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A notion of passivity gain and a generalization of the "secant condition" for stability

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Abstract— **The secant condition plays a useful role in stability studies, especially for biological models. This paper provides a generalization of that condition to nonlinear passive systems. A "secant gain" is introduced, which combines gain and phase information for each of the cascaded subsystems.**

I. Introduction

The *secant condition* for linear stability was introduced and proved by Tyson and Othmer [13] and Thron [12]. (See also [10] for recent remarks.) One way to express the condition is as follows: the matrix

$$\begin{pmatrix} -\alpha_1 & 0 & \dots & 0 & -\beta_1 \\ \beta_2 & -\alpha_2 & \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \beta_n & -\alpha_n \end{pmatrix}$$

(with $\alpha_i > 0$ and $\beta_i > 0$ for all i), is Hurwitz (all eigenvalues have negative real part) if:

$$\frac{\beta_1 \dots \beta_n}{\alpha_1 \dots \alpha_n} < \left(\sec \frac{\pi}{n}\right)^n .$$

It is useful to compare this restriction to the small-gain theorem, which would have a 1 in the right-hand side. The secant expression, on the other hand, is always bigger than one. It is singular at $n = 2$ –which it should be, since then the matrix is always Hurwitz– and it equals 8 for $n = 3$, 4 for $n = 4$, and ≈ 2.88 for $n = 5$, and tends monotonically to 1 as $n \to \infty$ (the bound is achieved exactly in the special case in which all the constants α_i's coincide). The secant takes into account simultaneously of phase and gain information on the open-loop system, at least for stable systems with distinct real eigenvalues and no zeros.

In this paper, we give a generalization of the secant condition to nonlinear systems, more precisely cascades of output strictly passive (OSP) systems. When each system is linear and one-dimensional, the known result is recovered. The generalization is based on systematic use of a "gain" associated to OSP systems, and it is possible that this type of gain might be useful for many other problems as well. Some details omitted from this conference version can be found in the full journal paper [11]. We also remark that, in the very recent work [1] with Murat Arcak, the reader may find a very different approach, based upon Lyapunov functions and Popov Criterion techniques instead of input/output methods. Although limited to finite dimensional continuous-time systems, this other approach is very powerful and it helps tighten up estimates for many examples, such as the inhibitory feedback loop with Michaelis-Menten kinetics mentioned in this paper.

II. Notations, Definitions, and Statement of Main Result

We use standard notations: $L^2_e(0,\infty)$ denotes the "extended" set of signals $w : [0,\infty) \to \mathbb{R}$ such that the restriction $w_T = w|_{[0,T]}$ belongs to $L^2(0,T)$ for each $T > 0$. For $w \in L^2_e(0,\infty)$ and $T > 0$, we denote by $\|w\|_T$ the L^2 the norm of the restriction w_T. For $v, w \in L^2_e(0,\infty)$ and any fixed $T > 0$, $\langle v, w\rangle_T$ is the inner product of v_T and w_T . In any Hilbert space, $\theta(v,w) \in [0,\pi]$ is the angle formed by v and w, that is:

$$\cos\theta(v,w) = \frac{\langle v, w\rangle}{\|v\|\,\|w\|}$$

(zero if $v = w = 0$). The angle between the restrictions of $v, w \in L^2_e(0,\infty)$ to $[0,T]$ is denoted by $\theta_T(v,w)$ (instead of $\theta(v_T, w_T)$).

Generally (but the theorems are proved in fact in more generality), we take continuous-time finite-dimensional systems $\dot{x} = f(x,u)$, $y = h(x)$ as usual in control theory (e.g. [9]), with scalar valued inputs and outputs (generalizations to vector inputs and outputs are just a matter of notations), and state space $\mathbb{R}^n$. For simplicity, we assume that the systems being considered are L^2-well-posed, meaning that for each $u \in L^2_e(0,\infty)$ and initial state $x(0) = 0$ there is a unique solution $x(\cdot)$ defined for all $t \geq 0$, and the corresponding output $y(t) = h(x(t))$ is also in $L^2_e(0,\infty)$, and in that case call (u,y) an *input/output (i/o) pair* of the system.

