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PID Stabilization of a Class of Time Delay Systems

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***Abstract*— In this paper we use the Hermite-Biehler theorem to establish results for the design of proportional plus integral plus derivative (PID) controllers concerning a class of time delay systems. Using the property of interlacing at high frequencies of the class of systems considered and linear programming we obtain the set of all stabilizing PID controllers.**

I. INTRODUCTION

The dynamic behavior of many industrial plants may be mathematically described by linear time invariant systems with time delays. The problem of stability of linear time invariant systems with time delays involves finding the location of roots of transcendental functions. An extension of the Hermite-Biehler theorem to cope with transcendental functions was first derived by Pontryagin [1]. In [1], necessary and sufficient conditions for the negativity of the real parts of all zeros of transcendental functions of the form $h(z, e^z)$, where $h(z, t)$ is a polynomial in two variables, are given. The functions $h(z, e^z)$ are usually called exponential polynomials or quasi-polynomials.

Recently, new results on the synthesis of PID controllers for a first-order and a class of arbitrary order plants with time delay using the extension of the Hermite-Biehler theorem derived by Pontryagin [1] were given in [2] and [3], respectively. In this paper, we obtain the set of stabilizing PID controllers also for an arbitrary order plant with time delay using the results on the proportional case derived in a previous paper [4]. In both [2] and [3] transcendental equations need to be solved. However, in the proposed method only linear inequalities need to be solved. The method is based on a signature assigned to the quasi-polynomial involved in the stabilizing problem using the fact that this quasi-polynomial in the class considered has a finite set of right half plane zeros.

II. PRELIMINARY RESULTS ON TIME DELAY SYSTEMS

Linear time-invariant systems with delays can be mathematically described by homogeneous, linear, difference-differential equations with constant coefficients as follows [5]

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}u^{(j)}(t-\tau_i) = 0. \tag{1}$$

where $a_{ij},\ i = 0, \cdots m,\ j = 0, \cdots, n$ are real numbers, m and n are positive integers, and $\tau_0, \tau_1, \cdots, \tau_m$ are real numbers satisfying $0 = \tau_0 < \tau_1 < \cdots < \tau_m$ called delays. If $a_{0n} \neq 0$ while the other $a_{in} = 0$, then (1) is an equation of retarded type. If $a_{0n} \neq 0$ and, if only for one $i > 0,\ a_{in} \neq 0$, then (1) is an equation of neutral type. If $a_{0n} = 0$, but, if only for one $i > 0,\ a_{in} \neq 0$, then (1) is an equation of advanced type.

The characteristic function associated to (1) is given by

$$F(s) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}s^j e^{-s\tau_i}. \tag{2}$$

Multiplying (2) by $e^{s\tau_m}$, we have

$$H(s) = e^{s\tau_m}F(s) = \sum_{i=0}^{m} p_i(s)e^{s(\tau_m-\tau_i)} \tag{3}$$

with

$$p_i(s) = \sum_{j=0}^{n} a_{ij}s^j, \quad i = 0, \cdots, m. \tag{4}$$

For $m \neq 0$, the function (3) belongs to a general class of quasi-polynomials [6]. In particular, if the delays are integer multiples of one another, the quasi-polynomial (3) can be written in the form studied by Pontryagin [1]

$$h(s,t) = s^n\mathcal{X}^{*(n)}(t) + \sum_{i=0}^{m}\sum_{j=0}^{n-1} a_{ij}t^i s^j \tag{5}$$

with $\mathcal{X}^{*(n)}(t) = \sum_{i=0}^{m} a_{in}t^i = a_{mn}t^m + a_{m-1\ n}t^{m-1} + \cdots + a_{1n}t + a_{0n}$. The term $a_{mn}t^m s^n$ is denoted the principal term if $a_{mn} \neq 0$.

The stability of the systems described in (1) depends on the location of the roots of (2). It is evident that H and F have the same zeros. Therefore, we now analyze the roots of the quasi-polynomial (3) which has infinite roots. From the fact that the quasi-polynomial (3) is an entire function, we have that, in any bounded region of the complex plane there is only a finite number of roots. The roots of (3) with $|s|$ sufficiently large can be assigned to a finite number of asymptotic chains. The quasi-polynomials corresponding to the retarded type equation contain asymptotic chain of roots which go "deep" into the left-half complex plane, while the one corresponding to the neutral type equation in addition to such chains of roots it also has at least one asymptotic chain of roots in a vertical strip of the complex plane. Finally, the quasi-polynomials corresponding to the advanced type equation contain at least one asymptotic chain of roots that goes “deep” into the right-half complex plane [7].

The quasi-polynomial (3) is said to be stable if and only if there exists a positive number ϵ such that the real part of all zeros of H are less than $-\epsilon$. It is worth mentioning that only quasi-polynomials corresponding to the retarded or neutral type equation may be stable and that stability of the former is equivalent to the negativity of the real parts of all zeros of H. For easy reference, let us now state the following results by [1].

