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Robust Adaptive Scheduled Switched Control

Khalid El Rifai and Kamal Youcef-Toumi
Massachusetts Institute of Technology
77 Massachusetts Ave. Room 3-348, Cambridge, MA 02139
elrifai@mit.edu, youcef@mit.edu

Abstract— **An algorithm for robust adaptive control design for a class of nonlinear single-input-single-output (SISO) switched systems is presented. The control scheme achieves prespecified exponential stability, bounded-input-bounded-state (BIBS) stability, and arbitrary switched system stability. This is achieved for any bounded time varying parametric uncertainty and disturbance without requiring a priori knowledge of such bounds. The algorithm is based on a standard adaptive control architecture with a scheduled periodic switching between a standard adaptation law and an exponentially stabilizing robust one. The results are illustrated through simulations.**

Index Terms— **switched control, robust adaptive control, hybrid systems.**

I. Introduction

The control of systems characterized by hybrid, i.e., continuous and discrete, dynamics has been attracting many research efforts in recent years. This is motivated by the need to achieve reliably, repeatable, and safe control schemes to handel complex systems with switching dynamics of large, rapid, and sudden changes in model characteristics due to either natural (physical) changes or controlled (decision making based) changes. Such systems arise in many application such as robotics, chemical processes, power and communications networks. There are two main issues with control of switched systems, which are stability and response of the switched system even when each subsystem is stable and known and the other is the robustness of stability with respect to uncertainties.

In terms of stability and response of switched systems, several results have been obtained in recent years, e.g. [6]. The most common approach to control of switched systems uses switching between linear-time-invariant (LTI) controllers. In this context, sufficient conditions for stability such as common Lyapunov functions and average dwell time [6] are the most commonly used tools. One class of results requires very sensitive adjustment to controller gains with each plant switch to guarantee stability for any switching speed. This is the case for common Lyapunov function based work, e.g., [10], [6], which requires switching control gains such that closed loop LTI system matrices are all stable and commute or are symmetric. Whereas another group of results shows stability if switching is slow on average, e.g., [1], [6], which limits possible plant variations to be dealt with and requires gains to be adjusted to guarantee the stability of each frozen configuration and some level of knowledge of system parameters to compute admissible switching speed, which is the average dwell time.

The other problem of interest is that of dealing with uncertainty. The existing results are categorized based on the architecture being either that of a fixed robust control or standard adaptive control. In this regard, controller switching is used to deal with large uncertainties in a plant belonging to a known family of plants [1], [10], [8]. The first approach usually uses switching between LTI controllers, where stability is either based on a common Lyapunov function condition [10] or an average dwell time condition [1]. The weak robustness of LTI controllers with respect to parametric uncertainty causes the problem of unstable frozen plant/controller configuration [1]. On the other hand, methods based on adaptive control [8], [4] enjoy better stability guarantees in theory for similarly parameterized plants, if no disturbances or noise are present, analogous to standard adaptive control systems. Yet as standard adaptive control, these methods do not allow for characterizing the dynamic response neither for a frozen configuration nor for the overall dynamics. As suggested by the authors [8], [4], re-initializing the adaptation by switching between fixed estimates or resetting the adaptive estimate, is solely for improving transients, which is possible only if such fixed estimates are good. These results display similar pros and cons of the robust and adaptive methods analogous to those in non-switching control designs.

This paper poses a solution to the following problem:

Find a *single control scheme* that guarantees :
1-Stability
2-Steady state and dynamic performance.
for a large class of *uncertain time varying switched* systems subject to:
(i) Large and time varying parametric uncertainty.
(ii) Disturbances and unstructured uncertainties.

More specifically, the question is to find a scheme that achieves 1 and yields design guidelines for addressing 2 as good as possible without employing situation specific information, estimation, and heuristics. Though, the use of situation specific information and logic-based codes can allow for improvements, the objective here is to find the best possible single design. Naturally, this will yield various trade-offs and performance limitations for which some generic design parameters should be adjusted according to the nature of the problem. If situation specific information is available then logic-based codes can be incorporated to

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optimize the performance based on that information.