We recall that a system is *output strictly passive (OSP)* (see e.g. [5], [14], [15]) if, for some constant $\gamma > 0$ it holds that

$$\|y\|_T^2 \;\leq\; \gamma\,\langle u, y\rangle_T \tag{1}$$

for every i/o pair (u,y) and all $T > 0$. (We only consider zero state responses when applying this definition, so we do not add a separate additive constant. Non-zero initial states will be dealt with separately.) If a system is OSP, There is a smallest such γ, since the set of γ's that satisfy (1) is a closed set, and we call it the *secant gain* of the system, denoted by γ_s. An equivalent definition of γ_s is as the smallest γ with the property that

$$\|y\|_T^2 \leq \gamma\,\|u\|_T\,\|y\|_T\,\theta_T(u,y),$$

or equivalently:

$$\|y\|_T \;\leq\; \gamma\,\|u\|_T\,\cos\theta_T(u,y) \tag{2}$$

for all $T > 0$ and all i/o pairs. Since (1) implies that $\langle u, y\rangle_T \geq 0$ for all i/o pairs and all T, for OSP systems we always think of the angle as lying in the interval $[0, \pi/2]$, and the cosine is nonnegative. Note that the Cauchy-Schwartz inequality applied to (1) gives $\|y\|_T \leq \gamma \|u\|_T \leq \gamma \|u\|$ for all $T > 0$; thus, $y \in L^2$ if $u \in L^2$, so an OSP system necessarily has finite L^2-induced (or "H_∞") gain $\gamma_\infty \leq \gamma_s$ (this inequality is in general a strict one). Just as the L^2 gain is the supremum of the expressions $\|y\|_T / \|u\|_T$ over all T and all i/o pairs with nonzero u, the secant gain is obtained by maximizing $\sec\theta_T(u,y)\, \|y\|_T / \|u\|_T$, hence our terminology. If $u \in L^2$, so that also $y \in L^2$, taking limits in (1) gives

$$\|y\|^2 \;\leq\; \gamma\langle u, y\rangle\,. \tag{3}$$

Conversely, if $u \in L^2 \Rightarrow y \in L^2$ and (3) is true for all $u \in L^2$, then (1) holds. This is a routine exercise in causality.

Our goal is to study the stability of the closed-loop system

$$\dot{x} = f(x, u - h(x))$$

obtained under negative unity feedback, and specifically starting from a cascade of n subsystems, as shown in the diagram in Figure 1 and subject to unity negative feedback.

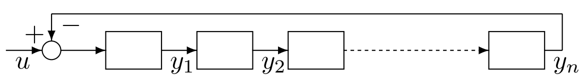

Fig. 1. Closed-loop system

Such cascades appear frequently in control theory as well as in biological applications, and, when components are one-dimensional, tend to have especially good dynamical properties such as the validity of the Poincaré-Bendixson Theorem ([6]). We will assume that the i-th system has a secant gain γ_i, and we write y_i for the output of the ith subsystem. We also assume well-posedness of the closed-loop. The main result is as follows:

Theorem. Suppose that

$$\gamma_1\gamma_2\ldots\gamma_n \;<\; \left(\sec\frac{\pi}{n}\right)^n\,.$$

Then the cascade is L^2-stable: there is a number c so that

$$\|y_n\|_T \;\leq\; c\,\|u\|_T$$

for all input/output pairs in the cascade and all $T > 0$.

Note that this property implies that every $\|y_i\|_T$ is bounded by some linear function of $\|u\|_T$, and that the signals y_i belong to L^2 if $u \in L^2$. In the special cases $n = 1$ and $n = 2$ (i.e., the secant is infinite), we interpret the inequality in the theorem as saying that the condition holds for any possible values of the γ_i's. For $n = 2$, therefore, the theorem is simply a restatement of the Passivity Theorem as given e.g. in [14], Theorem 2.2.15, Part a (using only the input u).

The assumption that the initial state of the cascade is $x(0) = 0$ is easy to dispose of, assuming appropriate reachability of the cascade, as routinely done in going from input/output stability to state space stability, and Barbălat's Lemma combined with either reachability or detectability arguments can be used to show convergence of internal states to zero. This is one such corollary:

Corollary. Suppose that the condition in the Theorem is verified, that the composite system shown in Figure 1 is zero-reachable and that each subsystem is input to state L^2-stable. Then the system with no inputs ($u = 0$) has the property that all solutions converge to $x = 0$.

