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A Notion of Controllability for Uncertain Linear Systems with Structured Uncertainty

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Abstract— **This paper introduces a notion of possible controllability for a class of uncertain linear systems with structured uncertainty described by averaged integral quadratic constraints. This notion relates to the question of when a state is controllable for some possible value of the uncertainty. This question may arise which considering modeling and realization theory for uncertain systems. The paper presents an algorithm for finding the possible controllability function which is related to the controllability Gramian for linear systems and the corresponding controllability cone.**

I. INTRODUCTION

The notion of controllability is one of the fundamental properties of a linear system; e.g., see [1]. One reason for considering the issue of controllability for uncertain systems might be to determine if a robust state feedback controller can be constructed for the system; e.g., see [2]. In this case, one would be interested in the question of whether the system is "controllable" for all possible values of the uncertainty.

For the case of linear systems, the issue of controllability is also central to realization theory. For example, it is known that if a linear system contains uncontrollable states, a reduced dimension realization of the system's input-output behavior can be obtained by considering only the controllable states. For the case of uncertain systems, a natural extension of this notion of controllability is to consider possibly controllable states which are "controllable" for some possible values of the uncertainty. The aim of this paper is to introduce a notion of possibly controllable states for uncertain systems which will provide insight into the structure of uncertain systems as it relates to questions of realization theory; e.g., see [3]. The notion of possible controllability introduced in this paper involves extending the definition of the controllability Gramian to the case of uncertain systems; see also [4]. The framework considered is similar to that of [5] where a dual problem of robust unobservability is considered.

As in the papers [6], [7], the uncertain systems considered in this paper will use an averaged integral quadratic constraint (IQC) uncertainty description. The paper shows that the notion possible controllability is related to the solution of a Riccati differential equation and a duality connection is established between the notion of possible controllability and the notion of robust unobservability considered in [5].

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II. PROBLEM FORMULATION

We consider the following linear time-varying uncertain system defined on the finite time interval $[-T, 0]$:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + \sum_{s=1}^{k} C_s(t)\xi_s(t); \\ z_s(t) &= K_s(t)x(t) + G_s(t)u(t),\ s = 1, 2, \ldots k \quad (1) \end{aligned}$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $u(t) \in \mathbf{R}^m$ is the *control input*, $z_1(t) \in \mathbf{R}^{h_1}$, $z_2(t) \in \mathbf{R}^{h_2}, \ldots, z_k(t) \in \mathbf{R}^{h_k}$ are the *uncertainty outputs*, $\xi_1(t) \in \mathbf{R}^{r_1}$, $\xi_2(t) \in \mathbf{R}^{r_2}, \ldots, \xi_k(t) \in \mathbf{R}^{r_k}$ are the *uncertainty inputs*, and $A(\cdot)$, $B(\cdot)$, $C_1(\cdot)$, $C_2(\cdot)$, $\ldots$, $C_k(\cdot)$, $K_1(\cdot)$, $K_2(\cdot)$, $\ldots$, $K_k(\cdot)$, $G_1(\cdot)$, $G_2(\cdot)$, $\ldots$, $G_k(\cdot)$ are bounded piecewise continuous matrix functions defined on $[-T, 0]$.

a) System Uncertainty: The uncertainty in the above system is described by a set of equations of the form:

$$\xi_s(t) = \phi_s(t, x(\cdot)|_0^t) \quad \text{for } s = 1, 2, \ldots, k. \quad (2)$$

Alternatively, the uncertainty inputs and outputs may be collected together into two vectors. That is, we define $\xi(t) \triangleq [\xi_1(t)'\ \xi_2(t)'\ \ldots\ \xi_k(t)'\ ]'$ and $z(t) \triangleq [z_1(t)'\ z_2(t)'\ \ldots\ z_k(t)'\ ]'$. Then (2) can be re-written in the more compact form

$$\xi(t) = \Phi(t, x(\cdot)|_0^t). \quad (3)$$

We consider finite sequences of uncertainty functions of the form (3) such that the following constraint is satisfied.

Definition 1: (Averaged Integral Quadratic Constraint) Let $d_1 > 0, d_2 > 0, \ldots, d_k > 0$, be given positive constants associated with the system (1). We will consider sequences of uncertainty functions $\mathcal{S} = \{\Phi^1(\cdot), \Phi^2(\cdot), \ldots \Phi^q(\cdot)\}$ of arbitrary length q. A sequence of uncertainty functions $\mathcal{S}$ is an admissible *uncertainty sequence* for the system (1) if the following conditions hold: Given any $\Phi^i(\cdot) \in \mathcal{S}$, any control input $u^i(\cdot) \in \mathbf{L}_2[-T, 0]$, and any corresponding solution $\{x^i(\cdot), \xi^i(\cdot)\}$ to equations (1), (3) defined on $[-T, 0]$, then $\xi^i(\cdot) \in \mathbf{L}_2[-T, 0]$ and

$$\frac{1}{q}\sum_{i=1}^{q}\int_{-T}^{0}(\|\xi_s^i(t)\|^2 - \|z_s^i(t)\|^2)dt \le d_s \quad (4)$$

for $s = 1, 2, \ldots, k$. Here $\mathbf{L}_2[-T, 0]$ denotes the set of square integrable vector functions defined on the set $[-T, 0]$ and $\|\cdot\|$ denotes the standard Euclidean norm. The class of all such admissible uncertainty sequences is denoted Ξ.