The algorithm developed in this paper is based on a standard adaptive control architecture for single-input-single-output (SISO) nonlinear systems with a scheduled periodic switching between a standard gradient adaptation law and an exponentially stabilizing robust one [2]. The approach used here bypasses the switched system stability problem since switching in plant parameters appear as step and impulse inputs to a continuous exponentially stable system [2]. Then a scheduled switching scheme is introduced solely for improving steady state tracking error. This is in contrast to most current schemes where logic-based codes are developed to estimate and switch between candidate controllers. However, some older papers [3], [7] for adaptive control of linear systems used a predetermined switching scheme between a set of candidate controllers. As suggested by the authors, despite promising potential the difficulty in finding these candidate controllers is a key limitation.

The remainder of the paper is organized as follows. Section II reviews the exponentially stable robust adaptive control system and classes of systems of interest. Section III presents the scheduled switched control system. A simulation example is given in section IV and conclusions are given in section V. The appendix provides a proof of the main result. In this paper, $\overline{\lambda}(.)$ and $\underline{\lambda}(.)$ denote the maximal and minimal eigenvalues of a matrix, $\|.\|$ the euclidian norm, and $diag(.,.,\ldots)$ denotes a block diagonal matrix.

II. Exponentially Stable Robust Adaptive Control

A. Standard Adaptive Control

Consider the following closed loop error dynamics commonly found in adaptive control:

$$\begin{aligned} \dot{e} &= f_e(e,\tilde{a},t)+d(t) \\ \dot{\tilde{a}} &= f_a(e,\hat{a},t) \end{aligned} \quad (1)$$

where e is a generalized tracking error, includes state estimation error in general output feedback problems, parameter estimation error $\tilde{a}=\hat{a}-a$ is the difference between parameter estimate $\hat{a}$ and actual parameter a, and d is the disturbance. The parameter vector a corresponds to a parametrization of the plant's modeled dynamics. The adaptation law in Equation (1) though not specified is usually a gradient adaptation law.

In standard adaptive control with constant parameters and no disturbances we have $\dot{a}=d=0$ and the Lyapunov analysis:

$$V(e,\tilde{a})=e^TPe+\tilde{a}^T\Gamma^{-1}\tilde{a} \Rightarrow \dot{V}(e,\tilde{a})=-2e^TCe \le 0 \quad (2)$$

where matrices $P,C>0$ are chosen matrices depending on the particular algorithm, e.g. choice of reference model and $\Gamma>0$ is adaptation gain matrix. This concludes Lyapunov stability of the system, with fixed point $(e,\tilde{a})=(0,0)$, and further analysis from Barbalat's lemma shows that $e(t)\to 0$ as $t\to\infty$.

B. Classes of Systems

The previous section describes a generic closed loop error dynamics and a Lyapunov analysis with quadratic Lyapunov functions without specifying how such a system is obtained and what classes of plants are considered. We will focus on linearly parameterized minimum phase SISO nonlinear systems for which there exists well established algorithms, which sum of which are briefly summarized next.

Consider the following n-dimensional single input linear system:

$$\begin{aligned} \dot{x} &= A(t)x+B(t)u+d \\ y &= C(t)x \end{aligned} \quad (3)$$

where A,B,C,d are piecewise continuous uniformly bounded. The objective is for the output y to follow the output of a reference model:

$$\begin{aligned} \dot{x}_m &= A_mx_m+B_mr \\ y_m &= C_mx_m \end{aligned} \quad (4)$$

where A_m,B_m,C_m are nonminimal realization of a stable reference model and and scalar r is a piecewise continuous bounded reference trajectory. Standard assumptions as in [9] follow with pointwise in time analogues for the output feedback matching condition and uniform exponential stability of the zero dynamics required. Similarly, the full state feedback model reference design can be incorporated using standard assumptions [9] with pointwise in time analogues for the full state feedback matching condition. Due to space limitations only an informal presentation on the classes of systems is done in this paper.

Another class of systems based on the adaptive control design in [11] are those globally transformable to the companion form given below:

$$y^{(r)}=\sum_{i=1}^{m}\theta_if_i(x)+b\beta(x)u+d \quad (5)$$

where $y^{(r)}$ is the r^{th} derivative of the targeted output, where r is the relative degree of the system, vector $\theta=[\theta_1,\theta_2,\ldots,\theta_m]^T$ and scalar b are piecewise continuous uniformly bounded unknown plant parameters, f_i, and $\beta(x)\neq 0$ are known functions, and d is the disturbance. The state $x=[y,y^{(1)},\ldots y^{(r-1)}]$ is measured, which reduces to output feedback for relative degree 1 plants. The sign of high frequency gain $b\neq 0$ is assumed to be known and constant. Uniformly exponentially stable zero dynamics are also assumed.