Let $f(z,u,v)$ be a polynomial in z, u and v written in the form

$$f(z,u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m} z^i\phi_i^{(j)}(u,v) \tag{6}$$

where $\phi_i^{(j)}(u,v)$ is a polynomial of degree j, homogeneous in u e v, that is, the sum of exponents in u and v is j. The principal term in the polynomial $f(z,u,v)$ is the term $z^i\phi_i^{(j)}(u,v)$ for which i and j simultaneously attain maximum values n and m, respectively. Let $\phi^{*(m)}(u,v)$ be the coefficient of z^n in $f(z,u,v)$ such that

$$\phi^{*(m)}(u,v)=\sum_{j=0}^{m}\phi_n^{(j)}(u,v) \tag{7}$$

and define $\Phi^{*(m)}(z)=\phi^{*(m)}(\cos z,\sin z)$.

Theorem 1: Consider real transcendental functions $f_r(\omega,\cos(\omega),\sin(\omega))$ and $f_i(\omega,\cos(\omega),\sin(\omega))$ such that $H(j\omega)=f_r(\omega,\cos(\omega),\sin(\omega))+jf_i(\omega,\cos(\omega),\sin(\omega))$. Assume that $f_r(\omega,u,v)$ and $f_i(\omega,u,v)$ are polynomials with principal terms of the form $\omega^n\phi_n^{(m)}(u,v)$. Let η be an appropriate constant such that $\phi^{*(m)}(u,v)$ in $f_r(\omega,u,v)$ and $f_i(\omega,u,v)$ do not vanish at $\omega=\eta$. Then for the equations $F_r(\omega):=f_r(\omega,cos(\omega),sin(\omega))=0$ or $F_i(\omega):=f_i(\omega,\cos(\omega),\sin(\omega))=0$ to have only real roots it is necessary and sufficient that in the interval $-2\pi\ell+\eta\le\omega\le 2\pi\ell+\eta$, $f_r(\omega,u,v)$ and $f_i(\omega,u,v)$ has exactly $4m\ell+n$ real roots starting with a sufficiently large ℓ.

Theorem 2: Let $h(z,t)$ be a polynomial of the form of (5) having principal term $a_{mn}t^m z^n$ with $a_{mn}\neq 0$ and consider $H(z)=h(z,e^z)$. If the function $\mathcal{X}^{*(n)}(e^z)$ has roots in the open right half plane, then $H(z)$ has an unbounded set of zeros in the open right half plane. If all the zeros of $\mathcal{X}^{*(n)}(e^z)$ lie in the open left half plane, then the function $H(z)$ has no more than a bounded set of zeros in the open right half plane.

Consider a special class of (2) appearing in control engineering problems

$$F(s)=D(s)+e^{-sL}N(s) \tag{8}$$

with $L>0$, $N(s)=b_m s^m+b_{m-1}s^{m-1}+\cdots+b_0$ and $D(s)=s^n+a_{n-1}s^{n-1}+\cdots+a_0$. Multiplying $F(s)$ in (8) by e^{sL}, we obtain

$$\delta(s)=\tilde{h}(s,e^{sL})=e^{sL}D(s)+N(s). \tag{9}$$

Performing the change of variables $z=sL$ in (8), we have

$$\delta(z)=\tilde{h}(\frac{z}{L},e^z)=e^zD(\frac{z}{L})+N(\frac{z}{L}). \tag{10}$$

The polynomial associated with the quasi-polynomial (10) can be written as

$$h(z,t)=tD(\frac{z}{L})+N(\frac{z}{L}). \tag{11}$$

This polynomial has the principal term $L^{-n}z^nt$ if $m<n$ or $L^{-n}z^n(t+b_n)$ if $m=n$. We make the following assumption

A1) $m<n$ or $m=n$ and $|b_n|<1$.

Under Assumption A1), note that the quasi-polynomial (9) is a quasi-polynomial of a retarded or neutral type equation. In [4] only the retarded quasi-polynomials type are considered. There are transcendental real functions $p(\omega)$ and $q(\omega)$ in the real variable ω associated with (9) such that $\delta(j\omega)=p(\omega)+jq(\omega)$, where $p(\omega)$ and $q(\omega)$ denote the real and imaginary parts of $\delta(j\omega)$. From the used terminology, we say that the zeros of the real functions $p(\omega)$ and $q(\omega)$ interlace, or alternate, along the ω-axis if each of the functions has no multiple zeros and between every two zeros of one of these functions there exists only one zero of the other and if the functions are never simultaneously equal to zero.

Lemma 1: Consider the quasi-polynomial given in (9). Let $p(\omega)$ and $q(\omega)$ be the real and the imaginary parts of $\delta(j\omega)$, respectively. Under Assumption A1), there exists $0<\omega_0<\infty$ such that in $[\omega_0,\infty)$ the functions $p(\omega)$ and $q(\omega)$ have only real roots and these roots interlace.