Given any uncertainty sequence $\mathcal{S} = \{\Phi^1(\cdot), \Phi^2(\cdot), \ldots \Phi^q(\cdot)\}$, we denote any corresponding sequence of control inputs $\mathcal{U} = \{u^1(\cdot), u^2(\cdot), \dots u^q(\cdot)\}$ where $u^i(\cdot) \in L_2[-T, 0]$ $\forall i = 1, 2, \dots, q$ and we write $\mathcal{U} \in \mathbf{L}_2^q[-T, 0]$.

Definition 2: The *possible controllability function* for the uncertain system (1), (4) is defined as

$$L_c(x_0) \triangleq \sup_{\epsilon>0} \inf_{\mathcal{S}\in\Xi} \inf_{\mathcal{U}\in\mathbf{L}_2^q[-T,0]} \frac{1}{q} \sum_{i=1}^{q} \left[\begin{array}{c} \frac{\|x^i(-T)\|^2}{\epsilon} \\ + \int_{-T}^{0} \|u^i(t)\|^2 dt \end{array} \right] \quad (5)$$

where $x(0) = x_0$ in (1).

This definition modifies the definition of controllability function given in [4] to the case of linear uncertain systems and is closely related to the standard definition of the controllability Gramian for linear systems.

Notation. $\mathcal{D} \triangleq \{d = [d_1\ d_2\ \dots\ d_k] : d_i > 0\ \forall i\}$.

Definition 3: A non-zero state $x_0 \in \mathbf{R}^n$ is said to be *possibly controllable* for the uncertain system (1), (4) if $\sup_{d\in\mathcal{D}} L_c(x_0) < \infty$. The set of all possibly controllable states for the uncertain system (1), (4) is referred to as the *possibly controllable cone* $\mathcal{C}$; i.e.,

$$\mathcal{C} \triangleq \left\{ x \in \mathbf{R}^n : \sup_{d\in\mathcal{D}} L_c(x) < \infty \right\}.$$

III. The Main Result

A. A Family of Unconstrained Optimization Problems.

For the uncertain system (1), (4), we define functions $V_\tau^\epsilon(x_0, \lambda)$, $V_\tau^\epsilon(x_0)$ and $V_\tau(x_0)$ as follows:

$$V_\tau^\epsilon(x_0, \lambda) \triangleq \inf_{[\xi(\cdot),u(\cdot)]\in\mathbf{L}_2[-T,\lambda]} \frac{\|x(-T)\|^2}{\epsilon} + \int_{-T}^{\lambda} \left(\begin{array}{c} \|u\|^2 \\ + \sum_{s=1}^{k} \tau_s \|\xi_s\|^2 \\ - \sum_{s=1}^{k} \tau_s \|z_s\|^2 \end{array} \right) dt \quad (6)$$

subject to $x(\lambda) = x_0$;

$$\begin{aligned} V_\tau^\epsilon(x_0) &\triangleq V_\tau^\epsilon(x_0, 0); \\ V_\tau(x_0) &\triangleq \sup_{\epsilon>0} V_\tau^\epsilon(x_0). \end{aligned}$$

Here $\tau_1 \geq 0$, $\tau_2 \geq 0$, ... , $\tau_k \geq 0$ are given constants.

B. An S-procedure Result.

For the system (1) a corresponding set $\Omega \subset \mathbf{L}_2[-T, 0]$ is defined as follows:

$$\Omega \triangleq \left\{ \begin{array}{l} \lambda(\cdot) = [x(\cdot)\ u(\cdot)\ \xi(\cdot)] : [\xi(\cdot), u(\cdot)] \in \mathbf{L}_2[-T, 0], \\ \{x(\cdot), u(\cdot), \xi(\cdot)\} \text{ satisfy (1) with } x(0) = x_0, \end{array} \right\}. \quad (7)$$

Also, we consider the following set of functionals mapping from Ω into $\mathbf{R}$:

$$\begin{array}{l} F_0(x(\cdot), u(\cdot), \xi(\cdot)), \\ \vdots \\ F_k(x(\cdot), u(\cdot), \xi(\cdot)). \end{array}$$

Lemma 1: Suppose that for any sequence contained in the set Ω $\{[x(\cdot), u(\cdot), \xi(\cdot)]^1,\ [x(\cdot), u(\cdot), \xi(\cdot)]^2, \dots, [x(\cdot), u(\cdot), \xi(\cdot)]^q\}$ such that

$$\begin{aligned} \sum_{i=1}^{q} F_1([x(\cdot), u(\cdot), \xi(\cdot)]^i) &\geq 0; \\ &\vdots \\ \sum_{i=1}^{q} F_k([x(\cdot), u(\cdot), \xi(\cdot)]^i) &\geq 0; \end{aligned} \quad (8)$$

we have

$$\sum_{i=1}^{q} F_0([x(\cdot), u(\cdot), \xi(\cdot)]^i) \geq 0. \quad (9)$$

Then there exist constants $\tau_0 \geq 0$, ... ,$\tau_k \geq 0$ such that $\sum_{i=0}^{k} \tau_i > 0$ and

$$\tau_0 F_0(x(\cdot), u(\cdot), \xi(\cdot)) \geq \sum_{i=0}^{k} \tau_i F_k(x(\cdot), u(\cdot), \xi(\cdot)) \quad (10)$$

for all $[x(\cdot), u(\cdot), \xi(\cdot)] \in \Omega$.