C. The Robust Adaptive Controller

In order to deal with time varying and switching dynamics as well as allow characterizing the dynamic response we propose the modified adaptation law:

$$\dot{\hat{a}}=f_a(e,\hat{a},t)-L(\hat{a}-a^*) \quad (6)$$

with $L > 0$ and a^* is a chosen estimate of a which yields the system:

$$\begin{aligned} \dot{e} &= f_e(e, \tilde{a}, t) + d(t) \\ \dot{\tilde{a}} &= f_a(e, \hat{a}, t) - L\tilde{a} + L(a^* - a) - \dot{a} \end{aligned} \tag{7}$$

The system above, as shown in [2], using the Lyapunov function in Equation (2), is an exponentially stable system with state $x = [e, \tilde{a}]^T$ and bounded-input-bounded-state (BIBS) stable driven by an input $\bar{v} = [d,\, L(a^* - a) - \dot{a}]^T$. Note that the adaptation law of Equation (6) is closely related to the notion of leakage, which is usually used to maintain Lyapunov stability in the presence of disturbances with a priori known bounds, which is not pursued here. This result is provided without proof, see [2], since the new result to be presented in this paper will reduce to this earlier result in a limiting case. However, we will consider the implications of this result to motivate the scheduled switching control scheme developed in this paper. The following Theorem gives a formal statement of the result.

Theorem 1: Consider the system given by Equation (1) and the Lyapunov analysis of Equation (2) then the system given by Equation (7) is :
(i) Uniformly internally exponentially stable.
(ii) BIBS stable with
$\|e(t)\| \leq c_1\|x(t_o)\|e^{-\alpha(t-t_o)} + c_2\int_{t_o}^{t} e^{\alpha(\tau-t)}\|v(\tau)\|\,d\tau$.
where c_1, c_2 are constants, $\alpha = \bar{\lambda}(diag(P^{-1}C, L))$, and $v = [P^{1/2}d, \Gamma^{-1/2}(L(a^* - a) - \dot{a})]^T$.
The following remarks summarize some key properties of the developed result:

- Exponential stability allows for shaping the transient response, e.g. settling time, and frequency response of the system to low/high frequency dynamics and inputs by adjusting the decay rate α independent of parametric uncertainty, see [2] for details.
- The system is *robustly stable* with respect to any bounded magnitude parametric uncertainty and bounded disturbance *without requiring any a priori knowledge of such bounds* since these affects enter as inputs to a BIBS stable system.
- Also note that plant parameter switching no longer affects internal dynamics and stability but enters as a step change in input $L(a^* - a)$ and an impulse in input $\dot{a}$ at the switching instant. Similarly, switching the estimate a^* yields a step change in input $L(a^* - a)$.
- An allowed arbitrary time variation and switching in the parameter vector a as a uniformly bounded piecewise continuous vector suggests that such changes in the controlled plant parameters are for a plant with the same assumed parameterized structure and within admissible values dictated by design assumptions. Such assumptions include constant and known sign of high frequency gain, nonzero high frequency gain, and uniformly exponentially stable zero dynamics. A more precise statement is due on class by class basis.

III. Robust Adaptive Scheduled Switching Controller

A. Rationale

This section introduces the scheduled switching control scheme. In this scheme it is proposed to periodically (or generally synchronously) switch between the standard adaptation law and the modified robust adaptation law. The motivation for such an idea can be seen by investigating its effect on the convolution integral for THE input $v(t)$ in Theorem 1 (ii). When we turn on the standard adaptive controller the convolution integral is zero, when $\dot{a} = d = 0$, since there are no inputs yet the system's dynamics is not exponentially stable. Whereas, if we switch back to the robust adaptation law (with exponentially stable system dynamics) for a very short period of time Δt^* then the convolution integral is very small since the integration period is very small. Therefore, repeating the process gives a system with convolution integral ≈ 0, i.e., $e \approx 0$ after transients, yet the average response of the system remains that of an exponentially stable system. Therefore, this scheme allows for a *reduction in the tracking error's sensitivity with respect to parametric uncertainty without high gains* as standard adaptive control while retaining *exponential stability and BIBS stability* of the robust adaptive scheme. As opposed to other schemes that switch between different controllers or reset parameter estimates, the switching here is for the purpose of optimizing performance independent of any additional knowledge. Whereas, adjustments to the estimate a^* based on multiple candidate models, as has been briefly discussed in [2] would be an additional capability that takes advantage of available information in a manner analogous to controller switching or resetting techniques in [1], [4], [10], [8].