III. Extensions

We formulated the results in terms of state-space systems only in order to be concrete. One could equally well consider arbitrary operators $L^2 \to L^2$, or even just relations R on $L^2 \times L^2$, where an "i/o pair" is by definition any element of R, and define secant gain γ_s as the smallest number so that (1) holds for all T and all i/o pairs. Nor is it needed for the inputs and outputs to be scalar-valued; one may consider values on arbitrary Hilbert spaces, with inner product and norms taken pointwise in that space. More generally, functions of time are not required: one could consider an arbitrary Hilbert space H and simply ask that u and y belong to H. (To be precise, one needs a Hilbert space together with a resolution of the identity, in order to be able to have a concept of "restriction" of u and y to subintervals; this is the formalism of *resolution spaces* developed in [8].)

Even more generally, if one has a system in which inputs u and outputs y are known to lie in a specific subset $S \subseteq H$, then γ_s can be defined in terms only of i/o pairs that lie in S; the validity of the main theorem is not affected, since it is just an algebraic statement about norms and inner products. One example of an operator defined only on subsets, which is of interest in biomolecular applications ("Michaelis-Menten kinetics"), is as follows. Suppose that S is the set of all L^2 maps $w : [0,\infty) \to [-a,\infty)$ with any fixed $a > 0$, and that we consider the function $\ell : [-a,\infty) \to \mathbb{R}$ given by

$$\ell(r) = \frac{Vr}{K + a + r} \tag{4}$$

(with $K, V > 0$ some constants) and the operator $u \mapsto y$ defined on S, where $y(t) = F(u)(t) = \ell(u(t))$. This is an example of a "sector" nonlinearity. The analysis of sector nonlinearities is routine in passivity theory. The operator F is OSP and has $\gamma_s = V/K$, because we have, for all $r \in [-a,\infty)$:

$$\begin{aligned}[]
[\ell(r)]^2 &= \frac{V}{K+a+r}\frac{Vr^2}{K+a+r} \\
&\leq \frac{V}{K}\frac{Vr^2}{K+a+r} \;=\; \frac{V}{K}r\ell(r)
\end{aligned}$$

(since $K + a + r \geq K$), and thus

$$\begin{aligned}
\|y\|_T^2 &= \int_0^T \ell(u(t))^2\,dt \\
&\leq \frac{V}{K}\int_0^T u(t)\ell(u(t))\,dt \;=\; \frac{V}{K}\langle u, y\rangle_T
\end{aligned}$$

so $\gamma_s \leq V/K$, and the equality is verified when $u(t) \equiv -a$.

We formulated our results in terms of stability in the L^2 sense, which is really appropriate only when dealing with equilibria associated to zero signals. However, there are easy extensions, which are of interest, particularly, when dealing with problems in biology and chemistry, where quantities represent concentrations of substances, and hence are always nonnegative. Suppose that one wishes to study a system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned}$$

under the feedback law $u=-y$, and that there is a steady state x^* for this closed-loop system:

$$f(x^*,-h(x^*))=0$$

whose stability is of interest to analyze. We assume that the states $x(t)$ evolve in some subset S of $\mathbb{R}^n$, for example the positive orthant

$$\mathbb{R}^n_+ = \{(x_1,\ldots,x_n),\, x_i \geq 0\,\forall i\},$$

and inputs u of the open-loop system take values on some set U. (We assume that $-h(S)\subseteq U$.) We make a change of variables $z=x-x^*$ which leads to the system

$$\begin{aligned} \dot{z} &= g(z,v) \;=\; f(z+x^*, v-h(x^*)) \\ w &= \ell(z) \;=\; h(z+x^*)-h(x^*), \end{aligned}$$

having states $z(t)\in\{x-x^*,\, x\in S\}$, inputs $v(t)$ in the input-value space $\{u+h(x^*),\, u\in U\}$, and outputs $w(t)$. We have that $g(0,0)=0$.