Proof. We first define a subset of $\mathbf{R}^{k+1}$

$$\mathcal{P} \triangleq \left\{ \left[\begin{array}{l} F_0(x(\cdot), u(\cdot), \xi(\cdot)) \\ F_1(x(\cdot), u(\cdot), \xi(\cdot)) \\ \dots \\ F_k(x(\cdot), u(\cdot), \xi(\cdot)) \end{array} \right] : [x(\cdot), u(\cdot), \xi(\cdot)] \in \Omega \right\}.$$

Then condition (8), (9) implies that this set satisfies the assumptions of Theorem 3.1 of [8]. From this theorem, (10) follows. □

Observation 1: If there exists an $[x(\cdot), u(\cdot), \xi(\cdot)] \in \Omega$ such that $F_1(x(\cdot), u(\cdot), \xi(\cdot)) > 0$, $F_2(x(\cdot), u(\cdot), \xi(\cdot)) > 0$, ..., $F_k(x(\cdot), u(\cdot), \xi(\cdot)) > 0$ and the assumptions of the above lemma hold, then τ_0 may be chosen as $\tau_0 = 1$ in (10); see Observation 3.1 in [8].

C. A Formula for the Possible Controllability Function.

We first introduce the following notation:

$$\mathcal{T} \triangleq \{\tau = [\tau_1\ \dots\ \tau_k] : \tau_1 \geq 0\ \dots\ \tau_k \geq 0\}.$$

Theorem 1: Consider the uncertain system (1), (4) and corresponding possible controllability function (5). Then for any $x_0 \in \mathbf{R}^n$,

$$\begin{aligned} L_c(x_0) &= \sup_{\epsilon>0} \sup_{\tau\in\mathcal{T}} \left\{ V_\tau^\epsilon(x_0) - \sum_{s=1}^{k} \tau_s d_s \right\}; \\ &= \sup_{\tau\in\mathcal{T}} \left\{ V_\tau(x_0) - \sum_{s=1}^{k} \tau_s d_s \right\}. \end{aligned} \quad (11)$$

Proof. Given any $\epsilon > 0$, any vector $\tau \in \mathcal{T}$, any admissible uncertainty sequence $\mathcal{S} \in \Xi$ and any input sequence $\mathcal{U} \in \mathbf{L}_2^q[-T, 0]$ for the uncertain system (1), (4) such that $x^i(0) = x_0$ for $i = 1, 2, \dots q$, we claim

$$\frac{1}{q} \sum_{i=1}^{q} \left[\frac{\|x^i(-T)\|^2}{\epsilon} + \int_{-T}^{0} \|u^i(t)\|^2 dt \right] \geq V_\tau^\epsilon(x_0) - \sum_{s=1}^{k} \tau_s d_s. \quad (12)$$

To establish this claim, we first note that it follows from the definition of $V_\tau^\epsilon(x_0)$ (6) that

$$\frac{\|x(-T)\|^2}{\epsilon}+\int_{-T}^{0}\left(\|u\|^2+\sum_{s=1}^{k}\tau_s\|\xi_s\|^2-\sum_{s=1}^{k}\tau_s\|z_s\|^2\right)dt \geq V_\tau^\epsilon(x_0)$$

for all $[x(\cdot),u(\cdot),\xi(\cdot)]\in\Omega$. In particular, this inequality holds for $[x^i(\cdot),u^i(\cdot),\xi^i(\cdot)]\in\Omega:\ i=1,2,\ldots,q$ corresponding to the given sequence $\mathcal{S}$. Hence,

$$\begin{array}{l}\frac{1}{q}\sum_{i=1}^{q}\left[\begin{array}{c}\frac{\|x^i(-T)\|^2}{\epsilon}\\ +\int_{-T}^{0}\left(\begin{array}{c}\|u^i\|^2+\sum_{s=1}^{k}\tau_s\|\xi_s^i\|^2\\ -\sum_{s=1}^{k}\tau_s\|z_s^i\|^2\end{array}\right)dt\end{array}\right]\\ \quad\geq \frac{1}{q}\sum_{i=1}^{q}V_\tau^\epsilon(x_0)\\ \quad= V_\tau^\epsilon(x_0).\end{array} \tag{13}$$

However, $\mathcal{S}\in\Xi$ implies that (4) is satisfied and hence from (13), we obtain

$$\frac{1}{q}\sum_{i=1}^{q}\left[\frac{\|x^i(-T)\|^2}{\epsilon}+\int_{-T}^{0}\|u^i(t)\|^2dt+\sum_{s=1}^{k}\tau_s d_s\right] \geq V_\tau^\epsilon(x_0).$$

Thus, (12) holds.

Now for any $\epsilon>0$, it follows from (12) that

$$\begin{array}{l}\inf_{\mathcal{S}\in\Xi}\inf_{\mathcal{U}\in\mathbf{L}_2^q[-T,0]}\frac{1}{q}\sum_{i=1}^{q}\left[\frac{\|x^i(-T)\|^2}{\epsilon}+\int_{-T}^{0}\|u^i\|^2dt\right]\\ \quad\geq V_\tau^\epsilon(x_0)-\sum_{s=1}^{k}\tau_s d_s\end{array} \tag{14}$$

for all $\tau\in\mathcal{T}$ where the inf on the left hand side of this inequality is subject to the constraints $x^i(0)=x_0:\ i=1,2,\ldots,q$ in (1).

