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Polynomial Feedback and Observer Design using Nonquadratic Lyapunov Functions

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Abstract— **In polynomial state feedback and observer design, it is often assumed that the corresponding Lyapunov functions are quadratic. This assumption allows to guarantee global stability and to use semidefinite programming and the sum of squares decomposition. In the present paper, state feedback and observer design strategies based on semidefinite programming and the sum of squares decomposition are proposed which can deal with nonquadratic Lyapunov functions without jeopardizing global stability. In particular, homogeneous Lyapunov functions and generalized Krasovskii-type Lyapunov functions are studied for state feedback design and Lyapunov functions which are nonquadratic with respect to the control system output are studied for observer design.**

I. INTRODUCTION

Nonlinear control design has been passed through a remarkable phase over the last two decades. Many descriptive concepts for nonlinear analysis turned into constructive design strategies [9]. However, nonlinear control design is far away from being solved. Many existing design strategies impose very particular assumptions on the control system which are hard to verify or simply not satisfied in many applications. Furthermore, the assumptions on the system structure are often only locally valid so that a global design is not feasible. Moreover, there exists only a few design strategies for nonlinear control design which can be efficiently and reliably solved on a computer. These facts are often the main stumbling blocks for applying these strategies to real world problems. Recently, polynomial control systems have gained considerable attention in nonlinear control [8], [15], [16], [5]. Polynomial control systems are control systems where the maps in the control system description are polynomial maps. This class of control systems includes the class of linear control systems and many nonlinear control problems can be formulated or approximated by polynomial control systems. One encounters this class of control systems in a wide range of applications. For example, in modeling of chemical processes and chemistry, electronic circuits and mechatronic systems, biological systems, to name only a few. In combination with semidefinite programming, in particular with the help of the sum of squares decomposition, many problems in polynomial control systems analysis and design have been attacked successfully, due to the fact that the sum of squares decomposition using semidefinite programming can be solved reliably and efficiently on a computer. In the present paper, computer-aided feedback and observer design strategies are proposed. The design strategies are able to deal with nonquadratic Lyapunov functions. The first strategy is a state feedback design strategy which is based on state dependend Riccati techniques. This approach was also used in [15]. However, there global stability can only be guaranteed using a quadratic Lyapunov function. In the present paper, it is shown that semidefinite programming and the sum of squares decomposition can be also used for the case of homogeneous Lyapunov functions. This is established using Euler's homogeneity relation for positive homogeneous functions. Furthermore, generalized Krasovskii-type Lyapunov functions are proposed in the spirit of state dependend Riccati techniques to allow to deal with another type of nonquadratic (nonconvex) Lyapunov functions. Notice that in [5], a state feedback design strategy was proposed which allows to use convex Lyapunov functions. The third strategy is an observer design strategy. In contrast to feedback design, observer design, in the context of polynomial control systems and semidefinite programming, was studied very rarely. In the present paper, it is shown that one can also use Lyapunov functions which are nonquadratic with respect to the output. Furthermore, it is shown that the proposed observer design is feasible in case of a certain passivity condition is satisfied and if the nonlinearities are Lipschitz nonlinearities with respect to the state but not necessarily with respect to the output. The proposed design strategies are illustrated on two examples. The main advantages of the proposed design strategies are that beside polynomial, no special system structure is imposed, that the Lyapunov functions are nonquadratic, that the results guarantee global stability, and that an simple and efficient computer-aided design is possible.

The remainder of the paper is organized as follows: In Section II, the feedback design strategies are presented. In Section III, the observer design strategy is presented as well as a feasibility condition is discussed. Two examples are given in Section IV to illustrate the proposed design strategies. A discussion and summary in Section V concludes the paper.