The feedback $v=-\ell(z)$ results in

$$\dot{z}=g(z,-\ell(z))=f(z+x^*,-h(z+x^*)).$$

Thus, for each solution $x(t)$ of $\dot{x}=f(x,-h(x))$, we have that $z(t)=x(t)-x^*$ satisfies $\dot{z}=g(z,-\ell(z))$, and each solution of the latter system arises from the former. We have reduced the analysis to the case treated in this paper, since all solutions of

$$\dot{x}=f(x,-h(x))$$

converge to x^* if and only if all solutions of the new system converge to $z=0$. For example, suppose that we wish to study a positive system, that is, a system whose state state space is $\mathbb{R}^n_+$ and inputs are also nonnegative. Furthermore, suppose that, as is often the case in biological feedback loops, one wishes to study an inhibitory feedback of the form

$$u=\frac{M}{K+x_n}$$

where M and K are some positive constants and x_n is the nth coordinate of the state, that is to say, we have $h(x)=-M/(K+x_n)$.

In terms of the variables z, we have the output

$$\begin{aligned} w &= \ell(z)=h(z+x^*)-h(x^*) \\ &= \frac{M}{K+x_n^*}-\frac{M}{K+(z_n+x_n^*)}=\frac{Vz_n}{K+x_n^*+z_n} \end{aligned}$$

which is the function in (4) with $a=x^*$ and $V=M/(K+x_n^*)$. Since $x_n(t)$ is nonnegative, the state variable $z_n(t)$ takes values in $[-x^*,\infty)$. Thus, the closed-loop system as obtained from cascading the original system (which may itself be a cascade of several subsystems) with the static system "$y=\ell(u)$", which has $\gamma_s=V/K$, so the previous analysis applies. This is all particularly simple for a linear system $\dot{x}=f(x,u)=Ax+Bu$. Positivity amounts to asking that all the off-diagonal entries of A as well as all entries of B are nonnegative (see e.g. [2], [4]). Since the system is linear and $Ax^*-Bh(x^*)=0$, we have that

$$g(z,v)=A(z+x^*)+B(v-h(x^*))=Az+Bv,$$

so the same open loop system results, except that now we are interested in the stability of $z=0$.

IV. Proof of Main Result

Given an external input u, the solutions of the closed-loop system with initial state zero are so that the signals y_i have the following properties:

$$\begin{aligned} \|y_1\|_T^2 &\leq \gamma_1\,\langle u+y_0,y_1\rangle_T \\ \|y_2\|_T^2 &\leq \gamma_2\,\langle y_1,y_2\rangle_T \\ &\vdots \\ \|y_n\|_T^2 &\leq \gamma_n\,\langle y_{n-1},y_n\rangle_T \end{aligned}$$

for every $T>0$, where we are writing $y_0=-y_n$. We expand $\langle u+y_0,y_1\rangle_T = \langle u,y_1\rangle_T+\langle y_0,y_1\rangle_T$, and use the Cauchy-Schwartz inequality for the first term, upper-bounding it by $\|u\|_T\,\|y_1\|_T$. Replacing now each $\langle y_{i-1},y_i\rangle_T$ by $\|y_{i-1}\|_T\,\|y_i\|_T\cos\theta_T(y_{i-1},y_i)$ and dividing by $\|y_i\|_T$ (assumed nonzero; otherwise, there will be nothing to prove), we have these estimates:

$$\begin{aligned} \|y_1\|_T &\leq \gamma_1\,\|y_0\|_T\cos\theta_T(y_0,y_1)\,+\,\gamma_1\,\|u\|_T \\ \|y_2\|_T &\leq \gamma_2\,\|y_1\|_T\cos\theta_T(y_1,y_2) \\ &\vdots \\ \|y_n\|_T &\leq \gamma_n\,\|y_{n-1}\|_T\cos\theta_T(y_{n-1},y_n) \end{aligned}$$

from which we conclude, by recursively substituting the estimates starting from the last one backward towards the first, that:

$$\|y_n\|_T \;\leq\; \kappa\,\|y_n\|_T \,+\, \alpha\,\|u\|_T$$

where

$$\alpha \;=\; \gamma_1\gamma_2\ldots\gamma_n\cos\theta_T(y_1,y_2)\ldots.\cos\theta_T(y_{n-1},y_n)$$

and

$$\kappa \;=\; \alpha\,\cos\theta_T(y_0,y_1)\,.$$

It is enough to show that $\kappa<1$, since then we can write $(1-\kappa)\,\|y_n\|_T \;\leq\; \alpha\,\|u\|_T$, and therefore the result holds with $c=\alpha/(1-\kappa)$. Let us fix T and write $\theta_i:=\theta_T(y_{i-1},y_i)$ for $i=1,\ldots,n$. We must show, then, that

$$\cos\theta_1\ldots\cos\theta_n \leq \left(\cos\frac{\pi}{n}\right)^n\,. \tag{5}$$

