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Stabilization of a 3D Axially Symmetric Rigid Pendulum

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Abstract— **Models for a 3D pendulum, consisting of a rigid body that is supported at a frictionless pivot, were introduced in a recent 2004 CDC paper [1]. Control problems were posed based on these models. A subsequent paper, in the 2005 ACC [2], developed stabilizing controllers for a 3D rigid pendulum assuming three independent control inputs. In the present paper, stabilizing controllers are developed for a 3D rigid pendulum assuming that the pendulum has a single axis of symmetry that is uncontrollable. This assumption allows development of a reduced model that forms the basis for controller design and closed loop analysis; this reduced model is parameterized by the constant angular velocity component of the 3D pendulum about its axis of symmetry. Several different controllers are proposed. The first controller, based on angular velocity feedback only, asymptotically stabilizes the hanging equilibrium. Then controllers are introduced, based on angular velocity and reduced attitude feedback, that asymptotically stabilize either the hanging equilibrium or the inverted equilibrium. These problems can be viewed as stabilization of a Lagrange top. Finally, if the angular velocity about the axis of symmetry is assumed to be zero, controllers are introduced, based on angular velocity and reduced attitude feedback, that asymptotically stabilize either the hanging equilibrium or the inverted equilibrium. This problem can be viewed as stabilization of a spherical pendulum.**

I. Introduction

Pendulum models have provided a rich source of examples that have motivated and illustrated many recent developments in nonlinear dynamics and control. Much of the published research treats 1D planar pendulum models or 2D spherical pendulum models or some multi-body version of these. In a recent paper [1], we summarized a large part of this published research, emphasizing control design results. In addition, we introduced a new 3D pendulum model that, seems not to have been studied in the prior literature.

A closely related paper [2], obtained controllers for a 3D asymmetric rigid pendulum. Controllers were introduced that were shown to provide asymptotic stabilization of a reduced attitude equilibrium. The reduced attitude of the 3D rigid pendulum is defined as the attitude or orientation of the 3D rigid pendulum, modulo rotation about a vertical axis. Stabilization results are provided in [2] for the hanging equilibrium, and for the inverted equilibrium.

The present paper continues our research on control of 3D rigid pendula. The pendulum is supported at a pivot that is assumed to be frictionless and inertially fixed. The rigid body is axially symmetric. The location of its center of mass is distinct from the location of the pivot. Forces that arise from uniform and constant gravity act on the pendulum. Two independent control moments are assumed to act about the two principal axes of the pendulum that are not the axis of symmetry; in other words, there is no control moment about the axis of symmetry of the pendulum. This is in contrast to the assumption of three independent control moments in [2].

We follow the development and notation introduced in [1]. In particular, the formulation of the models depends on construction of a Euclidean coordinate frame fixed to the pendulum with origin at the pivot and an inertial Euclidean coordinate frame with origin at the pivot. We assume that the pendulum fixed coordinate frame is selected to be coincident with the principal axes of the pendulum, and that the center of mass of the pendulum lies on the axis of symmetry of the rigid pendulum. We also assume that the inertial coordinate frame is selected so that the first two axes lie in a horizontal plane and the “positive” third axis points down. These assumptions are shown to guarantee that the angular velocity component about the axis of symmetry of the rigid pendulum is always constant. This conservation property allows development of reduced equations of motion for the 3D axially symmetric pendulum. The resulting reduced model is expressed in terms of two components of the angular velocity vector of the pendulum and the reduced attitude vector of the pendulum.

The control problems that are treated in this paper involve asymptotic stabilization of an equilibrium of the reduced equations of motion of the 3D pendulum; this corresponds to stabilization of a relative equilibrium of the 3D pendulum. The relative equilibrium corresponds to either the hanging reduced equilibrium or the inverted reduced equilibrium with a pure spin about the pendulum’s axis of symmetry.

The main contributions of this paper are the development of controllers that asymptotically stabilize the hanging relative equilibrium, the development of controllers that asymptotically stabilize the inverted relative equilibrium, and for the special case that there is zero angular velocity about the axis of symmetry of the pendulum, development of controllers that asymptotically stabilize either the hanging reduced equilibrium or the inverted reduced equilibrium. If the angular velocity component about the axis of symmetry

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is nonzero, our control results can be compared with other results in the literature on stabilization of Lagrange tops. If the angular velocity component about the axis of symmetry is zero, our control results can be compared with other results in the literature on stabilization of spherical pendula. In all of these cases, our proposed stabilization results are new.

The results are derived using novel Lyapunov functions that are suited to the geometry of the 3D axially symmetric pendulum. An important feature of the development is that the results are stated in terms of a global representation of the reduced attitude. In particular, we avoid the use of Euler angles and other non-global attitude representations. This work compares with [3], wherein PD control laws for systems evolving over a Riemannian manifold were proposed. In contrast with the PD-based laws in [3] that generally give a conservative domain of attraction, we provide almost global asymptotic stabilization results; see [2] for definition of almost global asymptotic stabilization. Finally, we note that results in this paper provide almost global asymptotic stabilization in a direct way using relatively simple nonlinear controllers. This approach avoids the artificial need to develop a "swing-up" controller, a locally asymptotically stabilizing controller, and a strategy for switching between the two as in [4], [5].