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NOTATIONS. A function $V:\mathbb{R}^n \to \mathbb{R}$ is called positive definite, if $V(0)=0, V(x)>0, \forall x \in \mathbb{R}^n \setminus \{0\}$ and positive semidefinite if $V(x) \geq 0, \forall x \in \mathbb{R}^n$. A matrix $P \in \mathbb{R}^{n\times n}$ is positive definite if $x^T P x > 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ and positive semidefinite if $x^T P x \geq 0$, $\forall x \in \mathbb{R}^n$. Furthermore, $P(x) > 0$ denotes a symmetric matrix function $P:\mathbb{R}^n \to \mathbb{R}^{n\times n}$ which is positive definite for all $x \in \mathbb{R}^n$. The row vector $\nabla V(x) = (\partial V/\partial x)(x)^T = V_x(x)^T$ denotes the derivative of V with respect to x and $\nabla^2 V(x)$ the Hessian of V. The derivative along the vector field $f:\mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\nabla V(x) f(x)$. A control Lyapunov function V of the control system $\dot{x} = f(x) + G(x)u$ is a radially unbounded positive definite function such that for every nonzero $x \in \mathbb{R}^n$ there exists a $u \in \mathbb{R}^p$ such that $\dot{V}(x) = \nabla V(x) f(x) + \nabla V(x) G(x) u < 0$.

II. Feedback Design

The following problem is considered in this section:

State Feedback Problem Given a polynomial control system of the form

$$\dot{x} = f(x) + G(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^p$ is the input. f is a polynomial vector field with $f(0)=0$ and G is a polynomial map, i.e., the components f_i, G_{ij} are polynomials in x. Find a polynomial state feedback $u = K(x)x$, such that the closed loop is globally asymptotically stable w.r.t. the origin $x=0$.

In the first part, the polynomial control system (1) in combination with the state dependend Riccati approach [3], [20] is considered, by factorizing the control system (1) as follows:

$$\dot{x} = A(x)x + G(x)u. \tag{2}$$

Note that the matrix $A(x)$ is not unique, i.e., there are different matrices $A(x)$ to write $f(x) = A(x)x$. However, one can show now very easily (cf. [3], [15]) that if there exists a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and a polynomial matrix function $M:\mathbb{R}^n \to \mathbb{R}^{p\times n}$ such that

$$A(x)Q + G(x)M(x) + QA^T(x) + M^T(x)G^T(x) < 0, \tag{3}$$

for all x, then $u = K(x)x$ with $K(x) = M(x)Q^{-1}$ is a globally asymptotically stabilizing state feedback. This follows by multiplying inequality (3) from left and right with Q^{-1}. Using the quadratic Lyapunov function $V(x) = \frac{1}{2}x^T Q^{-1} x$, one arrives at the Lyapunov inequality

$$x^T Q^{-1}(A(x) + G(x)K(x) + A^T(x) + K^T(x)G^T(x))x < 0. \tag{4}$$

The step from inequality (4) to inequality (3), is necesary and well-known in semidefinite programming [1] to obtain (matrix) inequalities which are linear (affine) in the unknown. Since the unknown Q, M appear linear in inequality (3), the inequality can be solved via semidefinite programming and the sum of squares decomposition (cf. [12]). Note that the affine appearance of the unknown is essential to apply semidefinite programming and the sum of squares decomposition. All the following results are basically driven by the wish to find inequalities which are linear (affine) in the unknown. Now, the problem which appears if one wants to extend the state dependend Riccati techniques to nonquadratic Lyapunov functions of the form $V(x) = \frac{1}{2}x^T Q(x)^{-1} x$, with $Q(x) > 0$ is the following. By replacing the constant positive definite matrix Q in inequality (3) by a positive definite matrix function $Q(x)$, one has to ensure that $Q(x)^{-1}x$ is a gradient of a positive definite function [7], [11], i.e., for global stability must hold: $\nabla V(x) \overset{!}{=} x^T Q(x)^{-1}$. Unfortunately, the constraint $\nabla V(x) \overset{!}{=} x^T Q(x)^{-1}$ is nonconvex and hence cannot be combined with semidefinite programming. An interesting but difficult questions is now, in how far is it possible to get away from quadratic Lyapunov functions. In a first step, it is shown that this is at least possible for homogeneous Lyapunov functions.

Definition 1: A function $V:\mathbb{R}^n \to \mathbb{R}$ is said to be a (positive) homogeneous (control) Lyapunov function of degree r, if V is a (control) Lyapunov function and if

$$V(\lambda x) \quad = \quad \lambda^r V(x) \tag{5}$$

holds for all x and all $\lambda \geq 0$.