We now claim there exists a $\tau\in\mathcal{T}$ such that

$$\begin{array}{l}\inf_{\mathcal{S}\in\Xi}\inf_{\mathcal{U}\in\mathbf{L}_2^q[-T,0]}\frac{1}{q}\sum_{i=1}^{q}\left[\frac{\|x^i(-T)\|^2}{\epsilon}+\int_{-T}^{0}\|u^i\|^2dt\right]\\ \quad\leq V_\tau^\epsilon(x_0)-\sum_{s=1}^{k}\tau_s d_s\end{array} \tag{15}$$

where the inf on the left hand side of this inequality is subject to the constraints $x^i(0)=x_0:\ i=1,2,\ldots,q$ in (1).

To establish this claim, we let

$$c\triangleq\inf_{\mathcal{S}\in\Xi}\inf_{\mathcal{U}\in\mathbf{L}_2^q[-T,0]}\frac{1}{q}\sum_{i=1}^{q}\left[\begin{array}{c}\frac{\|x^i(-T)\|^2}{\epsilon}\\ +\int_{-T}^{0}\|u^i(t)\|^2dt\end{array}\right]. \tag{16}$$

Also, we define the functionals in Lemma 1 as follows:

$$\begin{array}{rcl}F_0^\epsilon(x(\cdot),u(\cdot),\xi(\cdot)) &\triangleq& \frac{\|x(-T)\|^2}{\epsilon}+\int_{-T}^{0}\|u\|^2dt\\ && -c;\\ F_1(x(\cdot),u(\cdot),\xi(\cdot)) &\triangleq& -\int_{-T}^{0}\left(\|\xi_1\|^2-\|z_1\|^2\right)dt+d_1;\\ &\vdots&\\ F_k(x(\cdot),u(\cdot),\xi(\cdot)) &\triangleq& -\int_{-T}^{0}\left(\|\xi_k\|^2-\|z_k\|^2\right)dt+d_k.\end{array}$$

Now consider a sequence $\{[x(\cdot),u(\cdot),\xi(\cdot)]^i\}\subset\Omega$, such that

$$\begin{array}{rcl}\frac{1}{q}\sum_{i=1}^{q}F_1([x(\cdot),u(\cdot),\xi(\cdot)]^i) &\geq& 0,\\ &\vdots&\\ \frac{1}{q}\sum_{i=1}^{q}F_k([x(\cdot),u(\cdot),\xi(\cdot)]^i) &\geq& 0.\end{array}$$

If we recall Definition 1, it follows that this sequence corresponds to an admissible uncertainty sequence $\mathcal{S}\in\Xi$. Then, it follows from (16) that $\frac{1}{q}\sum_{i=1}^{q}F_0^\epsilon([x(\cdot),u(\cdot),\xi(\cdot)]^i)\geq 0$. Thus, the conditions of the S-procedure result, Lemma 1 are satisfied. Also note that since $d_1>0,\ d_2>0,\ldots,d_k>0$, then $F_1([x(\cdot),u(\cdot),\xi(\cdot)])>0,\ F_2([x(\cdot),u(\cdot),\xi(\cdot)])>0,\ldots,F_k([x(\cdot),u(\cdot),\xi(\cdot)])>0$ for any $[x(\cdot),u(\cdot),\xi(\cdot)]\in\Omega$ corresponding to $\xi(\cdot)\equiv 0$ and any $u(\cdot)\in\mathbf{L}_2[-T,0]$. Thus, it follows from Lemma 1 and Observation 1 that there exist constants $\tau_1\geq 0$, ... ,$\tau_k\geq 0$ such that

$$F_0^\epsilon(x(\cdot),u(\cdot),\xi(\cdot))\geq\sum_{s=1}^{k}\tau_s F_s(x(\cdot),u(\cdot),\xi(\cdot))$$

for all $[x(\cdot),u(\cdot),\xi(\cdot)]\in\Omega$. That is

$$\begin{array}{l}\frac{\|x(-T)\|^2}{\epsilon}+\int_{-T}^{0}\|u\|^2dt-c\\ \quad\geq\sum_{s=1}^{k}\tau_s\left[\int_{-T}^{0}\left(\|\xi_s\|^2-\|z_s\|^2\right)dt+d_s\right]\end{array}$$

for all $[x(\cdot),u(\cdot),\xi(\cdot)]\in\Omega$. Hence,

$$\begin{array}{l}\inf_{[x(\cdot),u(\cdot),\xi(\cdot)]\in\Omega}\frac{\|x(-T)\|^2}{\epsilon}\\ \quad+\int_{-T}^{0}\left(\begin{array}{c}\|u\|^2+\sum_{s=1}^{k}\tau_s\|\xi_s\|^2\\ -\sum_{s=1}^{k}\tau_s\|z_s\|^2\end{array}\right)dt\\ \quad\geq c+\sum_{s=1}^{k}\tau_s d_s.\end{array}$$