The angles θ_i all lie in $[0,\pi/2]$, for each $i=2,\ldots,n$, since each system is OSP; thus $\cos\theta_i\geq 0$ for all such i. However, it is possible that $\cos\theta_1<0$, since all that is known is that

$\langle u+y_0, y_1\rangle_T \geq 0$, not that $\langle y_0, y_1\rangle_T \geq 0$. But if $\cos\theta_1 < 0$, then (5) is true because the left-hand side is ≤ 0 and the right-hand side is positive. So, in order to prove (5), we may assume from now on that all $\theta_i \in [0, \pi/2]$.

We prove, more generally, this fact about Hilbert spaces: suppose given vectors $v_0, v_1, \ldots, v_n$ such that $\langle v_i, v_{i+1}\rangle \geq 0$, and $v_0 = -v_n$. Let $\theta_i \in [0, \pi/2]$ be the angle between v_{i-1} and v_i. Then (5) holds. Intuitively, the property that the start and end vector are at angle π means that the consecutive vectors cannot be too close in angle, and therefore at least some of the angles must be large, and hence have small cosine, and the largest possible value is achieved when all angles are the same.

To prove this general fact, without loss of generality, we may assume that all the v_i are unit vectors (since only angles matter). Notice that $\sum_i \theta_i \geq \pi$. This is because, for any three unit vectors, $\theta(u,v)+\theta(v,w) \geq \theta(u,w)$, since we can view the angle as the geodesic distance in a sphere, and apply the triangle inequality; inductively applied starting from v_0, we get that $\sum_i \theta_i \geq \theta(v_0, v_n) = \pi$. Now, we have also this algebraic fact:

$$\cos\theta_1 \ldots \cos\theta_n \leq \left(\cos\frac{\theta_1+\ldots+\theta_n}{n}\right)^n$$

which follows by noticing that the function $f(x) = -\ln\cos x$ is convex for $x \in [0, \pi/2)$, applying Jensen's inequality to obtain $f(\sum_i \theta_i/n) \leq (1/n)\sum_i f(\theta_i)$, and taking exponentials. Together with $\sum_i \theta_i \geq \pi$, using that $\pi/n \leq (\theta_1+\ldots+\theta_n)/n \leq \pi/2 < \pi$ (recall that each $\theta_i \in [0, \pi/2]$), and using that cos decreases on $[0, \pi]$, we conclude:

$$\left(\cos\frac{\theta_1+\ldots+\theta_n}{n}\right)^n \leq \left(\cos\frac{\pi}{n}\right)^n .$$

This completes the proof of the Theorem.

To prove the Corollary, we provide a standard argument, as done e.g. in [9], Theorem 33. Pick any initial state x_0 and consider the solution $x(\cdot)$ of the closed-loop system $\dot{x} = f(x, u-h(x))$ with input 0 and $x(0) = x_0$. Zero-reachability means that there is some finite-time input $u_0 : [0,T] \to \mathbb{R}$ such that, if $z_0(\cdot)$ solves the closed-loop equations $\dot{z} = f(z, u-h(z))$ with initial state $z_0(0) = 0$ and this input u_0 on the interval $[0,T]$, then $z_0(T) = x_0$. Consider now the input u obtained by the formula $u(t) = u_0(t)$ for $t \leq T$ and $u(t) \equiv 0$ for $t > T$, and let $z(\cdot)$ be the solution with initial state $z(0) = 0$ and this input u; by causality, $z(t) = z_0(t)$ for $t \leq T$, and hence $z(T) = x_0 = x(0)$, from which it follows that $z(t+T) = x(t)$ for all $t \geq 0$. Showing $x(t) \to 0$ as $t \to \infty$ is the same as showing $z(t) \to 0$ as $t \to \infty$. Let y_i be the outputs of the subsystems when using input u (and zero initial state). Since $u \in L^2$ and $\|y\| \leq c\,\|u\| < \infty$, we have that $y_i \in L^2$ for each of the intermediate outputs. Since each subsystem is input to state L^2-stable, meaning that L^2 inputs (and zero initial state) produces L^2 state trajectories, we have that the complete state z is in L^2. Finally, as z is a trajectory of a semiflow in finite dimensions, we must have that $z(t) \to 0$, by a Barbălat's Lemma type of argument (see e.g. [3]).

