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A Universal Control Approach for a Family of Uncertain Nonlinear Systems

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Abstract— **We study the problem of global adaptive stabilization by output feedback for nonlinear systems with unknown parameters. The class of uncertain systems under consideration is assumed to be dominated by a bounding system which is linear growth in the unmeasurable states but can be a polynomial function of the system output, with unknown growth rate. To achieve global stabilization in the presence of parametric uncertainty, we propose a non-identifier based output feedback control law using the idea of universal control integrated with the linear-like output feedback control scheme proposed recently. In particular, we explicitly design a *universal output* feedback controller which globally regulates all the states of the uncertain system while maintaining global boundedness of the closed-loop system.**

I. Introduction

In this paper we consider the problem of global adaptive stabilization by output feedback, for a family of single-input single-output uncertain nonlinear systems in the following triangular form

$$\begin{aligned} \dot{x}_1 &= x_2 + \phi_1(x_1, \theta) \\ \dot{x}_2 &= x_3 + \phi_2(x_1, x_2, \theta) \\ &\vdots \\ \dot{x}_n &= u + \phi_n(x_1, \cdots, x_n, \theta) \\ y &= x_1 \end{aligned} \tag{1.1}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, input and output, respectively, $\theta \in R^s$ is an unknown constant vector, $\phi_i : R^i \times R^s \to R$, $i = 1, \cdots, n$, are C^1 functions with $\phi_i(0, \cdots, 0, \theta) = 0$. They represent the system uncertainty and need not to be precisely known.

Under the assumption that ϕ_i is a C^1 function with $\phi_i(0, \theta) = 0$, it is straightforward to deduce the existence of smooth functions $b_i(x_1, \cdots, x_i) \geq 1$ and $\gamma_i(\theta) \geq 1$, such that (see Lemma 2.5 in [12])

$$|\phi_i(\cdot)| \leq (|x_1| + \cdots + |x_i|) b_i(x_1, \cdots, x_i) \gamma_i(\theta)$$

Denote $\Theta = \sum_{i=1}^{n} \gamma_i(\theta)$. Then,

$$|\phi_i(\cdot)| \leq (|x_1| + \cdots + |x_i|) b_i(x_1, \cdots, x_i) \Theta \tag{1.2}$$

for $i = 1, \cdots, n$. In other words, the uncertain system (1.1) always satisfies the constraint (1.2).

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In the paper [10], counter-examples were given indicating that if the system nonlinearity $\phi_i(\cdot)$ or, equivalently, the bounding function $b_i(x_1, \cdots, x_n)$ in (1.2) grows faster than a quadratic function with respect to the unmeasurable states $(x_2, \cdots, x_n)$, system (1.1) (even in the absence of the unknown parameters) may not be globally stabilized by smooth output feedback. With this in mind and in view of (1.2), it is natural to assume that the bounding function $b_i(x_1, \cdots, x_i) = b_i(x_1)$, i.e., depends only on the system output. This hypothesis, together with inequality (1.2), yields the existence of a smooth function $c(y)$ satisfying

$$|\phi_i(x_1, \cdots, x_i, \theta)| \leq (|x_1| + \cdots + |x_i|) c(y) \Theta. \tag{1.3}$$

In the case when Θ is a known constant, global output feedback stabilization of the uncertain nonlinear system (1.1) was shown to be possible [8], [18], as long as the growth condition (1.3) is fulfilled. In [15], it has been further shown that global stabilization of the uncertain system (1.1) can even be achieved by a linear-like output feedback controller with dynamic gain, under the extra requirement that $c(y)$ in (1.3) is a polynomial function of the system output, or, equivalently, the following condition:

Assumption 1.1: The bounding function $b_i(\cdot)$ in (1.2) is dominated by a polynomial function of the output, i.e.

$$b_i(x_1, \cdots, x_n) \leq c(y) = 1 + |y|^{\nu}, \qquad i = 1, \cdots, n,$$

where $\nu \geq 1$ is a known integer.

When Θ in (1.3) represents an *unknown* constant, global output feedback control of the uncertain system (1.1) becomes much more involved and difficult, due to the lack of effective adaptive observer design techniques [14]. In the existing literature, most of the global adaptive stabilization results via output feedback are only applicable to a class of uncertain nonlinear systems in the *parametric output feedback* form [5], [13], [4], [7], i.e., the system uncertainty $\phi_i(t, x, u)$ in (1.1) depends only on the output y multiplied by the unknown parameter θ. They cannot, however, be used to deal with nonlinear systems with unknown parameters beyond the parametric output feedback form, such as the uncertain system (1.1) satisfying Assumption 1.1, in which the unknown parameters appear not only in the front of the system output y but also in the front of the unmeasurable states $(x_2, \cdots, x_n)$. The latter makes the application of the conventional observer design method [9] or the high-gain observer [2], [6] difficult, because the constructed observer contains a copy of the original system with unknown parameters and hence is not implementable. For this reason, global adaptive stabilization of the uncertain system (1.1) by output feedback is a nontrivial problem, even under a linear growth condition, i.e., $c(y) = 1$ in the growth condition (1.3).