B. Main Result

We now formulate the problem precisely. In this scheduled switching scheme it is proposed to use the following adaptation law:

$$\dot{\hat{a}} = f_a(e, \hat{a}, t) - q(t)L(\hat{a} - a^*) \tag{8}$$

This yields the following closed loop error dynamics:

$$\begin{aligned} \dot{e} &= f_e(e, \tilde{a}, t) + d(t) \\ \dot{\tilde{a}} &= f_a(e, \hat{a}, t) - q(t)L(\hat{a} - a^*) - \dot{a} \end{aligned} \tag{9}$$

This differs from the system given by Equation (7) by only the scalar switching function $q(t)$, the discrete state variable of this hybrid system, which is shown in Figure 1. This is referred to as scheduled switching since the the transition of the discrete state variable is scheduled ahead of time as opposed to being reactive.

Analyzing the switching function in Figure 1 shows that the function takes the value 1 during the time period Δt^*, which is the *activation period of the robust adaptation law* of Section II. Whereas, during periods Δt^s we have the *standard gradient adaptation law* with $q = 0$. Let $\Delta t = \Delta t^s + \Delta t^*$ be the period of the pulse then the switching ratio $r = \Delta t^*/\Delta t$, where $r \in (0, 1)$. The switching ratio

r indicates the relative activation of standard and robust adaptation laws. In this analysis, we will assume fixed values for all these periods for simplicity and tractability of performance evaluation. The results apply in a similar way to the case where the pulse is of time varying width and period. The following Theorem states the main result and incorporates Theorem 1 as a special case.

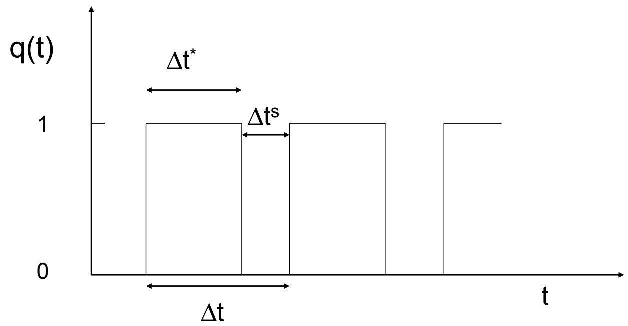

Fig. 1. Pulse Switching Function for Scheduled Switching.

Theorem 2: Consider the system given by Equation (1) and the Lyapunov analysis of Equation (2) then the system given by Equation (9) is :
(i) Uniformly internally exponentially stable and BIBS stable $\forall\ r \in (0, 1]$ with

$$\|e(t)\| \leq c_1 \|x(t_o)\| e^{-r\alpha(t-t_o)} + c_2 \int_{t_o}^{t} e^{r\alpha(\tau - t)} \|v(\tau)\|\, d\tau$$

(ii) The steady state tracking error $\forall\ r \in (0, 1)$ satisfies :

$$\begin{aligned} \|e(t)\|_{ss} &\leq e^* + e^o \\ e^o &= c_2 \int_{t_o}^{\infty} e^{r\alpha(\tau - t)} \|v^o(\tau)\|\, d\tau \\ e^* &= c_2 \frac{e^{r\alpha\Delta t^*} - 1}{r\alpha} \sup_{t \geq t_o} \|v^*\| \end{aligned}$$

where c_1 and c_2 are constants, $\alpha = \bar{\lambda}(diag(P^{-1}C, L))$, $r = \Delta t^*/\Delta t$, $v^* = [0, \Gamma^{-1/2} L(a^* - a)]$, $v^o = v - qv^*$, and $v = [P^{1/2} d, \Gamma^{-1/2}(L(a^* - a) - \dot{a})]^T$,