Then using (6) and (16), we have

$$\begin{array}{rcl}V_\tau^\epsilon(x_0) &\geq& \inf_{\mathcal{S}\in\Xi}\inf_{\mathcal{U}\in\mathbf{L}_2^q[-T,0]}\frac{1}{q}\sum_{i=1}^{q}\left[\begin{array}{c}\frac{\|x^i(-T)\|^2}{\epsilon}\\ +\int_{-T}^{0}\|u^i(t)\|^2dt\end{array}\right]\\ && +\sum_{s=1}^{k}\tau_s d_s\end{array}$$

That is (15) is satisfied. Combining (14) and (15) now leads to

$$\sup_{\tau\in\mathcal{T}}\left\{V_\tau^\epsilon(x_0)-\sum_{s=1}^k \tau_s d_s\right\}$$
$$=\sup_{\tau\in\mathcal{T}}\left\{V_\tau^\epsilon(x_0)-\sum_{s=1}^k \tau_s d_s\right\}$$
$$=\inf_{\mathcal{S}\in\Xi}\inf_{\mathcal{U}\in\mathbf{L}_2^q[-T,0]}\frac{1}{q}\sum_{i=1}^q\left[\begin{array}{c}\frac{\|x^i(-T)\|^2}{\epsilon}\\ +\int_{-T}^0\|u^i(t)\|^2dt\end{array}\right].$$

Hence, we obtain

$$\begin{aligned}L_c(x_0) &= \sup_{\epsilon>0}\inf_{\mathcal{S}\in\Xi}\inf_{\mathcal{U}\in\mathbf{L}_2^q[-T,0]}\frac{1}{q}\sum_{i=1}^q\left[\begin{array}{c}\frac{\|x^i(-T)\|^2}{\epsilon}\\ +\int_{-T}^0\|u^i(t)\|^2dt\end{array}\right]\\ &= \sup_{\epsilon>0}\sup_{\tau\in\mathcal{T}}\left\{V_\tau^\epsilon(x_0)-\sum_{s=1}^k \tau_s d_s\right\}\\ &= \sup_{\tau\in\mathcal{T}}\left\{V_\tau(x_0)-\sum_{s=1}^k \tau_s d_s\right\}.\end{aligned}$$

This completes the proof of the theorem. □

Corollary 1: If we define

$$\tilde{L}_c(x_0)\triangleq\sup_{d\in\mathcal{D}}L_c(x_0)$$

then

$$\tilde{L}_c(x_0)=\sup_{\epsilon>0}\sup_{\tau\in\mathcal{T}}V_\tau^\epsilon(x_0)=\sup_{\tau\in\mathcal{T}}V_\tau(x_0)$$

Proof. This result follows directly from the formula (11) and the definitions of the sets $\mathcal{T}$ and $\mathcal{D}$. □

Observation 2: From the above corollary, it follows immediately that the possibly controllable cone $\mathcal{C}$ can be written in the form:

$$\begin{aligned}\mathcal{C} &= \left\{x\in\mathbf{R}^n:\sup_{\epsilon>0}\sup_{\tau\in\mathcal{T}}V_\tau^\epsilon(x)<\infty\right\}\\ &= \left\{x\in\mathbf{R}^n:\sup_{\tau\in\mathcal{T}}V_\tau(x)<\infty\right\}.\end{aligned}$$

IV. Riccati Equation Solution to the Unconstrained Optimization Problems

In order to calculate $V_\tau^\epsilon(x_0)$, we first introduce some notation. Given $\tau=[\tau_1\ \tau_2\ \ldots\ \tau_k]$, let

$$C(t)=[C_1(t)\ C_2(t)\ \ldots\ C_k(t)];$$
$$G=\begin{bmatrix}G_1(t)\\G_2(t)\\ \vdots\\ G_k(t)\end{bmatrix};\quad K=\begin{bmatrix}K_1(t)\\K_2(t)\\ \vdots\\ K_k(t)\end{bmatrix};$$

$$\Lambda_\tau=\begin{bmatrix}\tau_1 I_{h_1} & & 0\\ & \ddots & \\ 0 & & \tau_k I_{h_k}\end{bmatrix};$$
$$\bar{\Lambda}_\tau=\begin{bmatrix}\tau_1 I_{r_1} & & 0\\ & \ddots & \\ 0 & & \tau_k I_{r_k}\end{bmatrix}.$$

Using this notation, it follows that the system (1) can be re-written as

$$\dot{x}(t)=A(t)x(t)+B(t)u(t)+C(t)\xi(t);\qquad(17)$$

Also, we will consider a dual system

$$\begin{aligned}\dot{x}(t) &= -A(t)'x(t)+K(t)'\xi(t);\\ y(t) &= B(t)'x(t)-G(t)'\xi(t);\\ z(t) &= C(t)'x(t)\end{aligned}\qquad(18)$$

This dual system will be used in the proof of the main result of this section. Also, we will show that the property of possible controllability for the original uncertain system is related to the property of robust unobservability for this dual system; see [5].