II. Models of the 3D Axially Symmetric Pendulum

In this section we introduce reduced models for the controlled 3D axially symmetric pendulum, and we summarize important stability properties of the uncontrolled 3D axially symmetric pendulum.

Since the pendulum is assumed to be axially symmetric, there is no loss of generality in assuming that the inertia matrix is $J = \mathrm{diag}(J_t, J_t, J_a)$. Let ρ denote the vector from the pivot to the center of mass of the pendulum; in the pendulum fixed coordinate frame it is a constant vector given by $\rho = (0, 0, \rho_s)^{\mathrm{T}}$, where ρ_s is a nonzero scalar. The angular velocity vector of the pendulum is denoted by $\omega = (\omega_x, \omega_y, \omega_z)^{\mathrm{T}}$, expressed in the pendulum fixed coordinate frame. As introduced in [1] the reduced attitude vector of the pendulum is formally defined as the unit vector pointing in the direction of gravity, expressed in the pendulum fixed coordinate frame. The reduced attitude vector is denoted by $\Gamma = (\Gamma_x, \Gamma_y, \Gamma_z)^{\mathrm{T}}$.

Euler's equations in scalar form for the rotational dynamics of the 3D axially symmetric pendulum, taking into account the moment due to gravity and the control moments, are

$$
\begin{aligned}
J_t \dot{\omega}_x &= (J_t - J_a)\omega_z \omega_y - mg\rho_s \Gamma_y + \tau_x, && (1)\\
J_t \dot{\omega}_y &= (J_a - J_t)\omega_z \omega_x + mg\rho_s \Gamma_x + \tau_y, && (2)\\
J_a \dot{\omega}_z &= 0. && (3)
\end{aligned}
$$

Here τ_x and τ_y denote the control moments. The rotational kinematics of the 3D pendulum can be expressed in terms of the reduced attitude vector according to the three scalar differential equations

$$
\begin{aligned}
\dot{\Gamma}_x &= \Gamma_y \omega_z - \Gamma_z \omega_y, && (4)\\
\dot{\Gamma}_y &= -\Gamma_x \omega_z + \Gamma_z \omega_x, && (5)\\
\dot{\Gamma}_z &= \Gamma_x \omega_y - \Gamma_y \omega_x. && (6)
\end{aligned}
$$

This model can be viewed as defining the motion of the 3D pendulum on the quotient space $T\mathrm{SO}(3)/S^1$. It is sufficient to view the motion of the 3D pendulum as evolving on $\mathbb{R}^3 \times S^2$, where $\Gamma \in S^2$, according to equations (1)–(6).

It is clear that the above equations cannot be asymptotically stabilized in any meaningful sense. Clearly equation (3) implies that the angular velocity component about the axis of symmetry of the pendulum is $\omega_z = c$ where c is a constant. Reduction in this case is easily achieved by ignoring equation (3) and substituting $\omega_z = c$ into equations (1), (2) and (4), (5). This leads to the reduced dynamics equations

$$
\begin{aligned}
J_t \dot{\omega}_x &= c(J_t - J_a)\omega_y - mg\rho_s \Gamma_y + \tau_x, && (7)\\
J_t \dot{\omega}_y &= c(J_a - J_t)\omega_x + mg\rho_s \Gamma_x + \tau_y, && (8)
\end{aligned}
$$

and the reduced kinematics equations

$$
\begin{aligned}
\dot{\Gamma}_x &= c\Gamma_y - \Gamma_z \omega_y, && (9)\\
\dot{\Gamma}_y &= -c\Gamma_x + \Gamma_z \omega_x, && (10)\\
\dot{\Gamma}_z &= \Gamma_x \omega_y - \Gamma_y \omega_x. && (11)
\end{aligned}
$$

We again note that this model can be viewed as defining the reduced motion of the 3D pendulum on $\mathbb{R}^2 \times S^2$, where $\Gamma \in S^2$ according to equations (7)–(11).

The uncontrolled equations (7)–(11) have two distinct equilibrium solutions, namely $\omega_x = \omega_y = 0$, $\Gamma = \Gamma_h = [0, 0, 1]$, and $\omega_x = \omega_y = 0$, $\Gamma = \Gamma_i = [0, 0, -1]$. The first equilibrium is referred to as the hanging equilibrium, since the center of mass of the pendulum is directly below the pivot. The second equilibrium is referred to as the inverted equilibrium, since the center of mass of the pendulum is directly above the pivot. Note that these are relative equilibrium solutions of the uncontrolled equations (1)–(6) corresponding to a pure spin of the pendulum about its axis of symmetry. It can be shown using standard Lyapunov analysis that the hanging equilibrium of (1)–(6) is stable in the sense of Lyapunov, and the inverted equilibrium of (1)–(6) is unstable.

We next present a result for the Lyapunov stability of the hanging and inverted equilibrium of (7)–(11).

Theorem 1: *Consider the 3D axially symmetric pendulum given by the equations (7)–(11). Then the hanging equilibrium is Lyapunov stable for all $c \in \mathbb{R}$ and the inverted equilibrium is Lyapunov stable if and only if $J_a^2 c^2 \geq 4mg\rho_s J_t$.*

Thus, the equilibrium of the uncontrolled system (7)–(11) is at best, Lyapunov stable. This background provides motivation for study of controllers that asymptotically stabilize either the hanging equilibrium or the inverted equilibrium.