C. Analysis of the Switching Function

The result of Section III.B simply states that the developed system retains exponential stability and BIBS stability from the robust scheme of Section II (part (i) of Theorem 2) yet allows the attenuation factor with respect parametric uncertainty $a^* - a$ to be improved for the same gains via adjusting the switching function $q(t)$. Let us examine the steady state tracking error from Theorem 2 (ii) and let $\dot{a} = d = 0$ since their affect can be superimposed, which is reflected by the term e^o in steady state upper bound in Theorem 2 (ii). Therefore, we have

$$\|e(t)\|_{ss} \leq c_2 \frac{e^{r\alpha\Delta t^*} - 1}{r\alpha} \sup_{t \geq t_o} \|v^*\| \tag{10}$$

This means that making Δt^* small allows for significantly attenuating parametric uncertainty for same system gains. In fact, using a Taylor expansion of $e^{r\alpha\Delta t^*}$ for a small $r\alpha\Delta t^*$ and substituting in Equation (10) we get:

$$\|e(t)\|_{ss} \leq c_2 \Delta t^* \sup_{t \geq t_o} \|v^*\| \tag{11}$$

which suggests that attenuation of parametric uncertainty scales with Δt^*. Since $\Delta t^* = r\Delta t$, we can reduce Δt^* by either reducing r or Δt. The first case, r small is obvious since it implies that adaptive control dominates the average response of the system, i.e. its turned on more, which should lead to small sensitivity to parametric uncertainty. However, it is important to note that the averaged response needs not to be dominated by adaptive control for reduced sensitivity with respect to parametric uncertainty. This means that we can have each activation period of the modified adaptation law larger than that of the standard adaptation law $\Delta t^* > \Delta t^s$, yet if Δt^* is small enough small tracking error can be achieved, which is in agreement with the discussion in Section III. A.

IV. Example Simulation

In this section, case study simulations are shown to demonstrate the key results presented in this paper. Consider a MRAC, see [9], [5] for control design details, for the following unstable 2^{nd} order plant of relative degree 1:

$$\begin{aligned} \dot{x}_1 &= b_1 x_1 + b_2 x_2 + b_3 u + b_3 d \\ \dot{x}_2 &= x_1 \\ y &= c_1 x_1 + c_2 x_2 + n \end{aligned} \tag{12}$$

where $b_1 = 3$, $b_2 = -2$, and $c_1 = c_2 = b_3 = 1$ are the nominal simulation values for which we will denote the vector a as the parameterized vector corresponding to these values. Whereas, u, d, and n are control signal, disturbance, and measurement noise, respectively. The measurement noise used in the simulations is of SN $1 : 1000$ and a pair of unmodeled complex poles at 22 rad/sec and 0.07 damping ratio are also included. The reference model used is of the following transfer function:

$$W_m(s) = \frac{a_m}{s + a_m}$$

Let us choose the nominal $a_m = 1$ and $L = I$, where I is the identity matrix. Also the nominal value of the adaptation gain Γ will be denoted $\Gamma_o = 100I$. Also $\Delta t^* = 0.0005$ seconds and $r = 0.5$ are used. The reference trajectory is a sine of amplitude 2 and frequency 0.3 rad/sec.

Figure 2 shows a comparative study for MRAC of the nominal LTI plant for disturbance $d = 0$ and constant parameters. The standard MRAC, denoted standard, shows poor and unpredictable transients, which were also observed when noise and unmodeled dynamics were removed but with less oscillations. The response of the robust adaptive algorithm of Section II.C, denoted robust, and switching algorithm are shown for a large parametric uncertainty $a^* =$

$100a$ yielding predesigned settling, 4 seconds, and better transients. Note that the modified scheme without switching yields larger steady state tracking error than the standard and the switching algorithms with the same adaptation gain Γ_o. Yet using a large adaptation gain $\Gamma = 10\Gamma_o$ yields similar tracking error. Therefore, the *switching scheme achieves the best performance trade-off* by achieving pre-specified good transient response as the robust scheme yet achieves small steady state tracking error as the standard adaptive controller without requiring larger gains. The lower part of Figure 2 shows the control signals used to achieve the aforementioned tracking responses. The control signal for the standard adaptive controller is clearly much more aggressive than all other signals during the transients. A key observation is that the switching scheme introduced here does not require any aggressive and oscillatory controls as most switching control schemes, this is in agreement with the comment made at the end of Section III.C.

