each end mass particle and the length of the rigid link of the dumbbell, respectively. The attitude control input N acts on each end mass normal to the connecting link, thereby generating a pure moment.

A. Equations of Motion

The Lagrangian of this system is given by

$$\mathcal{L} = T - V_g = m\Big(\dot{r}^2 + l^2\dot{\theta}^2 + 2l^2\dot{\theta}\dot{\nu} + l^2\dot{\nu}^2 + r^2\dot{\nu}^2 + \frac{\mu}{r}\Big(2 - \frac{l^2}{r^2}(1 - 3\cos^2\theta)\Big)\Big), \quad (25)$$

where μ denotes the strength of the gravitational potential. Note that ν is a cyclic variable for this Lagrangian, and the corresponding orbital angular momentum

$$p = \frac{\partial\mathcal{L}}{\partial\dot{\nu}} = 2m((r^2 + l^2)\dot{\nu} + l^2\dot{\theta}), \quad (26)$$

of the dumbbell body is conserved.

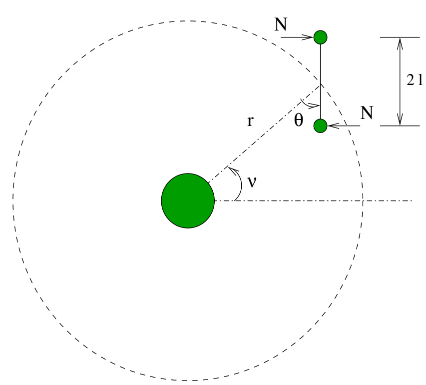

Fig. 1. Dumbbell body in planar orbit in central gravitational field.

We carry out classical Routh reduction ([2], [18]) to obtain reduced equations of motion. The Routhian is approximated using a second order expansion in $\frac{l}{r}$, to obtain

$$\mathcal{R}(r,\theta,\dot{r},\dot{\theta}) = m\Big(\dot{r}^2 + l^2\dot{\theta}^2\big(1 - \frac{l^2}{r^2}\big)\Big) + \frac{pl^2}{r^2}\dot{\theta}\times \Big(1 - \frac{l^2}{r^2}\Big) - \frac{p^2}{4mr^2}\Big(1 - \frac{l^2}{r^2}\Big) - V_g(r,\theta). \quad (27)$$

The quantity $V_a(r) = \frac{p^2}{4mr^2}\Big(1 - \frac{l^2}{r^2}\Big)$ is called the *amendment*, and $V_p(r,\theta) = V_a(r) + V_g(r,\theta)$ is the modified potential.

For convenience in averaging and numerical simulation, we scale the dynamics equations. Let R be the characteristic radius of orbit of the dumbbell body, and let us define:

$$\rho = \frac{r}{R}, \quad \Omega^2 = \frac{\mu}{R^3}, \quad \tau = \Omega t, \quad \text{and } \epsilon = \frac{l}{R}.$$

Note that τ here has a different meaning to that in Section 3, not be confused as a scaling of time by the large factor $1/\epsilon$; instead, the scaling here is by the (usually small) factor Ω. We also define the scaled angular momentum

$$\bar{p} = 2\{\rho^2\nu' + \epsilon^2(\theta' + \nu')\}, \tag{28}$$

which is conserved. The reduced equations of motion in these scaled coordinates are given in the form (3) below:

$$\begin{aligned} \rho'' &= \frac{\epsilon^4}{\rho^3}\Big(1 - \frac{2l^2}{r^2}\Big)\theta'^2 - \frac{\bar{p}\epsilon^2}{\rho^3}\Big(1 - \frac{2l^2}{r^2}\Big)\theta' \\ &\quad + \frac{\bar{p}^2}{4\rho^3}\Big(1 - \frac{2\epsilon^2}{\rho^2}\Big) - \frac{1}{\rho^2} + \frac{3\epsilon^2}{2\rho^4}\big(1 - 3\cos^2\theta\big) \end{aligned} \tag{29}$$

$$\begin{aligned} \theta'' &= -\frac{2\epsilon^2}{\rho^3}\Big(1 - \frac{\epsilon^2}{\rho^2}\Big)r'\theta' + \frac{\bar{p}}{\rho^3}\Big(1 - \frac{\epsilon^2}{\rho^2}\Big)r' \\ &\quad - \frac{3}{\rho^3}\Big(1 + \frac{\epsilon^2}{\rho^2}\Big)\cos\theta\sin\theta + \frac{\bar{N}}{\epsilon}\Big(1 + \frac{\epsilon^2}{\rho^2}\Big), \end{aligned} \tag{30}$$

where $\bar{N} = \frac{N}{mR\Omega^2}$. These equations have the same form as equation (8), with ϵ having the same role.