Proof: The proof of Lemma 1 follows from Theorem 2. As a matter of fact, under Assumption A1) the roots of $\delta(s)$ in (9) goes into the left hand plane for $|s|$ large so that interlacing occurs for ω_0 sufficiently large. ■

Remark 1: As a result of Lemma 1, under Assumption A1) the sufficiently large zeros of $p(\omega)$ and $q(\omega)$ interlace in $[\omega_0,+\infty)$, being the quasi-polynomial (9) stable or not.

III. Stabilization of a Class of Time Delay Systems

Consider a polynomial $P(s)$ and write $P(j\omega)=P_r(\omega)+jP_i(\omega)$. Let $\sigma(P):=$ number of zeros of $P(s)$ in the open left-half complex plane $-$ number of zeros of $P(s)$ in the open right-half plane zeros of $P(s)$ be the signature of $P(s)$ [8]. Let $\Delta_0^\infty\theta_P$ denote the net change in the argument $\theta_P(\omega):=\arctan[\frac{P_i(\omega)}{P_r(\omega)}]$ as ω increases from 0 to ∞. Then, we can state the following lemma by Gantmacher [9].

Lemma 2: Let $P(s)$ be a real polynomial with no imaginary axis roots. Then $\Delta_0^\infty\theta_P=\frac{\pi}{2}\sigma(P)$.

In the sequel, the key results on the design of proportional controllers for a class of time delay systems given in [4], which are completed with more detailed proofs of Lemmas 3 and 4, are reviewed.

Consider now the quasi-polynomial $\delta(s)$ as in (9) under Assumption A1). We shall analyze the roots of $\delta(j\omega)$ in the frequency range determined by ω_0. Differing from the polynomials, the quasi-polynomials have infinite roots. The results given in [8] deal with polynomials and make use of the number of roots to establish a procedure to design fixed order controllers. There, the number of roots of the polynomial is related to the real zeros of $P_r(\omega)$ and $P_i(\omega)$. For the quasi-polynomial case, let $\delta(j\omega) = p(\omega)+ jq(\omega)$, with $p(\omega)$ and $q(\omega)$ as before, and $0 = \omega_{q_0} < \omega_{q_1} < \omega_{q_2} < \cdots < \omega_{q_m}$ and $\omega_{p_1} < \omega_{p_2} < \cdots < \omega_{p_r}$ real, distinct finite zeros of $q(\omega)$ and $p(\omega)$, respectively. The following definition is crucial to select the frequency range to analyze the distribution of the roots of $\delta(s)$.

Definition 1: Let $m+1$ be the number of zeros of $q(\omega)$ in $[0, \omega_{q_m}]$ and r the number of zeros of $p(\omega)$ in $[0, \omega_{p_r}]$, with ω_{q_m} and ω_{p_r} as in Lemma 1 such that the zeros of $q(\omega)$ and $p(\omega)$ in $[\omega_{q_m}, \infty)$ and $[\omega_{p_r}, \infty)$ interlace. For $m + r$ even we define $\omega_0 = \omega_{q_m}$, otherwise we define $\omega_0 = \omega_{p_r}$.

Definition 2: Let $\delta(s)$ be a given quasi-polynomial described as in (9) with no $j\omega$ axis roots. For a sufficiently large ω_0 as in Definition 1, let $0 = \omega_{q_0} < \omega_{q_1} < \omega_{q_2} < \cdots < \omega_{q_m} \leq \omega_0$ and $\omega_{p_1} < \omega_{p_2} < \cdots < \omega_{p_r} \leq \omega_0$ be real, distinct finite zeros of $q(\omega)$ and $p(\omega)$, respectively. Then, the signature for $\delta(s)$ in $[0, \omega_0]$ which we denote as $\sigma_q(\delta)$ is given by

$$\sigma_q(\delta) = \begin{cases} \{sgn[p(\omega_{q_0})] - 2sgn[p(\omega_{q_1})] + 2sgn[p(\omega_{q_2})] + \\ \cdots + (-1)^{m-1}2sgn[p(\omega_{q_{m-1}})] \\ +(-1)^m sgn[p(\omega_{q_m})]\}.(-1)^{m-1}sgn[q(\omega^+_{q_{m-1}})] \\ \text{if } m+r \text{ is even} \\ \{sgn[p(\omega_{q_0})] - 2sgn[p(\omega_{q_1})] + 2sgn[p(\omega_{q_2})+ \\ \cdots + (-1)^m 2sgn[p(\omega_{q_m})]\}(-1)^m sgn[q(\omega^+_{q_m})] \\ \text{if } m+r \text{ is odd} \end{cases} \quad (12)$$

where sgn is the standard signum function and $sgn[q(\omega_i^+)]$ denotes the sign of $q(\omega)$ soon after the occurrence of the zero $q(\omega_i)$.

Remark 2: The signature $\sigma_q(\delta)$ as in Definition 2 is the counterpart to the signature of polynomials.

Lemma 3: Consider a stable quasi-polynomial $\delta(s)$ described as in (9) under Assumption A1). Let m and r be as already defined. Then, the signature for the quasi-polynomial $\delta(s)$ is given by $\sigma_q(\delta) = m + r$.