III. Stabilization of the Hanging Equilibrium of the Lagrange Top

In this section we assume that the constant angular velocity $c \neq 0$. In this case, the 3D axially symmetric pendulum described by equations (7)–(11) is effectively a Lagrange top; hence that terminology is used in this section. We propose two classes of feedback controllers that asymptotically stabilize the hanging equilibrium of the reduced model described by equations (7)–(11). The first result is based on feedback of the angular velocity of the top. The second result is based on feedback of both angular velocity and the reduced attitude of the top. In each case, we obtain almost global asymptotic stability.

We begin by considering controllers based on feedback of the angular velocity of the form

$$\tau_x = -\psi_x(\omega_x), \tag{12}$$
$$\tau_y = -\psi_y(\omega_y), \tag{13}$$

where $\psi_x : \mathbb{R} \mapsto \mathbb{R}$ and $\psi_y : \mathbb{R} \mapsto \mathbb{R}$ are C^1 functions satisfying the sector inequalities

$$\epsilon_1 |x|^2 \leq \{x\psi_x(x), x\psi_y(x)\} \leq \epsilon_2 |x|^2, \quad \forall x \in \mathbb{R}, \tag{14}$$

where $\epsilon_2 \geq \epsilon_1 > 0$ and $\psi_x'(0), \psi_y'(0) > 0$.

Lemma 1: *Consider the 3D axially symmetric pendulum given by equations (7)–(11). Let (ψ_x, ψ_y) be C^1 functions satisfying (14) and choose τ_x and τ_y as in (12)–(13). Then the hanging equilibrium of (7)–(11) is asymptotically stable. Furthermore, for every $\epsilon \in (0, 2mg\rho_s)$, all solutions of the closed loop system given by (7)–(11) and (12)–(13), such that $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathcal{H}_\epsilon$, where*

$$\mathcal{H}_\epsilon = \Big\{(\omega_x, \omega_y, \Gamma) \in (\mathbb{R}^2 \times S^2) : \frac{1}{2}\Big[J_t(\omega_x^2 + \omega_y^2) + mg\rho_s \|\Gamma - \Gamma_h\|^2\Big] \leq 2mg\rho_s - \epsilon\Big\} \tag{15}$$

satisfy $(\omega_x(t), \omega_y(t), \Gamma(t)) \in \mathcal{H}_\epsilon$, $t \geq 0$, and $\lim_{t\to\infty} \omega_x(t) = 0$, $\lim_{t\to\infty} \omega_y(t) = 0$ and $\lim_{t\to\infty} \Gamma(t) = \Gamma_h$.

Proof: Consider the closed loop system given by (7)–(11) and (12)–(13). We propose the following candidate Lyapunov function

$$V(\omega_x, \omega_y, \Gamma) = \frac{1}{2}\Big[J_t(\omega_x^2 + \omega_y^2) + mg\rho_s \|\Gamma - \Gamma_h\|^2\Big]. \tag{16}$$

Note that the Lyapunov function is positive definite on $\mathbb{R}^2 \times S^2$ and $V(0, 0, \Gamma_h) = 0$. Furthermore, the derivative $\dot{V}$ along a solution of the closed loop is given by

$$\dot{V}(\omega_x, \omega_y, \Gamma) = -\omega_x \psi_x(\omega_x, \omega_y) - \omega_y \psi_y(\omega_x, \omega_y), \\ \leq -\epsilon_1(\omega_x^2 + \omega_y^2) \leq 0,$$

where the last inequality follows from (14). Thus V is positive definite and $\dot{V}$ is negative semidefinite on $\mathbb{R}^2 \times S^2$.

Next, consider the sub-level set given by $\mathcal{H}_\epsilon = \big\{(\omega_x, \omega_y, \Gamma) \in (\mathbb{R}^2 \times S^2) : V(\omega_x, \omega_y, \Gamma) \leq 2mg\rho_s - \epsilon\big\}$. Note that the compact set $\mathcal{H}_\epsilon$ contains the hanging equilibrium $(0, 0, \Gamma_h)$. Since $\dot{V}(\omega_x, \omega_y, \Gamma) \leq 0$ on $\mathcal{H}_\epsilon$, all solutions such that $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathcal{H}_\epsilon$ satisfy $(\omega_x(t), \omega_y(t), \Gamma(t)) \in \mathcal{H}_\epsilon$ for all $t \geq 0$. Thus, $\mathcal{H}_\epsilon$ is an invariant set for solutions of the closed loop.

Next, from LaSalle's invariant set theorem, we obtain that the solutions satisfying $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathcal{H}_\epsilon$ converge to the largest invariant set in $\{(\omega_x, \omega_y, \Gamma) \in \mathcal{H}_\epsilon : (\omega_x, \omega_y) = (0, 0)\}$. Thus, $\omega_x \equiv \omega_y \equiv 0$ implies that $\Gamma_x = \Gamma_y = 0$ and $\dot{\Gamma}_z = 0$ and hence, $\Gamma_z = \pm 1$. Thus, as $t \to \infty$, either $\Gamma \to \Gamma_h$ or $\Gamma \to \Gamma_i$. However, since $(0, 0, \Gamma_i) \notin \mathcal{H}_\epsilon$, it follows that $\Gamma \to \Gamma_h$ as $t \to \infty$. Thus, $(0, 0, \Gamma_h)$ is an asymptotically stable equilibrium of the closed loop given by (7)–(11) and (12)–(13), with $\mathcal{H}_\epsilon$ as a domain of attraction. ∎

We next, strengthen the conclusions of Lemma 1 to show that the domain of attraction is almost global.