When $c(y) = 1$ in (1.3), the problem of global adaptive stabilization by output feedback has been addressed recently in [11], where the idea of the non-identifier based adaptive control [1], [3], [17], also referred as universal control, was exploited and coupled with the non-separation principle based output feedback design method [16], resulting in a solution to the output feedback control problem in the presence of unknown parameter θ. The novelty of the work [11] was the introduction of an observer gain update law which is motivated by the design of universal controllers [17], [3].

In this paper, we show that by taking advantage of the linear structure of the output feedback controller in [15], integrated with the universal output feedback design method developed in [11], it is possible to extend the adaptive output feedback control result of [11] to a larger class of nonlinear systems characterized by Assumption 1.1. To be precise, we shall prove that under Assumption 1.1, there is a smooth dynamic output compensator of the form

$$\begin{aligned} \dot{z} &= F(z, L, M, y), \qquad z \in \mathbb{R}^{\mathrm{n}} \\ \dot{L} &= H_1(z, L, M, y) \quad \text{and} \quad \dot{M} = H_2(M, y) \\ u &= g(z, L, M, y) \end{aligned} \tag{1.4}$$

such that all the solutions of the closed-loop system (1.1)-(1.4) are well-defined and globally bounded on $[0, +\infty)$. Moreover,

$$\lim_{t\to+\infty}(x(t), z(t)) = (0, 0).$$

The underlying philosophy here is, on one hand, to use the linear output feedback controller from [15] to deal with the uncertain nonlinear system (1.1) satisfying Assumption 1.1, and on the other hand, to employ the idea of universal control as done in [11] to handle the unknown parameter θ in (1.1) or Θ in (1.2). To achieve these two objectives simultaneously, the observer gain must be designed and tuned in a delicate manner. Compared with the work [15], [11], the observer gain in this paper is composed of two components M and L, both of them are not constant and need to be dynamically updated. The gain L is tuned in a universal manner, similar to the one suggested in [11]. The gain M, which plays a similar role as the one in [15], is updated on-line via a *linear*, rather than Riccati, differential equation driven by a nonlinear function of the output y. This is different from the gain update law introduced in [15] where the observer gain is tuned by a suitable Riccati differential equation. As we shall see in the next section, the introduction of the new dynamic gain update laws will be the key for the success of the proposed universal output feedback control scheme.

II. Universal Output Feedback Design

In this section we prove that under Assumption 1.1, it is possible to explicitly construct a universal output feedback controller of the form (1.4), achieving global boundedness and state regulation for the nonlinear system (1.1) with unknown parameters.

Theorem 2.1: Under Assumption 1.1, there is a $(n+2)$-dimensional output feedback compensator of the from (1.4) such that from any initial condition $(x_0, z_0) \in \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}$ and $M(0) = L(0) = 1$, the closed-loop system (1.1)-(1.4) has the following properties:

(i) all the states of the closed-loop system (1.1)-(1.4) are well-defined and globally bounded on $[0, +\infty)$;

(ii) $\lim_{t\to+\infty}(x(t), z(t)) = (0, 0)$.

Proof. The proof is constructive and carried out by explicitly designing a universal output feedback control law. While the construction of a high-gain observer is routine as done in [6], [2], the design of the observer gain update law is new. The gain update law for M bears a resemblance to the papers [15], [8], [18] but is distinct from them in the sense that M is tuned on-line through a linear ODE rather than a Riccati equation, enabling one to take full advantage of the linear structure of the output feedback control law. On the other hand, the gain update law for L is reminiscent of the work [11], where an adapted high gain observer was designed and the observer gain is tuned in a universal fashion.

As done in the paper [16], by discarding the uncertain terms $\phi_i(x_1, \cdots, x_i, \theta), 1 \le i \le n$, we design a high-gain observer for the uncertain nonlinear system (1.1)

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + (LM)a_1(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + (LM)^2 a_2(x_1 - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_n &= u + (LM)^n a_n(x_1 - \hat{x}_1) \end{aligned} \tag{2.1}$$

where $L \ge 1$ and $M \ge 1$ are two *dynamic gains* to be determined later on, and $a_i > 0,\ i = 1, \cdots, n$ are the coefficients of the Hurwitz polynomial $s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$.