Proof: See [4]. ■

Let the plant to be controlled be described by

$$G(s) = \frac{N(s)e^{-sL}}{D(s)}. \quad (13)$$

For this plant, the characteristic function of the feedback system with a controller $C(s) = k_p$ is thus given by

$$F(s, k_p) = D(s) + k_p e^{-sL} N(s) \quad (14)$$

where $D(s)$ and $N(s)$ are as in (8). Assuming $L > 0$ we obtain a quasi-polynomial of the form of (9). Now, multiplying (14) by e^{sL} results the quasi-polynomial of the form

$$\delta(s, k_p) = e^{sL}D(s) + k_p N(s). \quad (15)$$

Again we consider $\delta(s, k_p)$ under Assumption A1). In the stabilization problem we construct a quasi-polynomial of the form $\delta(s, k_p)N(-s)$ for which only the real part depends on k_p as we can write

$$\delta(jw, k_p)N(-jw) = p(\omega, k_p) + jq(\omega) \quad (16)$$

where

$$p(\omega, k_p) \quad = \quad p_1(\omega) + k_p p_2(\omega)$$

Lemma 4 below gives a frequency range signature for the product $\delta(s, k_p)N(-s)$ which is used to establish Theorem 3 in the stabilization problem. For a stabilizing k_p, we can associate to m and r the number of zeros of $\delta(s, k_p)$ in the frequency range determined by frequency ω_0 using the Hermite-Biehler theorem. Thus, the product $\delta(s, k_p)N(-s)$ introduces a finite number of zeros in the frequency range considered. In fact, it can be showed that the signature of this product is $[\sigma_q(\delta) - \sigma(N)]$, with $\sigma(N)$ the signature of the polynomial $N(s)$.

Lemma 4: Let $m + 1$ and r define the number of real, distinct and finite zeros of the imaginary and real parts of $\delta(j\omega, k_p)$ in (15), respectively, for a stabilizing k_p and a sufficiently large frequency ω_0 defined as before. Then, $\delta(s, k_p)$ is stable if and only if for any stabilizing k_p the signature for $\delta(s, k_p)N(-s)$ determined by the frequency ω_0 is given by $m + r - \sigma(N)$.

Based on the results given in [8] we introduce the following definition.

Definition 3: Let $0 = \omega_{q_0} < \omega_{q_1} < \omega_{q_2} < \cdots < \omega_{q_i}$ be real, distinct and finite zeros of $q(\omega)$. Then the set of strings A_I in a frequency range determined by the frequency ω_0 is defined as $A_I = \{z_0, z_1, \cdots, z_i\}$ with $z_t \in \{-1, 1\}$ with z_t identified to $sgn[p(\omega_{q_t})]$ in Definition 2.

Theorem 3: Consider $p(\omega, k_p)$ and $q(\omega)$ as the real and imaginary parts of $\delta(j\omega, k_p)N(-j\omega)$, respectively. Let $0 = \omega_{q_0} < \omega_{q_1} < \omega_{q_2} < \cdots < \omega_{q_i}$ be real, distinct and finite zeros of $q(\omega)$ in a frequency range. Assume that $N(-j\omega_{q_t}) \neq 0, t = 0, \cdots, i$ and $N(-s)$ has no zero at the origin. Suppose there exists a stabilizing k_p and choose ω_0 associated to the quasipolynomial $\delta(s, k_p)$ as in Definition 1. Then the set of all k_p such that $\delta(s, k_p)$ for a given plant $G(s)$ under Assumption A1) is stable are obtained using the following expression for the signature of $\delta(s, k_p)N^*(s)$

$$\sigma_q(\delta(s, k_p)N(-s) =$$

$$\begin{cases} \{z_0 - 2z_1 + 2z_2 + \cdots + \\ (-1)^{i-1}2z_{i-1} + (-1)^i z_i\}.(-1)^{i-1}sgn[q(\omega^+_{q_{i-1}})] \\ \text{if } m+r+m' \text{ is even} \\ \{z_0 - 2z_1 + 2z_2 + \cdots + (-1)^i 2z_i\}.(-1)^i sgn[q(\omega^+_{q_i})] \\ \text{if } m+r+m' \text{ is odd} \end{cases}$$

and

$$k_p = \cup k_{p_\ell}$$

where

$$k_{p_\ell} = (\max_{z_t \in A_I, z_t > 0}[-\frac{1}{G(j\omega_{q_t})}], \min_{z_t \in A_I, z_t < 0}[-\frac{1}{G(j\omega_{q_t})}])$$

with $\{z_0, z_1, \cdots, z_i\} \in A_I$ such that

$$\max_{z_t \in A_I, z_t > 0} [-\frac{1}{G(j\omega_{q_t})}] < \min_{z_t \in A_I, z_t < 0} [-\frac{1}{G(j\omega_{q_t})}])$$

and ℓ the number of feasible strings, $\delta(j\omega, k_p)N(-j\omega) = p_1(\omega) + k_p p_2(\omega) + jq(\omega)$, m' the degree of $N(s)$ and $\sigma_q(\delta(s, k_p)N(-s))$ given by $m + r - \sigma(N)$.