Theorem 2: *Consider the 3D axially symmetric pendulum given by equations (7)–(11). Let (ψ_x, ψ_y) be C^1 functions satisfying (14). Choose τ_x and τ_y as in (12)–(13). Then, all solutions of the closed loop system given by (7)–(11) and (12)–(13), such that $(\omega_x(0), \omega_y(0), \Gamma(0)) \in (T\mathrm{SO}(3)/TS^1)\backslash\mathcal{M}$ satisfy $\lim_{t\to\infty} \omega_x(t) = 0$, $\lim_{t\to\infty} \omega_y(t) = 0$ and $\lim_{t\to\infty} \Gamma(t) = \Gamma_h$. Here, $\mathcal{M}$, a set of Lebesgue measure zero, is the stable manifold of the closed loop inverted equilibrium.*

Proof: We present an outline of the proof. Denote

$$\mathcal{N} = \Big\{(\omega_x, \omega_y, \Gamma) \in (\mathbb{R}^2 \times S^2) : \frac{1}{2}\Big[J_t(\omega_x^2 + \omega_y^2) + mg\rho_s \|\Gamma - \Gamma_h\|^2\Big] \leq 2mg\rho_s\Big\} \tag{17}$$

Then, as in Lemma 2 in [2], it can be shown that all solutions of the closed loop (7)–(11) and (12)–(13), satisfying $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \partial\mathcal{N}\backslash\{(0, 0, \Gamma_i)\}$ enter the set $\mathcal{H}_\epsilon$ in Lemma 1, for some $\epsilon > 0$, in finite time. Next, from Lemma 1 and the definition of $\mathcal{N}$, we note that for every $\epsilon \in (0, 2mg\rho_s)$ and $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathcal{H}_\epsilon \bigcup \Big(\partial\mathcal{N}\backslash\{(0, 0, \Gamma_i)\}\Big)$, $\omega(t) \to 0$ and $\Gamma(t) \to \Gamma_h$ as $t \to \infty$. Now, since $\mathcal{N} = \bigcup_{\epsilon\in(0,2mg\rho_s)} \Big(\mathcal{H}_\epsilon \bigcup \partial\mathcal{N}\Big)$, it follows that all solutions satisfying $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathcal{N}\backslash\{(0, 0, \Gamma_i)\}$ converge to the hanging equilibrium.

Next, it can be shown that all solutions of the closed loop (7)–(11) and (12)–(13), enter the set $\mathcal{N}$ in finite time. Hence all solutions either converge to the inverted equilibrium, or the hanging equilibrium. Thus, it is sufficient to show that the stable manifold of the inverted equilibrium $(0, 0, \Gamma_i)$, has dimension less than the dimension of $T\mathrm{SO}(3)/TS^1$ i.e. four, since all other solutions converge to the hanging equilibrium.

Using linearization, it can be shown that the equilibrium $(0, 0, \Gamma_i)$ of the closed loop is unstable and hyperbolic with nontrivial stable and unstable manifolds. Hence, from Theorem 3.2.1 in [6], it follows that the dimension of the stable manifold is less than four, so that the domain of attraction in Theorem 2 is $(T\mathrm{SO}(3)/TS^1)\backslash\mathcal{M}$, where $\mathcal{M}$ is the stable manifold and hence, the result follows. ∎

Theorem 2 provides conditions under which the hanging equilibrium of the Lagrange top is made asymptotically stable by feedback of the angular velocity. Any controller of the form (12) and (13) can be viewed as providing damping injection. In Lemma 1, the hanging equilibrium of the closed loop has a domain of stability that is easily computed. In Theorem 2, the domain of attraction is almost global.

Next, we consider controllers based on feedback of the angular velocity and the reduced attitude. These controllers provide more design flexibility than the controllers that depend on angular velocity only; hence they can provide improved closed loop performance.

Theorem 3: *Consider the 3D axially symmetric pendulum given by equations (7)–(11) with $c \neq 0$. Let $\Phi : [0, 1) \mapsto \mathbb{R}$ be a C^1 function such that $\Phi(0) = 0$, $\Phi'(x) > 0$ if $x \neq 0$, and $\Phi(x) \to \infty$ as $x \to 1$. Furthermore, let (ψ_x, ψ_y) be C^1 functions satisfying the inequality given in (14). Choose*

$$\begin{aligned}\tau_x = &-\omega_x + \psi_x\Big((\Gamma_z - 1)\Gamma_y\Big) - c(J_t - J_a)\omega_y \\ &+ J_t(\Gamma_z - 1)(-c\Gamma_x + \Gamma_z\omega_x)\psi_x'\Big((\Gamma_z - 1)\Gamma_y\Big) \\ &+ (\Gamma_z - 1)\Gamma_y\Phi'\Big(\tfrac{1}{4}(\Gamma_z - 1)^2\Big) + mg\rho_s\Gamma_y, \qquad (18)\end{aligned}$$

$$\begin{aligned}\tau_y = &-\omega_y + \psi_y\Big((1 - \Gamma_z)\Gamma_x\Big) - c(J_a - J_t)\omega_x \\ &+ J_t(\Gamma_z - 1)(c\Gamma_y - \Gamma_z\omega_y)\psi_y'\Big((1 - \Gamma_z)\Gamma_x\Big) \\ &- (\Gamma_z - 1)\Gamma_x\Phi'\Big(\tfrac{1}{4}(\Gamma_z - 1)^2\Big) - mg\rho_s\Gamma_x. \qquad (19)\end{aligned}$$