Let $e_i = x_i - \hat{x}_i,\ i = 1, \cdots, n$ be the estimate errors. Then, the error dynamics can be expressed as

$$\begin{aligned} \dot{e}_1 &= e_2 - (LM)a_1 e_1 + \phi_1(x_1, \theta) \\ \dot{e}_2 &= e_3 - (LM)^2 a_2 e_1 + \phi_2(x_1, x_2, \theta) \\ &\vdots \\ \dot{e}_n &= -(LM)^n a_n e_1 + \phi_n(x, \theta). \end{aligned} \tag{2.2}$$

Similar to the work [15], we consider the following change of coordinates ($i = 1, \cdots, n$)

$$\begin{aligned} \varepsilon_i &= \frac{e_i}{(LM)^{i-1+\sigma}} \quad \text{and} \quad z_i = \frac{\hat{x}_i}{(LM)^{i-1+\sigma}}, \\ v &= \frac{u}{(LM)^{n+\sigma}} \end{aligned} \tag{2.3}$$

where σ is a constant satisfying

$$0 < \sigma < 1/(2\nu). \tag{2.4}$$

In the new coordinates, the closed-loop system (2.1)-(2.2) can be rewritten as

$$\begin{aligned} \dot{\varepsilon} &= LMA\varepsilon + \Phi(\cdot) - (\frac{\dot{L}}{L} + \frac{\dot{M}}{M})(D + \sigma I)\varepsilon \qquad (2.5) \\ \dot{z} &= LM(A_0 z + b_0 v) + LMa\varepsilon_1 - (\frac{\dot{L}}{L} + \frac{\dot{M}}{M})(D + \sigma I)z \end{aligned}$$

where $D = \text{diag}\{0, 1, \cdots, n-1\}$, $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)^T$, $z = (z_1, \cdots, z_n)^T$, $a = (a_1, \cdots, a_n)^T$, $b_0 = (0, \cdots, 0, 1)^T$ and

$$\Phi(\cdot) = \Big[\frac{\phi_1(x_1, \theta)}{(LM)^{\sigma}}, \ \frac{\phi_2(x_1, x_2, \theta)}{(LM)^{1+\sigma}}, \cdots, \frac{\phi_n(x, \theta)}{(LM)^{n-1+\sigma}}\Big]^T$$

$$A_0 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}.$$

Clearly, (A_0, b_0) is controllable. As a consequence, there is a real constant matrix $K = -[k_1 \ k_2 \ \cdots \ k_n]$ which assigns all the eigenvalues of the matrix $B := A_0 + b_0 K$ on the open left-half plane.

With this in mind, it is clear that the linear feedback controller $v = Kz = -[k_1 z_1 + k_2 z_2 + \cdots + k_n z_n]$, or, equivalently, the controller

$$u = -(LM)^{n+\sigma}[k_1 z_1 + k_2 z_2 + \cdots + k_n z_n] \tag{2.6}$$

yields

$$\begin{aligned} \dot{\varepsilon} &= LMA\varepsilon + \Phi(x, \theta, L, M) - (\frac{\dot{L}}{L} + \frac{\dot{M}}{M})D_\sigma \varepsilon \\ \dot{z} &= LM(Bz + a\varepsilon_1) - (\frac{\dot{L}}{L} + \frac{\dot{M}}{M})D_\sigma z \qquad (2.7) \end{aligned}$$

where $D_\sigma = D + \sigma I$.

Observer that A and B are Hurwitz matrices. Therefore, there exist $P = P^T > 0$ and $Q = Q^T > 0$, such that

$$\begin{aligned} &A^T P + PA \leq -I, \quad c_1 I \leq D_\sigma P + PD_\sigma \leq c_2 I \\ &B^T Q + QB \leq -2I, \quad c_3 I \leq D_\sigma Q + QD_\sigma \leq c_4 I \quad (2.8) \end{aligned}$$

where $c_i > 0, \ i = 1, \cdots, 4$ are real constants. The proof of inequality (2.8) can be found in [7], [15].

Using the matrices P and Q thus obtained, one can construct the Lyapunov function

$$V(\varepsilon, z) = (m_1 + 1)\varepsilon^T P\varepsilon + z^T Qz \tag{2.9}$$

where $m_1 = \|Q\|^2\|a\|^2$. The time derivative of $V(\varepsilon, z)$ along the trajectories of the closed-loop system is

$$\begin{aligned} \dot{V} \ \leq \ & -LM(m_1 + 1)\|\varepsilon\|^2 - 2LM\|z\|^2 - (\frac{\dot{L}}{L} + \frac{\dot{M}}{M}) \\ & \times\big[z^T(D_\sigma Q + QD_\sigma)z + (m_1 + 1)\varepsilon^T(D_\sigma P \\ & + PD_\sigma)\varepsilon\big] + 2LM\varepsilon_1 z^T Qa + 2(m_1 + 1)\varepsilon^T P\Phi(\cdot). \end{aligned}$$

Motivated by the papers [11] and [15], we design the following gain updated laws:

$$\begin{aligned} \dot{M} &= -\alpha M + \Delta(y), \qquad\qquad M(0) = 1 \quad (2.10) \\ \dot{L} &= M\varepsilon_1^2 = M^{1-2\sigma}(\frac{x_1 - \hat{x}_1}{L^\sigma})^2, \ L(0) = 1 \ (2.11) \end{aligned}$$

where α is a positive constant and $\Delta(y)$ is a positive continuous function with $\Delta(y) \geq \alpha$, both of them will be determined later.