Proof: Considering the frequency range determined by ω_0 and Lemma 4, the proof follows the same lines as for the polynomial case given in [8]. ■

We now present an example to illustrate the application of Theorem 3.

Example 1: Consider the stabilization of a given time delay system using a proportional controller. The system is a non-minimum phase, fifth order plus a single time delay in the input

$$G(s) = \frac{e^{-0.1s}(s^4 + 4s^3 + 23s^2 + 46s - 12)}{s^5 + 2s^4 + 23s^3 + 44s^2 + 97s + 98}.$$

In this case we have $N(s) = s^4 + 4s^3 + 23s^2 + 46s - 12$, $D(s) = s^5+2s^4+23s^3+44s^2+97s+98$ and $L = 0.1$. We use the Nyquist criterion and choose a stabilizing $k_p = 6$. Now, we choose $m = 4$ and $r = 4$ yielding $\sigma_q(\delta) = 8$ with $\omega_0 = \omega_{q_4} = 47.65$. We obtain $m + r + \deg(N(s)) = 12$, which is even. Finally, we must have $\sigma_q(\delta(s, k_p)N(-s)) = m + r - \sigma(N) = 6$, where $\sigma(N) = 2$. Writing $\delta(j\omega, k_p)N(-j\omega) = p_1(\omega) + k_p p_2(w) + jq(\omega)$ we find the zeros of $q(\omega)$ of $\delta(j\omega, k_p)N(-j\omega)$ as 0, 1.972, 4.126, 5.044, 14.199, 46.693. Figure 1 presents the plots $p(\omega)$ and $q(\omega)$ of $\delta(s, k_p)$ for $k_p = 6$ and the plots of $q(\omega)$ of $\delta(s, k_p)N(-s)$. To find the zeros of $q(\omega)$ one can use the function "*fzero*" of the Matlab. We now find the strings

$$\begin{aligned} A_I &= \{\{-1, 1, -1, 1, 1, 1\}, \{-1, 1, -1, -1, -1, 1\}, \\ &\quad \{-1, -1, -1, 1, -1, 1\}, \{-1, 1, 1, 1, -1, 1\}\} \end{aligned}$$

which satisfies $-z_0 + 2z_1 - 2z_2 + 2z_3 - 2z_4 + z_5 = 6$ and Theorem 3. Hence, we have the set of stabilizing gains as $k_p \in [-0.589, -0.508] \cup [2.967, 8.166]$.

IV. Stabilization using a PID controller

Initially, we consider a PI controller of the form $C(s) = k_p + \frac{k_i}{s}$. For the plant $G(s)$ in (13), the feedback system characteristic function takes the form

$$F(s, k_p, k_i) = sD(s) + (k_i + k_p s)e^{-s}N(s) \quad (17)$$

where $D(s)$ and $N(s)$ are as already defined. Again, assuming $L > 0$, after multiplying (17) by e^{sL}, we obtain a quasi-polynomial of the form

$$\delta(s, k_p, k_i) = e^{sL}sD(s) + (k_i + k_p s)N(s). \quad (18)$$

The problem of stabilization with a PI controller involves the determination of the set of (k_p, k_i) for which the quasi-polynomial $\delta(s, k_p, k_i)$ is stable. Following the proportional case, a new quasi-polynomial of the type $\delta(s, k_p, k_i)N(-s)$ is constructed such that its real part depends only on k_i and

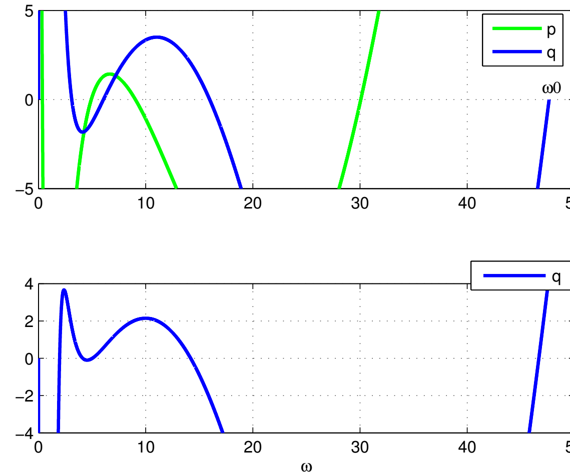

Fig. 1. Plots of $p(\omega)$ and $q(\omega)$ for $\delta(s, k_p)$(upper) and plot of $q(\omega)$ for $\delta(s, k_p)N(-s)$ (lower) with $k_p = 6$.

the imaginary part only on k_p. Substituting $s = j\omega$ in (18) we obtain

$$\delta(j\omega, k_p, k_i)N(-j\omega) = p(\omega, k_i) + jq(\omega, k_p) \quad (19)$$

where

$$\begin{aligned} p(\omega, k_i) &= p_1(\omega) + k_i p_2(\omega) \quad (20) \\ q(\omega, k_p) &= q_1(\omega) + k_p q_2(\omega) \quad (21) \end{aligned}$$

For every fixed k_p, the zeros of $q(\omega, k_p)$ do not depend on k_i and the results for the proportional case can be used to find k_i. Thus, the set of all stabilizing (k_p, k_i) for the system can be obtained by sweeping over all real k_p and solving the proportional case to find the corresponding range of k_i.