The function $V_\tau^\epsilon(x_0,\lambda)$ can be re-written as

$$V_\tau^\epsilon(x_0,\lambda)=\inf_{[u(\cdot),\xi(\cdot)]\in\mathbf{L}_2[-T,0]}J_\tau^\epsilon([u(\cdot),\xi(\cdot)])\qquad(19)$$

subject to $x(\lambda)=x_0$ in (17) where

$$J_\tau^\epsilon([u(\cdot),\xi(\cdot)]) = \frac{\|x(-T)\|^2}{\epsilon} + \int_{-T}^{\lambda}\left(\begin{array}{l}-x'K'\Lambda_\tau Kx\\ -2x'K'\Lambda_\tau Gu\\ +u'\left(I-G'\Lambda_\tau G\right)u\\ +\xi'\bar{\Lambda}_\tau\xi\end{array}\right)dt.\qquad(20)$$

If $\tau=[\tau_1\ \tau_2\ \ldots\ \tau_k]$ is such that $\tau_1>0$, ... , $\tau_k>0$, and the optimization problem (19) has a finite solution for all initial conditions, then it can be solved in terms of the following Riccati differential equation:

$$\begin{aligned}\dot{P}^\epsilon = &\\ & A'P^\epsilon+P^\epsilon A\\ & -(P^\epsilon B-K'\Lambda_\tau G)\left(I-G'\Lambda_\tau G\right)^{-1}(B'P^\epsilon-G'\Lambda_\tau K)\\ & -P^\epsilon C\bar{\Lambda}_\tau^{-1}C'P^\epsilon-K'\Lambda_\tau K;\quad P^\epsilon(-T)=I/\epsilon\end{aligned}\qquad(21)$$

which is solved forwards in time.

Lemma 2: Let $\tau=[\tau_1\ \tau_2\ \ldots\ \tau_k]$ be given such that $\tau_1>0$, $\tau_2>0$, ... , $\tau_k>0$ and

$$I-G'\Lambda_\tau G>0.\qquad(22)$$

Consider the corresponding system (17) and cost functional (20) with $\lambda\in(-T,0]$. Then

$$V_\tau^\epsilon(x_0,\lambda)>-\infty\ \forall x_0\in\mathbf{R}^n$$

if and only if the Riccati differential equation (21) has a solution $P_\tau^\epsilon(t)$ defined on $[-T,\lambda]$. In this case,

$$V_\tau^\epsilon(x_0,\lambda)=x_0'P_\tau^\epsilon(\lambda)x_0.\qquad(23)$$

Proof. This lemma follows directly from a standard result on the linear quadratic regulator problem; e.g., see page 55 of [9]. □

In order to calculate $V_\tau(x_0)$ using our Riccati equation approach, we will consider the following Riccati Differential Equations:

$$\begin{aligned}
-\dot{S}^\epsilon = &\\
& AS^\epsilon + S^\epsilon A' \\
& -(B - S^\epsilon K'\Lambda_\tau G)\left(I - G'\Lambda_\tau G\right)^{-1}\left(B' - G'\Lambda_\tau K S^\epsilon\right) \\
& -C\bar{\Lambda}_\tau^{-1}C' - S^\epsilon K'\Lambda_\tau K S^\epsilon; \quad S^\epsilon(-T) = \epsilon I; \qquad (24)
\end{aligned}$$

$$\begin{aligned}
-\dot{S} = &\\
& AS + SA' \\
& -(B - SK'\Lambda_\tau G)\left(I - G'\Lambda_\tau G\right)^{-1}\left(B' - G'\Lambda_\tau K S\right) \\
& -C\bar{\Lambda}_\tau^{-1}C' - SK'\Lambda_\tau K S; \quad S(-T) = 0 \qquad (25)
\end{aligned}$$

which are solved forward in time.

Theorem 2: Let $\tau = [\tau_1\ \tau_2\ \ldots\ \tau_k]$ be given such that $\tau_1 > 0$, $\tau_2 > 0$, ... , $\tau_k > 0$ and $I - G'\Lambda_\tau G > 0$. Also suppose there exists an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, all non-zero $x_0 \in \mathbf{R}^n$ and all $\lambda \in (-T, 0]$, then $V_\tau^\epsilon(x_0, \lambda) > 0$. Then for any $\epsilon \in (0, \epsilon_0)$, the Riccati equations (24) and (25) have solutions $S_\tau^\epsilon(t) > 0$ and $S_\tau(t) \geq 0$ defined on $[-T, 0]$ and for any $x_0 \neq 0$

$$V_\tau^\epsilon(x_0) = x_0' \left[S_\tau^\epsilon(0)\right]^{-1} x_0 > 0.$$

Also, if $S_\tau(0) > 0$ then

$$V_\tau(x_0) = x_0' \left[S_\tau(0)\right]^{-1} x_0 > 0.$$

Furthermore, if the matrix $S_\tau(0) \geq 0$ is singular and x_0 is not contained within the range space of $S_\tau(0)$, then

$$V_\tau(x_0) = \infty.$$

Proof. Under the conditions of the theorem, it follows from Lemma 2 that for all $\epsilon \in (0, \epsilon_0)$ the differential Riccati equation (21) has a solution P_τ^ϵ satisfying

$$V_\tau^\epsilon(x_0, \lambda) = x_0' P_\tau^\epsilon(\lambda) x_0 > 0$$

for all $x_0 \neq 0$ and for all $\lambda \in (-T, 0]$. That is, $P_\tau^\epsilon(t) > 0$ for $t \in [-T, 0]$. Now define

$$S_\tau^\epsilon(t) = P_\tau^\epsilon(t)^{-1} > 0$$

for $t \in [-T, 0]$. It follows via straightforward algebraic manipulations that $S_\tau^\epsilon(t)$ satisfies (24) on $[-T, 0]$. Thus, (24) has a solution on $[-T, 0]$ such that