By construction, it is easy to see that for all $t \in [0, +\infty)$, $L(t) \geq 1$ and $M(t) \geq 1$. Using inequality (2.8) and the fact that $\frac{\dot{M}}{M} + \frac{\dot{L}}{L} \geq \frac{\dot{M}}{M}$, we have

$$\begin{aligned} \dot{V} \ \leq \ & -(m_1 + 1)LM\|\varepsilon\|^2 - \frac{\dot{M}}{M}z^T(D_\sigma Q + QD_\sigma)z \\ & -2LM\|z\|^2 - (m_1 + 1)\frac{\dot{M}}{M}\varepsilon^T(D_\sigma P + PD_\sigma)\varepsilon \\ & +2LM\varepsilon_1 z^T Qa + 2(m_1 + 1)\varepsilon^T P\Phi(\cdot). \qquad (2.12) \end{aligned}$$

Now, let us estimate the last two terms on the right-hand side of (2.12). First of all, by the completion of squares, it is clear that

$$|2LM\varepsilon_1 z^T Qa| \leq LM(\|z\|^2 + \|Q\|^2\|a\|^2\varepsilon_1^2) \tag{2.13}$$

In addition, observe that

$$\Big|\frac{\phi_i(\cdot)}{(LM)^{i-1+\sigma}}\Big| \ \leq \ \Theta c(y)\sqrt{n}(||z|| + ||\varepsilon||).$$

Hence,

$$\begin{aligned} & |2(m_1 + 1)\varepsilon^T P\Phi| \\ \leq \ & 2(m_1 + 1)\Theta c(y)n\|\varepsilon\| \cdot \|P\|(\|z\| + \|\varepsilon\|) \\ \leq \ & \frac{c^2(y)}{M}(\|z\|^2 + \|\varepsilon\|^2) + mM\|\varepsilon\|^2. \qquad (2.14) \end{aligned}$$

where $m = 2\big(\Theta n(m_1 + 1)\|P\|\big)^2$.

Substituting (2.13) and (2.14) into (2.12) results in

$$\begin{aligned} \dot{V} \ \leq \ & -LM\|\varepsilon\|^2 - (m_1 + 1)\frac{\dot{M}}{M}\varepsilon^T(D_\sigma P + PD_\sigma)\varepsilon \\ & -LM\|z\|^2 - \frac{\dot{M}}{M}z^T(D_\sigma Q + QD_\sigma)z \\ & +mM\|\varepsilon\|^2 + \frac{c^2(y)}{M}(\|z\|^2 + \|\varepsilon\|^2). \end{aligned}$$

From (2.8) and (2.10), it follows that

$$\begin{aligned} \dot{V} \ \leq \ & -LM\|\varepsilon\|^2 + mM\|\varepsilon\|^2 + c_2(m_1 + 1)\alpha\|\varepsilon\|^2 \\ & -\frac{1}{M}\Big(c_1(m_1 + 1)\Delta(y) - c^2(y)\Big)\|\varepsilon\|^2 \\ & -LM\|z\|^2 + c_4\alpha\|z\|^2 \\ & -\frac{1}{M}\Big(c_3\Delta(y) - c^2(y)\Big)\|z\|^2. \qquad (2.15) \end{aligned}$$

Now, choose

$$\begin{aligned} & 0 < \alpha < \min\{\frac{1}{2c_2(m_1 + 1)}, \frac{1}{2c_4}\} \qquad (2.16) \\ & \Delta(y) = \beta c^2(y) + \alpha, \quad \beta \geq \frac{1}{c_1}(\frac{1}{m_1 + 1} + 1) + \frac{1}{c_3} \end{aligned}$$

Then,

$$\begin{aligned} \dot{V} \ \leq \ & -\frac{LM}{2}\|\varepsilon\|^2 + Mm\|\varepsilon\|^2 - \frac{LM}{2}\|z\|^2 \\ \leq \ & -\frac{M}{2}(L - 2m)\|\varepsilon\|^2 - \frac{1}{2}\|z\|^2 \qquad (2.17) \end{aligned}$$

Based on inequality (2.17), we can use a contradiction argument to prove Theorem 2.1. In particular, it can be shown that starting from any initial condition $(\varepsilon(0), z(0)) \in \mathbb{R}^n \times \mathbb{R}^n$, $L(0) = 1$ and $M(0) = 1$, the closed-loop system (2.7)-(2.10)-(2.11) has the following properties:

(i) All the states of (2.7)-(2.10)-(2.11) are well defined and globally bounded on $[0, +\infty)$;

(ii) Moreover, $\lim_{t\to+\infty}(z(t), \varepsilon(t)) = (0, 0)$.