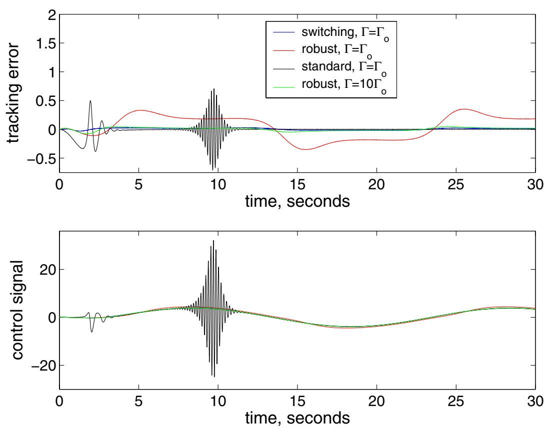

Fig. 2. Comparison of standard and modified MRAC for an LTI plant: Tracking error (top), control signal (bottom).

Figure 3 shows the response of the system for a time varying switching plant subject to disturbance $d = 10(\sin(2\pi 0.1t) + \sin(2\pi 1000t)$. Note that the response was found to be the same with or without the high frequency component of d due to pre-specified system roll-off, see [2] for details on related frequency response properties. Figure 3 shows 3 plant switches from the nominal value of b_1 to $2b_1$, $-2b_1$, and $2b_1$ at times $t = 10, 14, 21$ seconds and similarly for b_2. Another switch occurs at $t = 25$ seconds from $2b_1$ to a time varying parameter $12b_1 + 2b_1\cos(5t)$ and similarly for b_2. A fifth switch to the plant takes place at $t = 35$ seconds from $12b_1 + 2b_1\cos(5t)$ to $18b_1 + 2b_1\cos(5t)$ and similarly for b_2. As explained earlier, these switches correspond to step changes in $a^* - a$ and impulses in $\dot{a}$. Therefore, due to the attenuation of uncertainty, the system responds to these plant switches as impulse inputs and recovers to almost the same tracking error after a quick exponential decay. The maximum tracking error is about $35 - 40\%$ of the size of the reference, which occurs at two transition times $t \approx 10, 21$ seconds. Whereas a much smaller error of less than 5% is obtained elsewhere. Note that an increase in Γ, see Figure 3, can reduce the maximum overshoot at switching, to about 20%, since the effective input v acting on the system contains a term $\Gamma^{-1/2}\dot{a}$, see Theorem 2, which suggests reducing the size of the impulse in $\dot{a}$ corresponding to switching in a.

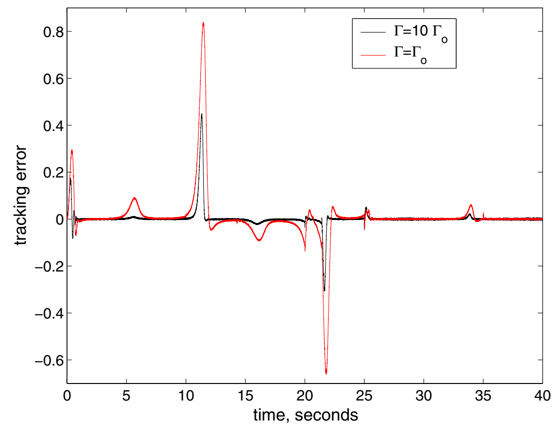

Fig. 3. Tracking error for scheduled MRAC for a switching linear plant with disturbances.

V. Conclusions

An algorithm for stable robust adaptive control for a class of SISO switched nonlinear systems has been presented. The control scheme guarantees robust exponential stability with respect to any bounded time varying and switching parametric uncertainty and bounded disturbance without requiring a priori knowledge of such bounds. The scheme is based on a standard adaptive control architecture with scheduled periodic switching between a standard gradient adaptation and an exponentially stabilizing one.

Appendix

A. Proof of Theorems 1 and 2

The proof of Theorem 2 is presented for which part (i) reduces to that of Theorem 1 by letting $r = 1$.