$$V_\tau^\epsilon(x_0, \lambda) = x_0' \left[S_\tau^\epsilon(\lambda)\right]^{-1} x_0 > 0$$

for $\lambda \in (-T, 0]$. In particular,

$$V_\tau^\epsilon(x_0) = x_0' \left[S_\tau^\epsilon(0)\right]^{-1} x_0 > 0.$$

It follows via some straightforward algebraic manipulations that the Riccati differential equation (24) can be re-written as

$$\begin{aligned}
-\dot{S}^\epsilon = &\\
& AS^\epsilon + S^\epsilon A' \\
& -(S^\epsilon K' - BG')\left(\Lambda_\tau^{-1} - GG'\right)^{-1}\left(KS^\epsilon - GB'\right) \\
& -C\bar{\Lambda}_\tau^{-1}C' - BB'; \quad S^\epsilon(-T) = \epsilon I; \qquad (26)
\end{aligned}$$

Now let $\Pi_\tau^\epsilon(t) = -S_\tau^\epsilon(t) < 0$ for $t \in [-T, 0]$. It follows that Π_τ^ϵ satisfies the Riccati differential equation

$$\begin{aligned}
\dot{\Pi}^\epsilon = &\\
& -A\Pi^\epsilon - \Pi^\epsilon A' \\
& -(\Pi^\epsilon K' + BG')\left(\Lambda_\tau^{-1} - GG'\right)^{-1}\left(K\Pi^\epsilon + GB'\right) \\
& -C\bar{\Lambda}_\tau^{-1}C' - BB'; \quad \Pi^\epsilon(-T) = -\epsilon I. \qquad (27)
\end{aligned}$$

This Riccati differential equation corresponds to a linear quadratic optimal control problem corresponding to the dual system (18) with cost functional

$$\begin{aligned}
J^\epsilon(\xi(\cdot)) \;=\; & -\epsilon\|x(-T)\|^2 \\
& + \int_{-T}^{\lambda} \left(-\|y\|^2 - z'\bar{\Lambda}_\tau^{-1}z + \xi'\Lambda_\tau^{-1}\xi\right) dt.
\end{aligned}$$

Indeed, if we let

$$W_\tau^\epsilon(x_0, \lambda) \triangleq \inf_{\xi(\cdot) \in \mathbf{L}_2[-T, \lambda]} J^\epsilon(\xi(\cdot))$$

subject to (18) and $x(\lambda) = x_0$, then it follows from a standard result on the linear quadratic regulator problem (e.g., see page 55 of [9]) and the existence of a solution to the Riccati differential equation (27) that $W_\tau^\epsilon(x_0, \lambda) > -\infty$ for all $x_0 \in \mathbf{R}^n$ and all $\lambda \in (-T, 0]$. Furthermore,

$$W_\tau^\epsilon(x_0, \lambda) = x_0' \Pi^\epsilon(\lambda) x_0 < 0.$$

Also, note that it follows from the form $J^\epsilon(\xi(\cdot))$ that for all $x_0 \in \mathbf{R}^n$, $W_\tau^\epsilon(x_0, 0)$ is monotone increasing as $\epsilon \to 0$. Hence, $S^\epsilon(0)$ is monotone decreasing (in a semi-definite sense) as $\epsilon \to 0$.

We now re-write the Riccati differential equation (25) with the substitution $\Pi(t) = -S(t)$:

$$\begin{aligned}
\dot{\Pi} = &\\
& -A\Pi - \Pi A' \\
& -(\Pi K' + BG')\left(\Lambda_\tau^{-1} - GG'\right)^{-1}\left(K\Pi + GB'\right) \\
& -C\bar{\Lambda}_\tau^{-1}C' - BB'; \quad \Pi(-T) = 0. \qquad (28)
\end{aligned}$$

This equation also corresponds to a linear quadratic optimal control problem for the dual system (18) with cost function

$$J(\xi(\cdot)) = \int_{-T}^{\lambda} \left(-\|y\|^2 - z'\bar{\Lambda}_\tau^{-1}z + \xi'\Lambda_\tau^{-1}\xi\right) dt.$$

Indeed, we define

$$W_\tau(x_0, \lambda) \triangleq \inf_{\xi(\cdot) \in \mathbf{L}_2[-T, \lambda]} J^\epsilon(\xi(\cdot))$$

subject to (18) and $x(\lambda) = x_0$. Comparing the cost functions $J(\xi(\cdot))$ and $J^\epsilon(\xi(\cdot))$, it follows that

$$W_\tau^\epsilon(x_0, \lambda) \leq W_\tau(x_0, \lambda)\ \forall x_0 \in \mathbf{R}^n\ \&\ \lambda \in (-T, 0].$$

Hence, we can conclude that $W_\tau(x_0, \lambda) > -\infty$ for all $x_0 \in \mathbf{R}^n$ and all $\lambda \in (-T, 0]$. Hence, again using the above result on the linear quadratic regulator problem, it follows that (28) has a solution on $[-T, 0]$ and

$$W_\tau(x_0, \lambda) = x_0' \Pi_\tau(\lambda) x_0\ \forall x_0 \in \mathbf{R}^n\ \&\ \lambda \in (-T, 0].$$