Assume that the property (i) does not hold. Then, the closed-loop system (2.7)-(2.10)-(2.11) has a solution $(L(t), M(t), \varepsilon(t), z(t))$ that is not well defined nor bounded on $[0, +\infty)$. In other words, there exists a finite escape time $T > 0$ such that $(L(t), M(t), \varepsilon(t), z(t))$ are only defined on the time interval $[0, T)$ but

$$\lim_{t\to T} \|(L(t), M(t), \varepsilon(t), z(t))^T\| = +\infty.$$

First of all, we prove that $L(t)$ must be bounded on $[0, T]$ and would not escape at $t = T$. If not, $\lim_{t\to T} L(t) = +\infty$. By construction, $\dot{L} = M\varepsilon_1^2 \geq 0$. Thus, $L(t) \geq 1$ is a monotone nondecreasing function. As a result, there exists a time $t^* > 0$, such that $\quad L(t) \geq 2m + 1, \quad \forall t \in [t^*, T)$.

This, in view of (2.17), yields

$$\dot{V} \leq -\frac{M}{2}\|\varepsilon\|^2, \qquad \forall t \in [t^*, T).$$

Hence, $$\int_{t^*}^{T} M\varepsilon_1^2 dt \leq \int_{t^*}^{T} M||\varepsilon||^2 dt \leq 2V(\varepsilon(t^*), z(t^*)),$$

which leads to

$$\begin{aligned} +\infty &= L(T) - L(t^*) = \int_{t^*}^{T} \dot{L}(t)dt = \int_{t^*}^{T} M\varepsilon_1^2(t)dt \\ &\leq 2V(\varepsilon(t^*), z(t^*)) = \text{constant}. \end{aligned}$$

This is of course not possible. Thus, the dynamic gain $L(t)$ is well defined and bounded on $[0, T]$. Since $\dot{L} = M\varepsilon_1^2$, $\int_0^T M\varepsilon_1^2 dt$ is bounded as well.

Next, we show that $z(t)$ is well defined and bounded on the interval $[0, T]$. To this end, consider the Lyapunov function $V_2(z) = z^T Q z$ for the z-dynamic system of (2.7). A simple calculation shows that

$$\dot{V_2}(z) \leq -\frac{LM}{2}\|z\|^2 + m_1 LM\varepsilon_1^2 \leq -\frac{M}{2}\|z\|^2 + m_1 L\dot{L}.$$

Consequently,
$$\begin{aligned} &\lambda_{\min}(Q)\|z(t)\|^2 - z(0)^T Q z(0) \\ &\leq -\frac{1}{2}\int_0^t M\|z(t)\|^2 dt + \frac{m_1}{2}[L^2(t) - 1], \end{aligned}$$

from which it follows that

$$\|z(t)\|^2 \leq \frac{1}{\lambda_{\min}(Q)}\Big(z(0)^T Q z(0) + \frac{m_1}{2}[L^2(t) - 1]\Big) \tag{2.18}$$

Since $L(t)$ is bounded on $[0, T]$, boundedness of $z(t)$ over $[0, T]$ follows from the last inequality. Moreover, $\int_0^t M||z(t)||^2 dt$ is also bounded $\forall t \in [0, T]$.

Finally, we claim that $\varepsilon(t)$ is bounded on $[0, T]$. To prove this claim, we introduce the change of coordinates

$$\xi_i = \frac{e_i}{(L^*M)^{i-1+\sigma}}, \quad i = 1, 2, \cdots, n \tag{2.19}$$

where L^* is a constant satisfying

$$L^* \geq \max\ \{L(T),\ 8m_2 + 1\}, \quad m_2 = \Theta^2 n^2 \|P\|^2.$$

In the new coordinates, the error dynamics (2.2) is represented as

$$\dot{\xi} = L^*M(A\xi + a\xi_1) - LM\Gamma a\xi_1 + \Phi^*(\cdot) - D_\sigma\xi\frac{\dot{M}}{M} \tag{2.20}$$

where $\Gamma = \text{diag}\{1, \frac{L}{L^*}, \cdots, (\frac{L}{L^*})^{n-1}\}$ and

$$\Phi^*(\cdot) = \left[\frac{\phi_1(x_1, \theta)}{(L^*M)^\sigma},\ \frac{\phi_2(x_1, x_2, \theta)}{(L^*M)^{1+\sigma}}, \cdots, \frac{\phi_n(x, \theta)}{(L^*M)^{n-1+\sigma}}\right]^T.$$