Proof:

(i) Let $x = [e, \tilde{a}]^T$ and $z = Sx$, where $S = diag(P^{1/2}, \Gamma^{-1/2})$ a symmetric positive definite matrix. Using the Lyapunov function $V(e, \tilde{a}) = e^T P e + \tilde{a}^T \Gamma^{-1}\tilde{a}$ and the result from Equation (2), we have for the system given by Equation (9):

$$\begin{aligned} \dot{V}(x) &= -2x^T SMSx + 2x^T S^2 \bar{v} \\ &= -2z^T Mz + 2z^T v \end{aligned}$$

where $M(t) = diag(P^{-1/2}CP^{-1/2}, \Gamma^{-1/2}qL\Gamma^{1/2})$, $\bar{v} = [d, qL(a^* - a) - \dot{a}]^T$ and $v = S\bar{v}$. But by comparison arguments as in [5] we have $V = \|z\|^2$, which means:

$$\frac{1}{2}\frac{d}{d\tau}\|z\|^2 = -z^T M(t) z + z^T v \leq -\bar{\alpha}(t)\|z\|^2 + \|z\|\|v\|$$

Hence

$$\frac{d}{d\tau}\|z\| \leq -\bar{\alpha}(t)\|z\| + \|v\|$$

where $\bar{\alpha} = \underline{\lambda}(M) = \underline{\lambda}(diag(P^{-1}C, qL))$ by similarity. Note that $\bar{\alpha} = \alpha$ if $q = 1$ and equals 0 otherwise, where $\alpha = \underline{\lambda}(diag(P^{-1}C, L))$. Using the integration factor $e^{\int_0^\tau \bar{\alpha}(\tau)d\tau}$ we then have:

$$\int_{t_o}^{t} \frac{d}{d\tau}\left(\|z(t)\|e^{\int_0^\tau \bar{\alpha}(\tau)d\tau}\right)d\tau \quad \leq \quad \int_{t_o}^{t} e^{\int_0^\tau \bar{\alpha}(\tau)d\tau}\|v(\tau)\|\,d\tau$$

Consider the integration factor used above, we have the current time $\tau = N\Delta t + T$, where N is the total number of switches and T is a time offset since the last switch. Then using the expression for τ and $\bar{\alpha}$ the integral:

$$\begin{aligned}\int_0^\tau \bar{\alpha}(\tau)d\tau &= \alpha N\Delta t^* + cT \\ &= r\alpha\tau + T(c - r\alpha)\end{aligned}$$

where c is the value $\bar{\alpha}$ takes during the period T, i.e. either 0 or α and $r = \Delta t^*/\Delta t$. Using the last two expressions we get:

$$\int_{t_o}^{t} \frac{d}{d\tau}\|z(\tau)\|e^{r\alpha\tau}\,d\tau \quad \leq \quad \int_{t_o}^{t} e^{r\alpha\tau}\|v(\tau)\|\,d\tau$$

where the constant term $e^{T(c-\alpha r)}$ is factored out on both sides. Then integrating we simply have:

$$\|z(t)\| \leq \|z(t_o)\|e^{-r\alpha(t-t_o)} + \int_{t_o}^{t} e^{r\alpha(\tau-t)}\|v(\tau)\|\,d\tau$$

By definition of $\|z\| = \|Sx\|$ we can get that:

$$\|x(t)\| \leq c_1\|x(t_o)\|e^{-r\alpha(t-t_o)} + c_2\int_{t_o}^{t} e^{r\alpha(\tau-t)}\|v(\tau)\|\,d\tau$$

where $c_1 = \|S\|\|S^{-1}\|$ and $c_2 = \|S^{-1}\|$. Internal exponential stability is shown by letting $v = 0$ above. BIBS stability is achieved by denoting $v_o = \sup_{t\geq t_o}\|v(t)\| < \infty$, then from the last expression we have:

$$\|x(t)\| \leq c_1\|x(t_o)\|e^{-r\alpha(t-t_o)} + \frac{c_2}{r\alpha}v_o$$

This proves part (i).