Then $(\omega_x, \omega_y, \Gamma) = (0, 0, \Gamma_h)$ is an equilibrium of the closed loop given by (7)–(11) and (18)–(19) that is asymptotically stable with $\mathbb{R}^2 \times \Big(S^2 \backslash \{\Gamma_i\}\Big)$ as a domain of attraction.

Proof: Consider the system represented by (7)–(11) and (18)–(19). We propose the following candidate Lyapunov function.

$$\begin{aligned}V(\omega_x, \omega_y, \Gamma) = &\frac{J_t}{2}\Big[\omega_x - \psi_x\Big((\Gamma_z - 1)\Gamma_y\Big)\Big]^2 \\ &+ \frac{J_t}{2}\Big[\omega_y - \psi_y\Big((1 - \Gamma_z)\Gamma_x\Big)\Big]^2 + 2\Phi\left(\frac{1}{4}(\Gamma_z - 1)^2\right).\end{aligned}$$

Note that the above Lyapunov function is positive definite on $\mathbb{R}^2 \times S^2$ with $V(0, 0, \Gamma_h) = 0$.

Suppose that $(\omega_x(0), \omega_y(0), \Gamma(0)) \neq (0, 0, \Gamma_i)$. Computing the derivative of the Lyapunov function along a solution of the closed loop, we obtain

$$\begin{aligned}\dot{V}(\omega_x, \omega_y, \Gamma) \leq &-\Big[\omega_x - \psi_x\Big((\Gamma_z - 1)\Gamma_y\Big)\Big]^2 \\ &-\Big[\omega_y - \psi_y\Big((1 - \Gamma_z)\Gamma_x\Big)\Big]^2 \\ &-\epsilon_1\Phi'\left(\frac{1}{4}(\Gamma_z - 1)^2\right)(\Gamma_z - 1)^2(\Gamma_x^2 + \Gamma_y^2) \leq 0. \qquad (20)\end{aligned}$$

Thus, $\dot{V}$ is negative semidefinite and hence, each solution remains in the compact invariant set $\mathcal{K} = \{(\omega_x, \omega_y, \Gamma) \in \mathbb{R}^2 \times S^2 : V(\omega_x, \omega_y, \Gamma) \leq V(\omega_x(0), \omega_y(0), \Gamma(0))\}$.

Since $\dot{V}$ is negative semidefinite and Φ satisfies $\Phi'(x) > 0$ if $x \neq 0$, we obtain that, $(\Gamma_z - 1)\Gamma_y \to 0$, $(\Gamma_z - 1)\Gamma_x \to 0$, $\omega_x \to \psi_x(0) = 0$ and $\omega_y \to \psi_y(0) = 0$ as $t \to \infty$. The last two limit properties follow from the properties of the function $\psi_i(\cdot)$, $i \in \{1, 2\}$ and Sandwiching theorem for the limit of a function.

Furthermore, by LaSalle's invariant set theorem, each solution converges to the largest invariant set $\mathcal{M} \subseteq \{(\omega_x, \omega_y, \Gamma) \in \mathcal{K} : \omega_x = \omega_y = 0,\ (\Gamma_z - 1)\Gamma_y = 0,\ (\Gamma_z - 1)\Gamma_x = 0\}$. Since, any closed-loop solution of (7)–(11) in $\mathcal{M}$ satisfies $\omega_x \equiv \omega_y \equiv 0$, we obtain that the solution also satisfies $\Gamma_z =$ constant.

Next, $(\Gamma_z - 1)\Gamma_y \equiv (\Gamma_z - 1)\Gamma_x \equiv 0$ yields either $\Gamma_z = 1$, in which case $\Gamma = \Gamma_h$, or it yields $\Gamma_x = 0$ and $\Gamma_y = 0$, and hence, $\Gamma = \Gamma_h$ or $\Gamma = \Gamma_i$. However, since $V(\omega_x(t), \omega_y(t), \Gamma(t)) \leq V(\omega_x(0), \omega_y(0), \Gamma(0))$, therefore $\Gamma(t) \neq \Gamma_i$ for all $t > 0$. Thus, $\Gamma_i \notin \mathcal{M}$. Hence, $\Gamma = \Gamma_h$. Thus, the only solution of the closed-loop contained in the invariant set $\mathcal{M}$ is $\omega_x = \omega_y = 0$ and $\Gamma = \Gamma_h$. ∎

Theorem 3 provides conditions under which the hanging equilibrium of the Lagrange top can be made almost-globally asymptotically stable by feedback of the angular velocity and feedback of the reduced attitude of the top. Any controller of the form (18) and (19) requires knowledge of the axial and transverse principal moments of inertia, mass, location of the center of mass, and spin rate of the Lagrange top.