Now, choose the Lyapunov function $V_3(\xi) = \xi^T P\xi$ for system (2.20). The time derivative of V_3 along the trajectories of (2.20) satisfies

$$\begin{aligned} \dot{V}_3 \leq\ & -L^*M\|\xi\|^2 + 2\xi_1 L^*Ma^T P\xi - 2\xi_1 LMa^T\Gamma P\xi \\ & +2\Phi^{*T}(\cdot)P\xi - \frac{\dot{M}}{M}\xi^T(D_\sigma P + PD_\sigma)\xi. \end{aligned}$$

Observe that

$$\begin{aligned} \left|2\xi_1 L^*Ma^T P\xi\right| &\leq 8L^*M\|a^T P\|^2\xi_1^2 + \frac{L^*M}{8}\|\xi\|^2 \\ \left|2\xi_1 LMa^T\Gamma P\xi\right| &\leq 8LM\|a^T\Gamma P\|^2\xi_1^2 + \frac{LM}{8}\|\xi\|^2. \end{aligned}$$

Moreover, by (2.3), (2.19) and Assumption 1.1, one has

$$\begin{aligned} \left|\frac{\phi_i(\cdot)}{(L^*M)^{i-1+\sigma}}\right| &\leq \Theta c(y)\sqrt{n}(\|z\| + \|\xi\|) \\ \left|2\Phi^{*T}(\cdot)P\xi\right| &\leq \frac{c^2(y)}{M}\|\xi\|^2 + 2Mm_2(\|z\|^2 + \|\xi\|^2). \end{aligned}$$

Putting the estimations above together, we have (note that $\xi_1^2 = \varepsilon_1^2(\frac{L^{2\sigma}}{(L^*)^{2\sigma}}) \leq c_0\varepsilon_1^2$)

$$\begin{aligned} \dot{V}_3 \leq\ & -\frac{3L^*M}{4}\|\xi\|^2 - \frac{1}{M}\Big(c_1\Delta(y) - c^2(y)\Big)\|\xi\|^2 \\ & +2Mm_2(\|z\|^2 + \|\xi\|^2) + c_2\alpha\|\xi\|^2 \\ & +8(L^*\|a^T P\|^2 + L\|a^T\Gamma P\|^2)M\xi_1^2 \\ \leq\ & -\frac{M}{4}(L^* - 8m_2)\|\xi\|^2 + \mu M\|z\|^2 + \mu M\varepsilon_1^2 \\ \leq\ & -\frac{1}{4}\|\xi\|^2 + \mu M\|z\|^2 + \mu M\varepsilon_1^2 \end{aligned} \tag{2.21}$$

where $\mu > 0$ is a suitable constant depending on the unknown parameter Θ.

From (2.21) it follows that

$$\begin{aligned} & \lambda_{\min}(P)\|\xi(t)\|^2 - \xi(0)^T P\xi(0) \\ \leq\ & V_3(\xi(t)) - V_3(\xi(0)) \\ \leq\ & -\frac{1}{4}\int_0^t \|\xi(t)\|^2 dt + \mu\int_0^t M\|z\|^2 dt + \mu\int_0^t M\varepsilon_1^2 dt. \end{aligned} \tag{2.22}$$

By the boundedness of $\int_0^T M\|z\|^2 dt$ and $\int_0^T M\varepsilon_1^2 dt$, we concludes from (2.22) that $\int_0^t \|\xi\|^2 dt$ and $\xi(t)$ are bounded on $[0, T]$. Consequently, boundedness of $\int_0^t \|\varepsilon\|^2 dt$ and $\varepsilon(t)$ $\forall t \in [0, T]$ follows from (2.19) and (2.3),

To complete the proof, we show now that $M(t)$ is bounded on $[0, T]$. First, it is easy to see that $\frac{x_1}{M^\sigma}$ is bounded on $[0, T]$. As a matter of fact,

$$\left|\frac{x_1}{M^\sigma}\right| \leq L^\sigma(|z_1| + |\varepsilon_1|) \leq C \qquad \forall t \in [0, T]$$

because $L(t), z_1(t)$ and $\varepsilon_1(t)$ are bounded on $[0, T]$, where C is a real constant. As a consequence,

$$|x_1| \le CM^{\sigma} \qquad \forall t \in [0, T]. \tag{2.23}$$

With this in mind, together with the choice of $\Delta(y)$ given in (2.16), we have

$$\begin{aligned} \dot{M} = -\alpha M + \Delta(y) &\le -\alpha M + \beta(|y|^{\nu} + 1)^2 + \alpha \\ &\le -\alpha M + 2\beta C^{2\nu} M^{2\sigma\nu} + 2\beta + \alpha \end{aligned}$$

where α and β are positive constants defined in (2.16).