B. Stability and Controllability of Relative Equilibria

We identify the relative equilibria of this dumbbell body in central gravity from the free reduced dynamics ($N = 0$). The relative equilibria, $x_e = (r_e, \theta_e)$, $\dot{x} = 0$, $\ddot{\nu} = 0$, are the critical points of the modified potential:

$$\theta_e = n\pi, \quad n \in \mathbb{Z}, \quad \nu_e'^2 = \frac{1}{\rho_e^3} + \frac{3\epsilon^2}{\rho_e^5}, \tag{31}$$

$$\theta_e = (n + \frac{1}{2})\pi, \quad n \in \mathbb{Z}, \quad \dot{\nu}_e^2 = \frac{1}{\rho_e^3} - \frac{3\epsilon^2}{2\rho_e^5}. \tag{32}$$

From our previous work in [10], [11], we know that the first set of relative equilibria given by (31) is stable; this result is obtained from the Routh stability criterion ([2], [18]). The second set of relative equilibria given by (32) is unstable.

In [11], the reduced equations were linearized about these equilibria, and their controllability properties obtained. We repeat part of that analysis here. If we denote the vector of reduced configuration perturbations by $\delta x = [\delta r \ \delta\theta]^\top$, then these linearized equations of motion can be expressed as a second order differential equation of the form

$$M\delta\ddot{x} + C\delta\dot{x} + K\delta x = BN, \tag{33}$$

where M is a symmetric positive definite inertia matrix, C is a skew-symmetric matrix representing gyroscopic terms, K is a symmetric "stiffness" matrix, and B is a control influence vector. The system (33) is completely controllable if and only if the controllability rank condition ([19])

$$\text{rank}[\lambda^2 M + \lambda C + K, \ B] = 2 \tag{34}$$

holds for all λ that satisfies $\det[\lambda^2 M + \lambda C + K] = 0$. This controllability condition gives the following result.

Proposition 3. *The linearized equations of motion for the reduced dynamics are completely controllable if only the attitude is actuated.*

V. Vibrational Control of the Dumbbell Body

A. Stabilization of Unstable Relative Equilibrium

We stabilize the above system about its unstable (relative) equilibria given by (32), using a control law of the form (9)

$$\bar{N}(y,\tau) = \epsilon\bar{N}_f(y) + \bar{N}_a(\alpha,\tau), \quad \bar{N}_f = -10\big((\theta - \frac{\pi}{2}) + \theta'\big). \tag{35}$$

The first term $\bar{N}_f(y)$ linearly stabilizes the directly actuated variable to its equilibrium value $\theta_e = \frac{\pi}{2}$. The second term is used to feedback stabilize the averaged system according to Theorem 2. The modified drift and control vector fields are:

$$X_S = \begin{bmatrix} \rho' \\ \theta' \\ f1 \\ f2 \end{bmatrix}, \quad Y^{\text{lift}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 + \frac{\epsilon^2}{\rho^2} \end{bmatrix},$$

where f_1 and f_2 are the right sides of equations (29) and (30) respectively, with $\bar{N}$ replaced by $\bar{N}_f$. The coordinate expressions for the Lie brackets and symmetric products in the second order expansion (16), and evaluated for this system, are given in [13]. For this system, the vector field $\langle Y^{\text{lift}} : \langle Y^{\text{lift}} : Y^{\text{lift}}\rangle\rangle$ is negligible. The vector fields $\bar{Y}_1 = [Y^{\text{lift}}, X_S]$, $\bar{Y}_2 = \langle Y^{\text{lift}} : Y^{\text{lift}}\rangle$, $\bar{Y}_3 = [[Y^{\text{lift}}, X_S], X_S]$, and $\bar{Y}_4 = [\langle Y^{\text{lift}} : Y^{\text{lift}}\rangle, X_S]$ evaluated at almost all points of TS span $\mathbb{R}^4$. Note that, although two of these brackets are "bad" brackets in the sense of Sussmann ([20]), their coefficients given by this vibrational control scheme (see Figure 2) are not always of the same sign. Hence, for feedback using this scheme, these brackets cannot be considered as "bad" since they provide additional control directions in the state space.

Now we feedback stabilize the averaged system corresponding to (29)-(30) with the vibrational control

$$\bar{N}_a(\tau) = \alpha\omega(1 - \cos(2\omega\tau)) + \omega\sin(\omega\tau). \tag{36}$$

The averaged coefficients of $\bar{Y}_1$, $\bar{Y}_2$, $\bar{Y}_3$, and $\bar{Y}_4$ are evaluated for the control law (36). These coefficients are provided in Figure 2. We choose the parameter value $\epsilon = \frac{l}{R} = 0.05$, and the frequency of the vibrational control $\omega = 50$. Tthe control (36) is a feedback law depending on the averaged state, $\alpha = \alpha(z)$, as follows

$$\alpha = -K(z - y_e) = -k_1\big(\theta - \frac{\pi}{2}\big) - k_2\theta'. \tag{37}$$

Note that we only use attitude and attitude rate feedback; effectively using the coupling between the attitude and radial degrees of freedom to asymptotically stabilize the system without feedback on the dumbbell's position.