IV. Stabilization of the Inverted Equilibrium of the Lagrange Top

As in the previous section, we assume that the constant $c \neq 0$, so that the 3D axially symmetric pendulum described by equations (7)–(11) is effectively a Lagrange top. We now propose feedback controllers that almost-globally asymptotically stabilize the inverted equilibrium of the reduced equations (7)–(11). The result is based on feedback of both angular velocity and the reduced attitude of the top.

Theorem 4: *Consider the 3D axially symmetric pendulum given by equations (7)–(11) with $c \neq 0$. Let $\Phi : [0, 1) \mapsto \mathbb{R}$ be a C^1 function such that $\Phi(0) = 0$, $\Phi'(x) > 0$ if $x \neq 0$, and $\Phi(x) \to \infty$ as $x \to 1$. Furthermore, let (ψ_x, ψ_y) be C^1 functions satisfying the inequality given in (14). Choose*

$$\begin{aligned}\tau_x = &-\omega_x + \psi_x\Big((1 + \Gamma_z)\Gamma_y\Big) - c(J_t - J_a)\omega_y \\ &+ J_t(\Gamma_z + 1)(-c\Gamma_x + \Gamma_z\omega_x)\psi_x'\Big((1 + \Gamma_z)\Gamma_y\Big) \\ &+ (\Gamma_z + 1)\Gamma_y\Phi'\Big(\tfrac{1}{4}(\Gamma_z + 1)^2\Big) + mg\rho_s\Gamma_y, \qquad (21)\end{aligned}$$

$$\begin{aligned}\tau_y = &-\omega_y + \psi_y\Big(-(\Gamma_z + 1)\Gamma_x\Big) - c(J_a - J_t)\omega_x \\ &+ J_t(\Gamma_z + 1)(c\Gamma_y - \Gamma_z\omega_y)\psi_y'\Big(-(\Gamma_z + 1)\Gamma_x\Big) \\ &- (\Gamma_z + 1)\Gamma_x\Phi'\Big(\tfrac{1}{4}(\Gamma_z + 1)^2\Big) - mg\rho_s\Gamma_x. \qquad (22)\end{aligned}$$

Then $(\omega_x, \omega_y, \Gamma) = (0, 0, \Gamma_i)$ is an equilibrium of the closed loop given by (7)–(11) and (21)–(22) that is asymptotically *stable with* $\mathbb{R}^2 \times \Big(S^2 \backslash \{\Gamma_h\}\Big)$ *as a domain of attraction.*

Proof: Consider the system represented by (7)–(11) and (21)–(22) and consider the following Lyapunov function.

$$V(\omega_x, \omega_y, \Gamma) = \frac{J_t}{2}\Big[\omega_x - \psi_x\Big((\Gamma_z + 1)\Gamma_y\Big)\Big]^2 + \frac{J_t}{2}\Big[\omega_y - \psi_y\Big(-(\Gamma_z + 1)\Gamma_x\Big)\Big]^2 + 2\Phi\left(\frac{1}{4}(\Gamma_z + 1)^2\right). \tag{23}$$

Note that the above Lyapunov function is positive definite and proper on $\mathbb{R}^2 \times S^2$ with $V(0, 0, \Gamma_i) = 0$.

Suppose that $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathbb{R}^2 \times \Big(S^2 \backslash \{\Gamma_h\}\Big)$. Computing the derivative of the Lyapunov function along a solution of the closed loop, we obtain

$$\begin{aligned}\dot{V}(\omega_x, \omega_y, \Gamma) \leq &-\Big[\omega_x - \psi_x\Big((\Gamma_z + 1)\Gamma_y\Big)\Big]^2 \\ &- \Big[\omega_y - \psi_y\Big(-(\Gamma_z + 1)\Gamma_x\Big)\Big]^2 \\ &- \epsilon_1\Phi'\left(\frac{1}{4}(\Gamma_z + 1)^2\right)(\Gamma_z + 1)^2(\Gamma_x^2 + \Gamma_y^2) \leq 0.\end{aligned} \tag{24}$$

Thus, $\dot{V}$ is negative semidefinite and hence, each solution remains in the compact invariant set $\mathcal{K} = \{(\omega_x, \omega_y, \Gamma) \in \mathbb{R}^2 \times S^2 : V(\omega_x, \omega_y, \Gamma) \leq V(\omega_x(0), \omega_y(0), \Gamma(0))\}$.

The remainder of the proof follows exactly the arguments used in Theorem 3. The only solution of the closed-loop system of (7)–(11) and (21)–(22) such that $(\Gamma_z + 1)\Gamma_y \to 0$, $(\Gamma_z + 1)\Gamma_x \to 0$, $\omega_x \to \psi_x(0) = 0$ and $\omega_y \to \psi_y(0) = 0$ as $t \to \infty$ is the inverted equilibrium $(\omega_x, \omega_y, \Gamma) = (0, 0, \Gamma_i)$. ■

Theorem 4 provides conditions under which the inverted equilibrium of the Lagrange top can be made asymptotically stable by feedback of the angular velocity and feedback of the reduced attitude of the top. Any controller of the form (21) and (22) requires knowledge of the axial and transverse principal moments of inertia, mass, location of the center of mass, and spin rate of the Lagrange top. The inverted equilibrium of the top is guaranteed to have an almost-global domain of attraction. These results can be compared with the extensive literature on stabilization of Lagrange tops; see for example [7], [8]. The results in Theorem 4 are substantially different from any of these cited results on stabilization of a Lagrange top.