Using (2.4) and Young's inequality, it is not difficult to deduce from the last inequality that

$$\dot{M} \le -\frac{\alpha}{2} M + b \tag{2.24}$$

where $b > 0$ is a suitable constant. Obviously, (2.24) implies the boundedness of $M(t)$ on $[0, T]$.

In summary, we have shown that $L(t)$, $M(t)$, $\varepsilon(t)$ and $z(t)$ are well-defined and all bounded on $[0, T]$. This conclusion contradicts to the hypothesis that $\lim_{t \to T} \|(L(t), M(t), \varepsilon(t), z(t))^T\| = +\infty$. Therefore, the property (i) must be true. That is, all the states of the closed-loop system (2.7)-(2.10)-(2.11) are well defined and bounded on $[0, +\infty)$.

Using the boundedness of $\int_0^{\infty} \|z\|^2 dt$, $\int_0^{\infty} \|\varepsilon\|^2 dt$ and $(L(t), M(t), \varepsilon(t), z(t))$ on $[0, +\infty)$, it is straightforward to deduce that $\varepsilon \in L_2$, $\dot{\varepsilon} \in L_{\infty}$ and $z \in L_2$, $\dot{z} \in L_{\infty}$. By the well-known Barbalat's Lemma,

$$\lim_{t \to +\infty} z(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \varepsilon(t) = 0. \qquad \blacksquare$$

Remark 2.2: From the proof of Theorem 2.1, it is clear that a universal output feedback controller can be explicitly constructed. It is composed of the high-gain observer (2.1), the controller (2.6) and the dynamic gain update laws (2.10)-(2.11). The parameters of the gain update laws, i.e. α and $\Delta(y)$, are given by (2.16).

Due to the nature of the feedback domination design, it is easy to see that Theorem 2.1 also holds for the uncertain system

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \phi_i(t, x, \theta), \quad i = 1, \cdots, n \\ y &= x_1, \qquad\qquad\qquad x_{n+1} := u, \end{aligned} \tag{2.25}$$

as long as

$$|\phi_i(t, x, \theta)| \le (|x_1| + \cdots + |x_i|) c(y) \Theta. \tag{2.26}$$

Indeed, using the exactly same argument one can prove the following result that is an extension of Theorem 2.1.

Theorem 2.3: For the uncertain nonlinear system (2.25) satisfying the growth condition (2.26), there is a dynamic output feedback compensator consisting of the high-gain observer (2.1), the controller (2.6), and the dynamic gain update laws (2.10)-(2.11)-(2.16), such that from any initial condition all the states of the closed-loop system are well-defined and globally bounded. Moreover,

$$\lim_{t \to +\infty} (x(t), \hat{x}(t)) = (0, 0). \qquad \blacksquare$$

III. Two Illustrative Examples

We now give two examples to illustrate the applications of Theorem 2.1. The examples also demonstrate how the control scheme proposed in the previous section can lead to a design of universal output feedback controllers.

Example 3.1: Consider the SISO uncertain nonlinear system [8]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + \theta x_1^2 x_3 \\ y &= x_1. \end{aligned} \tag{3.1}$$

where θ is an unknown constant.

When the bound of the unknown parameter θ is known, global stabilization of the nonlinear system (3.1) is solvable by the work [15], [18], [8]. However, in the case of when θ is completely unknown, how to globally regulate all the states of system (3.1) by output feedback is an unsolved problem, as pointed out in [8].

On the other hand, systems (3.1) satisfies Assumption 1.1 with $\nu = 2$. By Theorem 2.1, one can construct a universal-like output feedback controller solving the global regulation problem of system (3.1). In fact, it is not difficult to show that the output feedback controller (2.1)-(2.6)-(2.10)-(2.11) does the job, with the choice of the parameters $(a_1, a_2, a_3) = (3, 3, 1)$, $(k_1, k_2, k_3) = (1, 3, 3)$, $\sigma = \frac{1}{8}$ and

$$\Delta(y) = (1 + y^2)^2 + \frac{1}{20}.$$

The simulation shown in Fig. 1 confirms the conclusion. The simulation was conducted with the initial conditions $(x_1(0), x_2(0), x_3(0)) = (1, 2, 2)$, $(\hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0)) = (-2, -1, -1)$ and $L(0) = M(0) = 1$.