Hence (25) has a solution on $[-T, 0]$ and

$$W_\tau(x_0, 0) = x_0'\Pi_\tau(0)x_0 = -x_0'S_\tau(0)x_0$$

Moreover, setting $\xi(\cdot) \equiv 0$ in the optimal control problem defining $W_\tau(x_0, \lambda)$, it is clear that $W_\tau(x_0, \lambda) \le 0$ for all $x_0 \in \mathbf{R}^n$ and all $\lambda \in [-T, 0]$. Hence, $S_\tau(\lambda) \ge 0$ for all $\lambda \in [-T, 0]$ and in particular $S_\tau(0) \ge 0$

We now consider the case in which $S_\tau(0) > 0$. In this case, it follows from the continuity of solutions to the Riccati differential equation that $S_\tau^\epsilon(0) \to S_\tau(0)$ as $\epsilon \to 0$ and furthermore, as mentioned above $S_\tau^\epsilon(0)$ is monotone decreasing as $\epsilon \to 0$. Hence, for a given $x_0 \in \mathbf{R}^n$,

$$x_0'P_\tau^\epsilon(0)x_0 = x_0'S_\tau^\epsilon(0)^{-1}x_0 \to x_0'S_\tau(0)^{-1}x_0 \text{ as } \epsilon \to 0.$$

Furthermore, $x_0'P_\tau^\epsilon(0)x_0$ is monotone increasing as $\epsilon \to 0$. Therefore,

$$V_\tau(x_0) = \sup_{\epsilon>0} x_0'P_\tau^\epsilon(0)x_0 = x_0'S_\tau(0)^{-1}x_0$$

as required.

We now consider the case in which the matrix $S_\tau(0)$ is singular and suppose x_0 is not in the range of $S_\tau(0)$. We claim that in this case $V_\tau(x_0) = \infty$. We establish this claim by contradiction. Indeed, suppose there exists an $M > 0$ such that

$$V_\tau(x_0) = \lim_{\epsilon\to 0} x_0'S_\tau^\epsilon(0)^{-1}x_0 \le M.$$

Now define a sequence $\{y_0^k\}_{k=1}^\infty$ so that $y_0^k = \left[S_\tau^{\frac{\epsilon_0}{k}}(0)\right]^{-\frac{1}{2}} x_0$. Hence,

$$x_0 = \left[S_\tau^{\frac{\epsilon_0}{k}}(0)\right]^{\frac{1}{2}} y_0^k \tag{29}$$

for all k. Also, we can conclude

$$\|y_0^k\|^2 \le M$$

for all $k = 1, 2, \ldots$. That is, the sequence $\{y_0^k\}_{k=1}^\infty$ is contained in a compact set. From this it follows that this sequence has a convergent subsequence $\{\tilde{y}_0^k\}_{k=1}^\infty$: $\tilde{y}_0^k \to \bar{y}_0$. Now considering equation (29) for the subsequence $\{\tilde{y}_0^k\}_{k=1}^\infty$ and taking the limit as $k \to \infty$, it follows that

$$x_0 = [S_\tau(0)]^{\frac{1}{2}}\bar{y}_0$$

and thus x_0 must be in the range of $S_\tau(0)$ which is the desired contraction. Thus, we have established the claim. This completes the proof of the theorem. □

Remark The Riccati equation (25) which is used to characterize possible controllability in the above theorem can be used to illustrate the duality between the notion of possible controllability considered in this paper and the notion of robust unobservability considered in the paper [5]. Indeed, we consider the uncertain system defined by the dual system (18) and the averaged integral quadratic constraints

$$\begin{aligned}
\frac{1}{q}\sum_{i=1}^{q}\int_0^T \left(\|\xi_1^i(t)\|^2 - \|z_1^i(t)\|^2\right)dt \quad &\le \quad d_1; \\
&\vdots \\
\frac{1}{q}\sum_{i=1}^{q}\int_0^T \left(\|\xi_k^i(t)\|^2 - \|z_k^i(t)\|^2\right)dt \quad &\le \quad d_k.
\end{aligned} \tag{30}$$

where

$$\xi(t) \triangleq [\xi_1(t)'\ \xi_2(t)'\ \ldots\ \xi_k(t)'\ ]'$$

and

$$z(t) \triangleq [z_1(t)'\ z_2(t)'\ \ldots\ z_k(t)'\ ]'.$$

Also, $z_1(t) \in \mathbf{R}^{r_1}$, $z_2(t) \in \mathbf{R}^{r_2}, \ldots, z_k(t) \in \mathbf{R}^{r_k}$ and $\xi_1(t) \in \mathbf{R}^{h_1}$, $\xi_2(t) \in \mathbf{R}^{h_2}, \ldots, \xi_k(t) \in \mathbf{R}^{h_k}$. In [5] it is shown that the robustly unobservable cone for this system can be determined by solving the Riccati differential equation:

$$\begin{aligned}
\dot{P} = \\
&-AP - PA' \\
&-(PK' + BG')\left(\Lambda_\tau - GG'\right)^{-1}(KP + GB') \\
&-C\bar{\Lambda}_\tau C' - BB'; \quad P(T) = 0.
\end{aligned} \tag{31}$$

However, making the substitution $S(t) = -P(-t)$ in (25) and replacing τ_s by τ_s^{-1} leads to precisely this Riccati differential equation. This illustrates the duality between the notion of robust unobervability considered in [5] and the notion of possible controllability considered in this paper.

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