V. Stabilization of the Inverted Equilibrium of the Spherical Pendulum

In this section we assume that the angular velocity ω_z is a constant $c = 0$. In this case, the 3D axially symmetric pendulum described by equations (7)–(11) is effectively a spherical pendulum; hence that terminology is used in this section. We propose feedback controllers that asymptotically stabilize the inverted equilibrium of the reduced model described by equations (7)–(11). Since $\omega_z = c = 0$ it corresponds to an equilibrium manifold of the complete model (1)–(6). The result is based on feedback of both angular velocity and the reduced attitude of the spherical pendulum. The development in this section is easily modified to provide an almost globally stabilizing controller for the hanging equilibrium.

Theorem 5: *Consider the 3D axially symmetric pendulum given by equations (7)–(11) with* $c = 0$*. Let* $\Phi : [0, 1) \mapsto \mathbb{R}$ *be a* C^1 *function such that* $\Phi(0) = 0$*,* $\Phi'(x) > 0$ *if* $x \neq 0$*, and* $\Phi(x) \to \infty$ *as* $x \to 1$*. Furthermore, let* (ψ_x, ψ_y) *be* C^1 *functions satisfying the inequality given in (14). Assume* $\omega_z(0) = c = 0$*, and let*

$$y_1 \triangleq (1 + \Gamma_z)\Gamma_y, \tag{25}$$

$$y_2 \triangleq (1 + \Gamma_z)\Gamma_x. \tag{26}$$

$$\tau_x = mg\rho_s\Gamma_y + J_t\psi_x'(y_1)\dot{y}_1 - (\omega_x - \psi_x(y_1)) + y_1\Phi'\left(\frac{1}{4}(\Gamma_z + 1)^2\right), \tag{27}$$

$$\tau_y = -mg\rho_s\Gamma_x + J_t\psi_y'(y_2)\dot{y}_2 - (\omega_y - \psi_y(y_2)) + y_2\Phi'\left(\frac{1}{4}(\Gamma_z + 1)^2\right), \tag{28}$$

where $\dot{y}_1$ *and* $\dot{y}_2$ *are obtained by differentiating (25) and (26) and substituting from (9)–(11). Then* $(0, 0, \Gamma_i)$ *is an equilibrium of the closed loop given by (7)-(11) and (27)-(28) that is asymptotically stable with* $\mathbb{R}^2 \times \Big(S^2 \backslash \{\Gamma_h\}\Big)$ *as a domain of attraction.*

Proof: Consider the system given by (7)–(11) and (27)-(28). We propose the following candidate Lyapunov function.

$$V(\omega, \Gamma) = \frac{J_t}{2}[\omega_x - \psi_x(y_1)]^2 + \frac{J_t}{2}[\omega_y - \psi_y(y_2)]^2 + 2\Phi\left(\frac{1}{4}(\Gamma_z + 1)^2\right). \tag{29}$$

Note that the above Lyapunov function is positive definite on $\mathbb{R}^2 \times S^2$ and $V(0, \Gamma_i) = 0$. Furthermore, $V(\omega_x, \omega_y, \Gamma)$ is a proper function on $\mathbb{R}^2 \times S^2$. Next, computing the derivative of the Lyapunov function along a solution of the closed loop, we obtain

$$\dot{V}(\omega, \Gamma) \leq -(\omega_x - \psi_x(y_1))^2 - (\omega_y - \psi_y(y_2))^2 - \epsilon_1\Phi'\left(\frac{1}{4}(\Gamma_z + 1)^2\right)(y_1^2 + y_2^2) \leq 0. \tag{30}$$

Thus, $\dot{V}$ is negative semidefinite and hence, each solution remains in the compact invariant set $\mathcal{K} = \{(\omega_x, \omega_y, \Gamma) \in \mathbb{R}^2 \times S^2 : V(\omega_x, \omega_y, \Gamma) \leq V(\omega_x(0), \omega_y(0), \Gamma(0))\}$. Next, since $\dot{V}$ is negative semidefinite and from properties of $\Phi(\cdot)$, we obtain that, $y_1 \to 0$, $y_2 \to 0$, $\omega_x \to \psi_x(0) = 0$ and $\omega_y \to \psi_y(0) = 0$ as $t \to \infty$. The last two limit equalities follow from the Sandwiching theorem for the limit of a function.

Furthermore, by LaSalle's invariant set theorem, the solution converges to the largest invariant set $\mathcal{M} \subseteq \{(\omega_x, \omega_y, \Gamma) \in \mathcal{K} : \omega_x = \omega_y = 0,\ y_1 = 0,\ y_2 = 0\}$. Since, any closed-loop solution in $\mathcal{M}$ satisfies $\omega_x \equiv \omega_y \equiv 0$, we obtain that the solution also satisfies $\Gamma =$ constant.