Finally, we review in the present context a weaker version that applies when $n = 2$, basically part of the statement of the classical Passivity Theorem. Suppose that the first system is OSP but the second system is only known to be passive, in the sense that no estimate $\|y_2\|_T^2 \leq \gamma_2\langle y_1, y_2\rangle_T$ may hold, but we do know that $\langle y_1, y_2\rangle_T \geq 0$ for all $T > 0$. Then, $y_0 = -y_2$ implies that:

$$\begin{aligned}\|y_1\|_T^2 &\leq \gamma_1\langle u+y_0, y_1\rangle_T \\ &= \gamma_1\langle u, y_1\rangle_T - \gamma_1\langle y_2, y_1\rangle_T \;\leq\; \gamma_1\langle u, y_1\rangle_T\end{aligned}$$

and so the system *with output* y_1 is OSP, and in particular, L^2 stable. If, in addition, the second system is also L^2 stable, then stability to y_2 holds as well.

V. Linear Systems

The condition that a system be OSP is of course a restrictive one, but the concept of OSP system is thoroughly well-studied, and examples of passive systems abound, especially, but not only, for linear systems. We collect here some facts, mostly well-known, regarding the linear case.

For a stable linear system with transfer function $G(s)$, the secant gain can be characterized as the smallest γ such that

$$|G(i\omega)|^2 \;\leq\; \gamma\,\mathrm{Re}\,G(i\omega) \quad \forall\,\omega \in \mathbb{R}\,. \tag{6}$$

A proof is as follows. First of all, squaring the expression below and expanding $\langle y-(\gamma/2), y-(\gamma/2)\rangle_T$, one easily sees that the definition of OSP system is equivalent to the requirement that

$$\|y-(\gamma/2)u\|_T \;\leq\; (\gamma/2)\,\|u\|_T \tag{7}$$

for all i/o pairs and all T, which means γ_s is the smallest number such that the L^2-induced norm of $u \mapsto y-(\gamma/2)u$ is $\leq \gamma/2$. For linear systems, induced L^2-induced norm corresponds to H_∞ gain, that is to say, γ_s is the smallest number so that $\sup_{\omega\in\mathbb{R}} |G(i\omega)-(\gamma/2)| \leq \gamma/2$. Writing $|G(i\omega)-(\gamma/2)|^2 = (G(i\omega)-(\gamma/2))(\overline{G(i\omega)}-(\gamma/2))$ and expanding, one has (6).

An equivalent formulation of (6) is via the following analog of the estimate (2):

$$|G(i\omega)| \;\leq\; \gamma\,\cos\theta(G(i\omega)) \quad \forall\,\omega \in \mathbb{R}\,. \tag{8}$$

where we are denoting now by $\theta(\mu)$ the argument of a complex number μ. Since G is analytic on $\mathrm{Re}\,\lambda \geq 0$ (stability), the maximum modulus principle for analytic functions implies that same estimate is obtained when maximizing not merely over $\lambda = i\omega$ purely imaginary, but also over all complex numbers with nonnegative real part.

If we write $G(s) = p(s)/q(s)$ as a quotient of two polynomials, condition (6) can be also written as

$$|p(i\omega)|^2 \;\leq\; \gamma\,\mathrm{Re}\,[p(i\omega)\overline{q(i\omega)}]\,.$$

For example, for a one-dimensional system $\dot{x} = -\alpha x + \beta u$ with output $y = x$, the transfer function is $\beta/(s+\alpha)$, so that $p(i\omega) = \beta$ and $\mathrm{Re}\,[p(i\omega)\overline{q(i\omega)}] = \alpha\beta$ for any ω, from which it follows that $\gamma_s = \beta/\alpha$, and the classical result is obtained. On the other hand, as is well-known for OSP systems, $G(s)$ must have relative degree at most one (the condition $\operatorname{Re} G(i\omega) \geq 0$ is otherwise violated). Therefore, cascades, as studied here, of two or more such one-dimensional systems are not OSP themselves.