(ii) To prove this part let $v^* = [0, \Gamma^{-1/2}L(a^* - a)]^T$ and $v^o = v - qv^*$ and recall that from part (i) that :

$$\|x(t)\| \leq c_1\|x(t_o)\|e^{-r\alpha(t-t_o)} + c_2\int_{t_o}^{t} e^{r\alpha(\tau-t)}\|v(\tau)\|\,d\tau$$

Therefore,

$$\begin{aligned}\lim_{t\to\infty}\|e(t)\| &\leq c_2\int_{t_o}^{\infty} e^{r\alpha(\tau-t)}\|v\|\,d\tau \\ &\leq c_2\int_{t_o}^{\infty} e^{r\alpha(\tau-t)}(\|v^o\| + q\|v^*\|)d\tau\end{aligned}$$

Now consider the following term from above:

$$\begin{aligned}\int_{t_o}^{t} e^{r\alpha\tau}q\|v^*\|d\tau &\leq \sum_{i=1}^{N}\int_{t_i}^{t_i+\Delta t^*} e^{r\alpha\tau}\|v^*\|\,d\tau + c_3 \\ &\leq \frac{e^{r\alpha\Delta t^*}-1}{r\alpha}\sum_{i=1}^{N} e^{r\alpha t_i}\sup_{t\geq t_o}\|v^*\| + c_3\end{aligned}$$

where $\int_0^T e^{r\alpha\tau}q\|v^*\|d\tau \leq c_3$ but $t = t_o + N\Delta t + T$ and $t_i = i\Delta t$, which means:

$$e^{-r\alpha t}\sum_{i=1}^{N} e^{r\alpha t_i} = e^{-r\alpha(t_o+T)}\sum_{i=1}^{N} e^{-r\alpha\Delta t(N-i)}$$

The sequence $\{e^{-r\alpha\Delta t(N-i)}\}$ is monotonic and bounded by 1, which means its convergent. In fact, we have

$$\lim_{N\to\infty}\sum_{i=1}^{N} e^{-r\alpha\Delta t(N-i)} = 1$$

Using the fact that as $t \to \infty$ we have $N \to \infty$ and substituting the last three expressions into the expression for $\lim_{t\to\infty}\|e(t)\|$ we have :

$$\begin{aligned}\lim_{t\to\infty}\|e(t)\| &\leq e^* + e^o \\ e^o &= c_2\int_{t_o}^{\infty} e^{r\alpha(\tau-t)}\|v^o\|\,d\tau \\ e^* &= c_2\frac{e^{r\alpha\Delta t^*}-1}{r\alpha}\sup_{t\geq t_o}\|v^*\|\end{aligned}$$

By denoting $\|e(t)\|_{ss} = \lim_{t\to\infty}\|e(t)\|$ in the last expression part (ii) is proved.

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References

[1] Anderson, B.D.O., Brinsmead, T.S., Liberzon,D., and Morse, A.S. Multiple model adaptive control with safe switching. *Int. J. Adaptive Control Signal Process.*, 15, 445-470, 2001.

[2] El Rifai, K., El Rifai, O., and K. Youcef-Toumi. On Robust Adaptive Swicthed Control, *American Control Conference 2005, Portland, OR*.

[3] Fu, M. and B. R. Barmish . Adaptive stabilization of linear systems via switching control, *IEEE transactions on Automatic Control*, 31, 1097-1103, 1986.

[4] Kalkkuhl, J., Johansen, T.A., and J. Ldemann. Improved Transient Performance of Nonlinear Adaptive Backstepping Using Estimator Resetting Based on Multiple Models, *IEEE transactions on Automatic Control*, 47(1), 136-140, 2002.

[5] Khalil, H. Nonlinear Systems, 2nd Ed.,*Prentice-Hall*, 1995.

[6] Liberzon, D., and A.S. Morse. Basic Problems in Stability and Design of Switched Systems . *IEEE Control Systems Magazine*,19, pp. 59-70, 1999.

[7] Miller, D. E. and E. J. Davison. An adaptive controller which provides Lyapunov stability, *IEEE Transactions on Automatic Control*, 34, 599609, 1989.

[8] Narendra, K.S., Driollet, O.A., Feiler, M. , and K. George. Adaptive control using multiple models, switching and tuning.*Int. J. Adaptive Control Signal Process.*, 17, 87102, 2003.

[9] Tao, G. Adaptive Control Design and Analysis,*Wiley and Sons*, 2003.

[10] Zhivoglyadov, P. V. , Middleton, R. H., and M. Fu Localization Based Switching Adaptive Control for Time-Varying Discrete-Time Systems.*IEEE Transactions on Automatic Control*, 45 (4),752-755, 2000.

[11] Slotine, J.J.E., and Li, W. Applied Nonlinear Control, *Prentice-Hall*, 1991.