Coefficient	Value
$T^1(\alpha)$	$\epsilon(1+\pi\alpha)$
$T^2(\alpha)$	$-\frac{\epsilon}{2}\left(\frac{3}{2}+2\pi\alpha+\frac{5\alpha^2}{8}+\frac{4\pi^2\alpha^2}{3}\right)$
$T^3(\alpha)$	$\epsilon^2\left(\frac{3}{4}+\pi\alpha+\frac{5\alpha^2}{16}+\frac{2\pi^2\alpha^2}{3}-\frac{\pi(1+\pi\alpha)}{\omega}\right)$
$T^4(\alpha)$	$-\frac{\epsilon^2}{2}\left\{\frac{100\alpha+\pi(36+\alpha(3\alpha+16\pi(2+\pi\alpha)))}{24\omega}-\frac{\pi}{\omega}\left(\frac{3}{2}+2\pi\alpha+\frac{5\alpha^2}{8}+\frac{4\pi^2\alpha^2}{3}\right)\right\}$

Fig. 2. Coefficients for the first four brackets in the second order expansion of the averaged controlled dumbbell dynamics.

B. Simulation Results

We present simulation results for this feedback scheme used to stabilize an unstable relative equilibrium of the dumbbell body given by $y_e = (\rho_e, \theta_e, \rho'_e, \theta'_e) = (0.99826, \frac{\pi}{2}, 0, 0)$, $\bar{p} = 2.00102$. The scalar gains k_1 and k_2 are chosen so that the condition in Theorem 2 is satisfied. We evaluate all the matrices defined in Theorem 2 for the second order averaged system linearized about y_e. Choosing $k_1 = 0.7$, and $k_2 = -2.1$, we find that the eigenvalues of $\bar{A} = A - \frac{1}{\epsilon}BGK$ have negative real parts. Thus, the unstable relative equilibrium is asymptotically stabilized.

For a simulation of the dumbbell body in central gravity with this averaging-based feedback control scheme, we give an initial perturbation to the unstable relative equilibrium with $\rho_0 = 1.00027$, $\theta_0 = 1.6008$, $\nu'_0 = 1.001555$, $\rho'_0 = \theta'_0 = 0$. The simulation results are given by Figure 3, for both actual and averaged systems. The averaged system (which

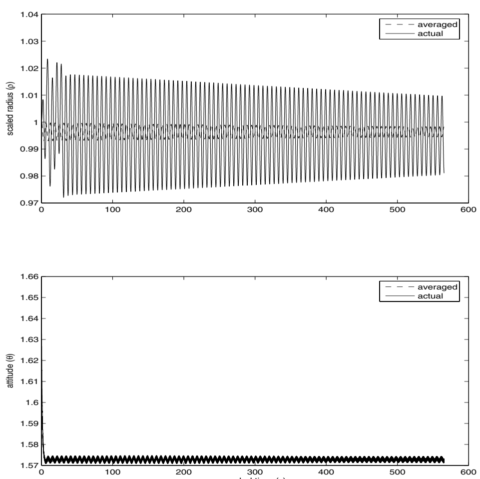

Fig. 3. Simulation results for stabilization of unstable equilibrium.

is not Lagrangian) was integrated using MATLAB's ode45, while the actual system was integrated by a variational integration scheme given in [13]. The simulation is for a duration of $\tau = 180\pi$ (about 90 orbits around the central body). The results show that the orbit radius asymptotically approaches that of the desired relative equilibrium, and the attitude angle has small oscillations around $\theta = \frac{\pi}{2}$ for both the averaged and the actual system. Since the coupling from attitude to orbit is weak (of order ϵ^2), the convergence in the orbital radius is slow.

VI. Conclusions

We present an averaging-based control scheme for under-actuated mechanical systems with cyclic coordinates. This scheme is applied to a dumbbell body in planar motion in a central gravitational field, with attitude actuation. High frequency periodic inputs are used to stabilize the dumbbell body at one of its unstable relative equilibria. Asymptotic stabilization of the orbital radius is slow, due to the small coupling between the attitude and orbit degrees of freedom. These results demonstrate the possibility of using attitude and/or internal actuation for control of the motion of complex mechanical systems in a potential field.

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