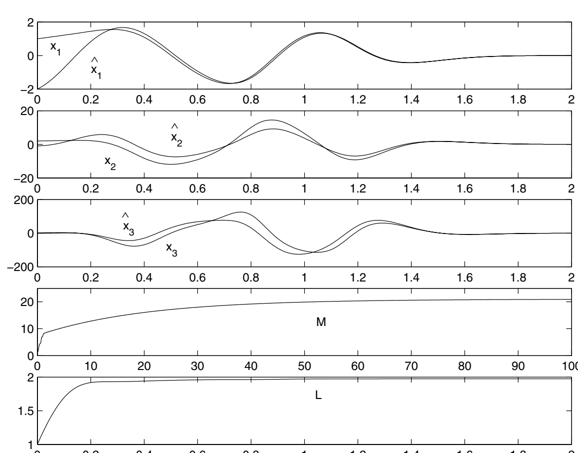

Fig. 1. The transient response of the closed-loop system (3.1) and (2.1)-(2.6)-(2.10)-(2.11) when $\theta = 2$

Example 3.2: Consider the nonlinearly parameterized system

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \frac{x_1^2}{(1+\theta_1 x_3)^2 + x_3^2} \\
\dot{x}_2 &= x_3 + x_2 \sin x_2 \, \ln(1+\theta_2^2 x_1^2) \\
\dot{x}_3 &= u \\
y &= x_1
\end{aligned} \tag{3.2}
$$

where θ_1 and θ_2 are *unknown* constants.

It should be noticed that (3.2) is not in a triangular form (1.1) nor in the parametric output feedback form. For this reason, global adaptive stabilization by output feedback is an open problem. However, it is easy to show that the uncertain system (3.2) satisfies (2.26). Indeed,

$$
\left| \frac{x_1^2}{(1+\theta_1 x_3)^2 + x_3^2} \right| \quad \leq \quad \Theta(1+|x_1|^2)|x_1|
$$

$$
\left| x_2 \sin x_2 \, \ln(1+x_1^2\theta_2^2) \right| \quad \leq \quad \Theta(1+|x_1|^2)|x_2|
$$

$\Theta = \max\{2(1+\theta_1^2), |\theta_2|\}$. Thus, according to Theorem 2.3, one can construct the output feedback controller (2.1)-(2.6)-(2.10)-(2.11), with the observer parameters $a = [3\ 3\ 1]^T$, controller parameters $k = [1\ 3\ 3]^T$, $\sigma = \frac{1}{8}$ and

$$
\Delta(y) = (1+y^2)^2 + 0.2.
$$

A numerical simulation is given in Fig. 2, illustrating that all the states of system (3.2) are globally regulated. The simulation was conducted with the system parameters $\theta_1 = 1$ and $\theta_2 = 2$. The initial conditions are $(x_1(0), x_2(0), x_3(0)) = (1, 2, 2)$, $(\hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0)) = (-2, -1, -10)$ and $L(0) = M(0) = 1$.

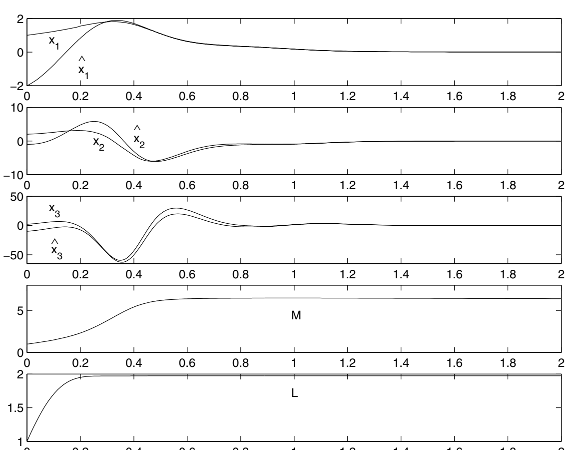

Fig. 2. Transient response of the closed-loop system (3.2)-(2.1)-(2.6)-(2.10)-(2.11)

IV. Conclusion

By integrating the universal output feedback control scheme [11] and the linear output feedback stabilization method with dynamic gain [15], we have shown that the problem of global adaptive regulation by output feedback can be solved under Assumption 1.1, for a class of nonlinear systems with unknown parameters beyond the parametric output feedback form.

The solution presented in this paper relies crucially on the linear structure of the dynamic output compensator (2.1)-(2.6)-(2.10)-(2.11), and hence does not seem to be extendable to the non-polynomial $c(y)$ case considered in [8], [18], where a nonlinear dynamic output compensator, instead of a linear one with dynamic gain, must be designed. Keeping this in mind, it appears that a new non-identifier based, adaptive output feedback control scheme is needed in order to deal with a larger family uncertain nonlinear systems (1.1) or (2.25) characterized by Assumption 1.1, in which $c(y) \geq 0$ is only a continuous function of y, rather than a *polynomial* function of y with a known order. This is certainly an interesting subject for future research.

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