For linear systems, a sufficient condition for a system to be OSP is that its transfer function $G(s)$ be strictly positive real (SPR), meaning that $G(s-\varepsilon)$ is positive real for some $\varepsilon > 0$, or equivalently (see e.g. [5], Lemma 10.1) that it be stable (all poles have negative real part) and satisfy $\operatorname{Re} G(i\omega) > 0$ for all $\omega \in \mathbb{R}$ and $\lim_{\omega\to\infty} \omega^2 \operatorname{Re} G(i\omega) > 0$. (Note that our transfer functions are strictly proper, by definition, since we are considering state-space systems with no direct i/o term; for non-strictly proper transfer functions, the condition is slightly different.) This provides a large class of examples; for instance, any transfer function of the form $(s+\alpha)/(s^2 + as + b)$ with $b > 0$ and $0 < a < 2\sqrt{b}$ is SPR if and only if $0 < \alpha < a$ ([5], Exercise 10.1). That SPR implies OSP can be proved using the Kalman-Yakubovich-Popov (KYP) Lemma. The converse implication does not hold: $s/(s^2 + s + 1)$ is not SPR, since it fails the test just quoted with ($a = b = 1$, $\alpha = 1/2$) or just by noting that there is an imaginary axis zero, since $\operatorname{Re} G(0) = 0$, but it is OSP, since $|p(i\omega)| / \operatorname{Re}\left[p(i\omega)\overline{q(i\omega)}\right] \equiv 1 < \infty$.

More generally, for not necessarily linear systems, if there exists some nonnegative definite smooth function V on states with the property that, for some $\gamma > 0$,

$$\nabla V(x).f(x,u) \;\leq\; -y^2 + \gamma\, uy$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y = h(x)$, then the system is OSP. Indeed, integrating along solutions corresponding to $x(0) = 0$, and using that V is nonnegative definite (so that $V(0) = 0$ and $V(x(T)) \geq 0$), one has that

$$\begin{aligned} 0 \;&\leq\; V(x(T)) - V(0) \\ &\leq\; -\int_0^T y(s)^2\, ds + \gamma \int_0^T u(s)y(s)\, ds \end{aligned}$$

and thus $\|y\|_T^2 \leq \gamma \langle u, y\rangle_T$ as claimed. This property can be checked by means of nonlinear versions of the KYP Lemma, see e.g. [5], [14].

Yet another way of stating the estimate (1) is in terms of integral quadratic constraints (IQC's), cf. [7]: one may equivalently write "$w^T M w \geq 0$" in L^2 for i/o pairs $w = (u, y)'$ and where:

$$M \;=\; \begin{pmatrix} 0 & \gamma/2 \\ \gamma/2 & -1 \end{pmatrix}$$

The powerful tools for analysis of IQC's, based on LMI's, as developed by Megretski and Rantzer and others, should thus be useful for the study of secant gains. (We wish to thank R. Sepulchre for suggesting this reformulation.)

We pointed out that the induced L^2 gain γ_∞ is upper bounded by the secant gain γ_s. In general, one has the strict inequality $\gamma_\infty < \gamma_s$. For example consider the linear system with transfer function

$$G(s) \;=\; \frac{2s+1}{s^2+s+1}\,.$$

This is a scalar multiple of $(s + 1/2)/(s^2 + s + 1)$, so it is SPR by the criterion mentioned earlier, and hence OSP. Explicitly:

$$\gamma_s \;=\; \sup_{\omega\in\mathbb{R}} \frac{|p(i\omega)|^2}{\operatorname{Re}\left[p(i\omega)\overline{q(i\omega)}\right]} \;=\; \sup_{\omega\in\mathbb{R}} \frac{1+4\omega^2}{1+\omega^2} \;=\; 4$$

and

$$\begin{aligned} \gamma_\infty \;&=\; \sup_{\omega\in\mathbb{R}} \left|\frac{1+2i\omega}{1-i\omega-\omega^2}\right| \;=\; \sup_{\omega\in\mathbb{R}} \sqrt{\frac{1+4\omega^2}{1-\omega^2+\omega^4}} \\ &=\; \sqrt{2+(2/3)\sqrt{21}} \;\approx\; 2.25 \;<\; 4 \end{aligned}$$

(the maximum value is achieved at $\omega = 1/2\,\sqrt{\sqrt{21}-1}$).

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