The search for the range of k_i for a fixed stabilizing k_p can be reduced by finding the real breakaway points on the root loci of $q_1(\omega) + k_p q_2(\omega) = 0$. The breakaway points correspond to a real multiple root and satisfy $\frac{dk_p}{d\omega} = 0$ for $q(\omega, k_p) = 0$. Furthermore, note that a necessary condition for the existence of a stabilizing k_i is that $q(\omega, k_p)$ must have at least $\frac{m+r-\sigma(N(s))}{2}$ or $\frac{m+r-\sigma(N(s))+1}{2}$ real, non-negative, distinct finite zeros of odd multiplicities, accordingly as $m' + m + r$ is even or odd, respectively. This condition eliminates the need to sweep over all the ranges of stabilizing k_p.

Example 2: As in [10], consider the problem of finding stabilizing PI gains for the plant

$$G(s) = \frac{e^{-s}}{4s + 1} \quad (22)$$

where $N(s) = 1$, $D(s) = 4s + 1$, $L = 1$ and the controller is given by $C(s) = k_p + \frac{k_i}{s}$. To obtain the signature for the corresponding quasi-polynomial we first encounter a value for k_p and k_i which stabilizes the feedback system. To find them, one could use the Nyquist criterion. We use the stabilizing set of (k_p, k_i) given in [10] and choose $k_p = 3$ and $k_i = 1$. We now plot $p(\omega, k_i)$ and $q(\omega, k_p)$ to obtain, for $m = 4$ and $r = 4$, $\omega_{e_1} = 0.5$, $\omega_{e_2} = 1.6$, $\omega_{e_3} = 4.8$,

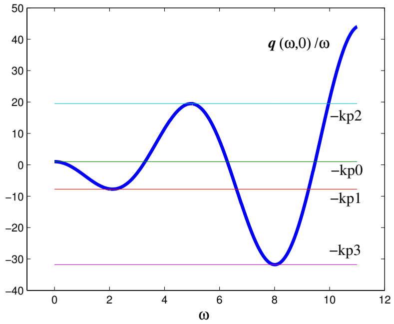

Fig. 2. Distribution of the real zeros of $q(\omega, k_p)$.

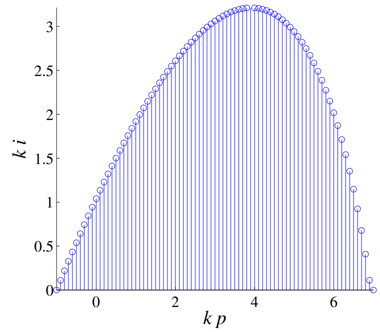

Fig. 3. Stabilizing region for k_p and k_i.

$\omega_{e_4} = 7.8$, $\omega_{o_0} = 0$, $\omega_{o_1} = 1$, $\omega_{o_2} = 3$, $\omega_{o_3} = 6.4$, $\omega_{o_4} = 9.3$. This yields $\sigma_q(\delta) = 8$ and $\omega_0 = \omega_{o_4} = 9.3$. To find the real breakaway points on the root loci we write

$$q(\omega, k_p) = -4\omega^2 \sin(\omega) + \omega\cos(\omega) + k_p\omega \quad (23)$$

and

$$\frac{q_1(\omega)}{q_2(\omega)} = -4\omega\sin(\omega) + \cos(\omega) = -k_p. \quad (24)$$

We find

$$\frac{dk_p}{d\omega} = 5\sin(\omega) + 4\omega\cos(\omega). \quad (25)$$

The positive zeros of $\frac{dk_p}{d\omega}$ are 0, 2.1064, 4.9593, 8.0088. From (24) the corresponding values for k_p are found as $k_{p_0} = -1.0000$, $k_{p_1} = 7.7560$, $k_{p_2} = -19.4800$, $k_{p_3} = 31.8062$. Now, from the plots of $\frac{q_1(\omega)}{q_2(\omega)}$, $-k_{p_0}$, $-k_{p_1}$, $-k_{p_2}$, $-k_{p_3}$ we can find the distribution of the non-negative real zeros of $q(\omega, k_p)$ in the interval $[0, \omega_0]$, except for the zero at the origin. In the example we have to look for $\frac{m+r}{2} = 4$ real non-negative distinct zeros of $q(\omega, k_p)$. The plots are shown in Figure 2. Therefore, the acceptable values for k_p are in between $k_{p_0} = -1$ and $k_{p_1} = 7.756$ where there are at least 4 real non-negative distinct zeros of $q(\omega, k_p)$. Sweeping over $k_p \in (-1, 7.756)$, the stabilizing range $-1 < k_p < 6.93$ is found, which is in agreement with the results found in [10] using another method. The stabilizing region in the $k_p k_i$ $-$plane is shown in Figure IV.