Next, $y_1 \equiv y_2 \equiv 0$ yields either $\Gamma_z = -1$, in which case $\Gamma = \Gamma_i$, or it yields $\Gamma_x = 0$ and $\Gamma_y = 0$ which implies that $\Gamma = \Gamma_i$ or $\Gamma = \Gamma_h$. However, since $V(\omega_x(t), \omega_y(t), \Gamma(t)) \leq V(\omega_x(0), \omega_y(0), \Gamma(0))$, therefore $\Gamma(t) \neq \Gamma_h$ for all $t \geq 0$. Thus, $(0, 0, \Gamma_h) \notin \mathcal{M}$. Hence, $\Gamma = \Gamma_i$. Thus, the only solution of the closed loop contained in the invariant set $\mathcal{M}$, is $\omega_x = \omega_y = 0$ and $\Gamma = \Gamma_i$. ■

Theorem 5 provides conditions under which the inverted equilibrium of the spherical pendulum can be made asymptotically stable by feedback of the angular velocity and feedback of the reduced attitude of the spherical pendulum. Any controller of the form (27) and (28) requires knowledge of the transverse (but not the axial) principal moment of inertia, the mass, and the location of the center of mass of the spherical pendulum. Theorem 5 provides a means for stabilizing the inverted equilibrium of the spherical pendulum with an almost global domain of attraction.

This is a new result for stabilization of the spherical pendulum. The results in Theorem 5 are substantially different from similar results on stabilization of spherical pendulums that have appeared in prior literature [5], [9], [10]. Our results provide an almost-globally stabilizing controller that avoids the need to construct a swing-up controller, a locally stabilizing controller, and a switching strategy between the two. In this comparative sense, our results are direct and simple.

VI. Simulation Results

In this section, we present simulation results for specific controllers that stabilize the inverted equilibrium of the Lagrange top and the spherical pendulum. Consider the model (7)–(11), where $m = 140$ kg, $\rho = (0, 0, 0.5)^{\mathrm{T}}$ m and $J = \mathrm{diag}(40, 40, 50)$ kg-m^2. We choose $\Phi(x) = -5\ln(1-x)$, $\psi_x(u) = 3u$, and $\psi_y(u) = 3u$.

Consider a Lagrange top with spin rate about its axis of symmetry $c = 1$ rad/sec. The controller is given by (21) and (22) with the above specifications, so that it stabilizes the inverted equilibrium, $\Gamma_i = (0, 0, -1)$. The initial conditions are $\omega(0) = (1, 3, 1)^{\mathrm{T}}$ rad/s and $\Gamma(0) = (0.1, 0.5916, 0.8)^{\mathrm{T}}$. Simulation results in Figure 1 illustrate that $\omega_x(t) \to 0$, $\omega_y(t) \to 0$ and $\Gamma(t) \to \Gamma_i$ as $t \to \infty$.

Now consider a spherical pendulum with controller given by (27) and (28) with the above specifications, so that it stabilizes the inverted equilibrium. The initial conditions are $\omega(0) = (1, 3, 0)^{\mathrm{T}}$ rad/s and $\Gamma(0) = (0.1, 0.5916, 0.8)^{\mathrm{T}}$. Simulation results in Figure 2 illustrate that $\omega_x(t) \to 0$, $\omega_y(t) \to 0$ and $\Gamma(t) \to \Gamma_i$ as $t \to \infty$.

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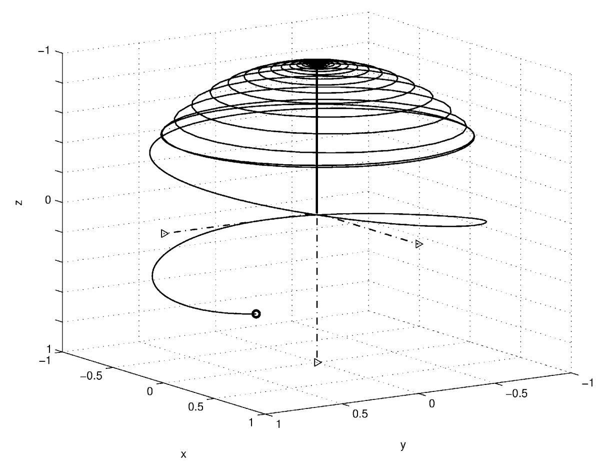

Fig. 1. Swing-up motion of the vector between the pivot and the center of mass of the Lagrange top in the inertial frame.

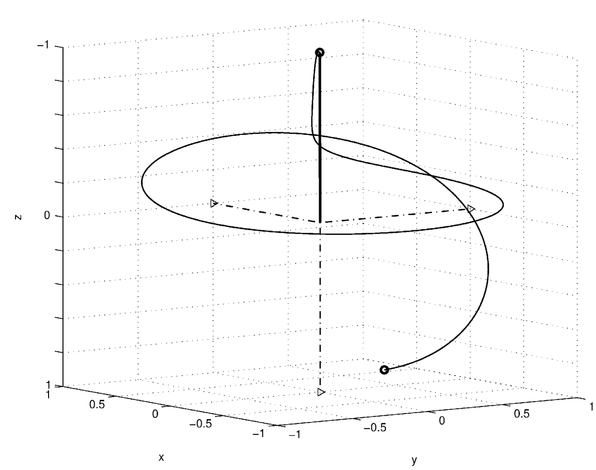

Fig. 2. Swing-up motion of the vector between the pivot and the center of mass of the spherical pendulum in the inertial frame.