The result in Example 2 can be checked using the Nyquist criterion via the Nyquist plot of $G(s)$. However, we give in this note an analytical characterization of all the stabilizing PI gains and this is useful to the design of optimal solutions considering various performance criteria such as the H_2 and H_∞ norms of certain closed loop transfer functions. In what follows, we yield an algorithm to search for the stabilizing (k_p, k_i) values by sweeping over a range for k_p.

Algorithm 1 (PI controller):
Step 1) Adopt a value for the pair (k_p, k_i) to stabilize the given plant $G(s)$. Select m and r and choose ω_0 as in Definition 1.
Step 2) Enter functions for $p_1(\omega)$ and $p_2(\omega)$ as in (20).
Step 3) In the frequency range determined by ω_0, find the zeros of $q(\omega, k_p)$ defined by (19) denoted as $\omega_{o_j}, j = 0, 1, \cdots$ for $k_p(1)$ using $\frac{q_1(\omega)}{q_2(\omega)} = -k_p(1)$.
Step 4) Initialize $n = 2$.
Step 5) Obtain $\min k_i(n-1)$ and $\max k_i(n-1)$ using Theorem 3 for $p_1(\omega)$, $p_2(\omega)$ in Step 2. If $\max k_i(n-1) < \min k_i(n-1)$ make
$\max k_i(n-1) = 0$ and $\max k_i(n-1) = 0$.
Step 6) If $n < size(k_p(n)) + 1$ find $\omega_{o_j}, j = 0, 1, \cdots, i$ for $k_p(n)$ using $\frac{q_1(\omega)}{q_2(\omega)} = -k_p(n)$.
Step 7) Make $n = n + 1$.
Step 8) If $n < size(k_p(n) + 1$ go to Step 5. End.

The development of PID controllers follows the same lines as the PI design with the new characterization of a signature for $\delta(s)$ as in (12). For a PID controller we have

$$C(s) = k_p + \frac{k_i}{s} + k_d s. \quad (26)$$

The corresponding characteristic function is

$$F(s, k_p, k_i, k_d) = sD(s) + (k_i + k_p s + k_d s^2)e^{-sL}N(s) \quad (27)$$

and, as before we obtain

$$\delta(s, k_p, k_i, k_d) = se^{sL}D(s) + (k_i + k_p s + k_d s^2)N(s). \quad (28)$$

Now, we consider the same approach used for the PI stabilization problem. Substituting $s = j\omega$ in (28) we obtain

$$\delta(j\omega, k_p, k_i, k_p)N(-j\omega) = p(\omega, k_i, k_d) + jq(\omega, k_p) \quad (29)$$

where

$$p(\omega, k_i, k_d) = p_1(\omega) + (k_i - k_d\omega^2)p_2(\omega) \quad (30)$$
$$q(\omega, k_p) = q_1(\omega) + k_p\ q_2(\omega) \quad (31)$$

For every fixed k_p, note that $q(\omega, k_p)$ does not depend on k_i and k_d. Because of that, we can use the constant gain stabilization approach to obtain the stabilizing k_i and k_d by solving a linear programming problem for each k_p.

Lemma 5: Let $m + 1$ and r define the number of real, distinct and finite zeros of the imaginary and real parts of $\delta(j\omega, k_p, k_i, k_d)$ in (28), respectively, for a stabilizing value for (k_p, k_i, k_d) and a sufficiently large frequency ω_0 defined as before. Then, $\delta(s, k_p, k_i, k_d)$ is stable if and only if for any stabilizing set (k_p, k_i, k_d) the signature for

$\delta(s, k_p, k_i, k_d)N(-s)$ determined by the frequency ω_0 is given by $m + r - \sigma(N)$.

Theorem 4: Consider $p(\omega, k_i, k_d)$ and $q(\omega, k_p)$ as the real and imaginary parts of the quasipolynomial $\delta(j\omega, k_p, k_i, k_d)N(-j\omega)$, respectively. Suppose there exists a stabilizing set (k_p, k_i, k_d) for a given plant $G(s)$ satisfying Assumption A1). Choose ω_0 associated to $\delta(s, k_p, k_i, k_d)$ as in Definition 1. For a fixed k_p, let $0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_i$ be real, distinct and finite zeros of $q(\omega, k_p)$ in a frequency range. Then, the (k_i, k_d) values such that $\delta(s, k_p, k_i, k_d)$ is stable are obtained by solving the following linear programming problem for $z_t \in A_I$ such that the signature for $\delta(s, k_p, k_i, k_d)N(-s)$ equals $m + r - \sigma(N)$) and $t = 0, 1, ..., i$

$$p_1(\omega_t) + (k_i - k_d\omega_t^2)p_2(\omega_t) \quad > \quad 0, \text{ for } z_t = 1 \quad (32)$$
$$p_1(\omega_t) + (k_i - k_d\omega_t^2)p_2(\omega_t) \quad < \quad 0, \text{ for } z_t = -1 \quad (33)$$

Algorithm 2 (PID controller):

Step 1) Adopt a value for the set k_p, k_i, k_d to stabilize the given plant $G(s)$. Select m and r and choose ω_0 as in Definition 1.

Step 2) Enter functions for $p_1(\omega)$ and $p_2(\omega)$ as in (30).

Step 3) In the frequency range determined by ω_0 find the zeros of $q(\omega, k_p)$ defined by (29) for a fixed k_p.

Step 4) Using Definition 2 for $\delta(s, k_p, k_i, k_d)N(-s)$, find the strings A_I that satisfy $\sigma_q(\delta(s, k_p, k_i, k_d)N(-s)) = m + r - \sigma(N))$.

Step 5) Apply Theorem 4 to obtain the inequalities (32) and (33).

Example 3: As in [3], consider the problem of finding the set of PID gains to stabilize the plant $G(s) = \frac{e^{-sL}N(s)}{D(s)}$ with $N(s) = s^3 - 4s^2 + s + 2$, $D(s) = s^8 + 8s^4 + 32s^3 + 46s62 + 46s + 17$ and $L = 1$. For $k_p = 1$, $k_i = 1$ and $k_d = 0$ we plot

$$qw = inline('3.*w.^3 + 3*w - (w.^6 - 32*w.^4 + 46*w.^2)$$
$$.*sin(w) + (8*w.^5 - 46*w.^3 + 17*w).*cos(w)','w')$$

to choose $m = 8$ and $r = 8$ to obtain $\sigma_q(\delta) = 16$ and $w_0 = 16.2095$. As $m + 4 - deg(N(s)) = 17$, which is odd, the strings in A_I must satisfy $z_0 - 2z_1 + 2z_2 - 2z_3 + 2z_4 - 2z_5 + 2z_6 - 2z_7 + 2z_8 = 17$. The imaginary part of $\delta(j\omega, k_p, k_i, k_d)N(-j\omega)$ denoted $q(w, k_p)$ is affine in k_p and thus for a fixed k_p its zeros can be obtained with the MATLAB function inline

$$inline(['w.*(-w.^8 + 65*w.^6 - 246*w.^4 + 22*w.^2 + 34)$$
$$.*cos(w) + (-12*w.^8 + 180*w.^6 - 149*w.^4 - 75*w.^2)$$
$$.*sin(w) + kp*(w.^7 + 14*w.^5 + 17*w.^3 + 4*w)'],'w','kp')$$

Thus, the zeros of $q(\omega)$ of $\delta(j\omega, k_p)N(-j\omega)$ are found as 0, 0.5377, 1.1764, 2.5880, 4.3155, 6.6001, 9.1734, 12.0136, 14.9422. From (30) we obtain $p_1(w)$ and $p_2(w)$ as follows

$$p_1 = inline(['-w.*(-w.^8 + 65*w.^6 - 246*w.^4 + 22*w.^2 + 34).*sin(w) + (-12*w.^8 + 180*w.^6 - 149*w.^4 - 75*w.^2).*cos(w)'],'w')$$
$$p_2 = inline('w.^6 + 14*w.^4 + 17*w.^2 + 4','w').$$

Finally, the inequalities affine in k_i and k_d obtained using Theorem 4 are found as $k_i > 0$ and

$$k_i - 0.289135k_d < 3.11984; k_i - 1.38400k_d > -4.83381$$
$$k_i - 6.69759k_d < 27.1568;\ k_i - 18.6231k_d > -84.5249$$
$$k_i - 43.5614k_d < 271.696;\ k_i - 84.1517k_d > -732.245$$
$$k_i - 144.328k_d < 1669.5\ ;\ k_i - 223.271k_d > -3247.82$$

and the solution is shown in Figura 4.

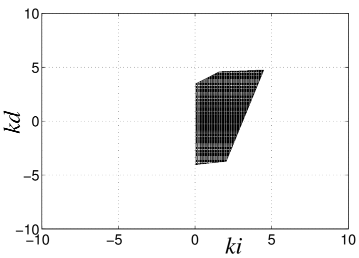

Fig. 4. Stabilizing values (k_i, k_d) for $k_p = 1$ and $L = 1$.

V. CONCLUSION

In this paper we use a extension of results of stabilization of linear time invariant systems to a class of time delay systems using the Hermite-Biehler Theorem. A signature derived for the quasi-polynomial case was used in the problem of stabilizing PID controllers. To obtain the value of the signature for a quasi-polynomial a known stabilizing value for the set (k_p, k_i, k_d) is used. The proposed approach yields the set of stabilizing PID controllers and can be applied to any time delay system which obeys the interlacing property at high frequencies, being it stable or not.

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