An important property of homogeneous functions is expressed by an appealing property, namely, by Euler's homogeneity relation (Euler's identity) [18]:

Theorem 1: (Euler's homogeneity relation). V is a homogeneous function of degree r, if and only if V satisfies

$$\nabla V(x)x \quad = \quad rV(x). \tag{6}$$

The proof is quite simple and follows by differentiation of (5) w.r.t. λ and by setting $\lambda = 1$. Another useful relation is:

Corollary 1: Let V be a homogeneous function of degree r, then V satisfies

$$x^T \nabla^2 V(x) x \quad = \quad (r-1)V(x). \tag{7}$$

The proof is quite simple again and follows by differentiation of (6) w.r.t. x and using (6). Using the relations (6), (7), the first result of this paper can be established:

Theorem 2: If there exists a polynomial matrix function M and a homogeneous positive definite matrix function Q such that

$$A(x)Q(x) + G(x)M(x) + Q(x)A^T(x) + M^T(x)G^T(x) < 0, \tag{8}$$

for all x, then $u = K(x)x$ with $K(x) = M(x)Q(x)^{-1}$ is a globally asymptotically stabilizing state feedback for the control system (1).

Proof. Suppose

$$A(x)Q(x) + G(x)M(x) + Q(x)A^T(x) + M^T(x)G^T(x) < 0,$$

is satisfied for all x. Then, by multiplication the above matrix inequality from left with $x^T Q(x)^{-1}$, and from right with $Q(x)^{-1}x$, one gets the scalar inequality

$$2x^T Q(x)^{-1}(A(x)+G(x)K(x))x < 0. \quad (9)$$

with $K(x) = M(x)Q(x)^{-1}$. Finally, by multiplication of the inequality (9) with $det(Q(x))$, one arrives at

$$2x^T adj(Q(x))(A(x)+G(x)K(x))x < 0. \quad (10)$$

with the adjugate matrix $adj(Q(x)) = det(Q(x))Q(x)^{-1}$. It has to be shown now that $x^T adj(Q(x))$ is related with a gradient function. First, notice that the function $V^\star(x) = \frac{1}{r-1}x^T adj(Q(x))x$ is homogeneous and positive definite. Therefore, using Euler's homogeneity relation (6), one can write

$$V^\star(x) = \frac{1}{r-1}x^T adj(Q(x))x = \frac{1}{r}\nabla V^\star(x)x. \quad (11)$$

Hence, together with (7), the desired property $\nabla V^\star(x) = \frac{r}{r-1}x^T adj(Q(x))$ and $adj(Q(x)) = \nabla^2 V^\star(x)$ follows. Finally, notice that the constant factor $\frac{r}{r-1}$ is irrelevant and can be easily eliminated. □

It is hard to say in how far homogeneous Lyapunov functions are more useful than quadratic Lyapunov functions. However, Theorem 2 may be of particular interest for the class homogeneous (polynomial) control systems, which is gaining more and more interest in the literature (see [17], [21] and reference therein). Also in combination with homogenization techniques and generalized homogeneity notions [21], further investigations are necessary in this direction.

The next theorem uses now quadratic Lyapunov functions. As a result, a state dependend Riccati-type result [3], [20], [15], is recovered.

Theorem 3: If there exists a polynomial matrix function M and a positive definite matrix Q such that

$$y^T(A(x)Q+G(x)M(x))y \quad < \quad 0, \quad (12)$$

for all nonzero x,y. Then $u = K(x)x$ with $K(x) = M(x)Q^{-1}$ is a globally asymptotically stabilizing state feedback for the control system (1).

First notice that inequality (12) is not written as matrix inequality anymore. The reason for this is to avoid to write long symmetric matrix inequalities. Second, as already mentioned, the matrix $A(x)$ in (2) is not unique, i.e., there are different matrices $A(x)$ to write $f(x) = A(x)x$. However, instead of this factorization, one could think to use the Jacobian f_x of f. In the following, Krasovskii methods [6] is used as a design strategy with Lyapunov functions of the type $V(x) = f(x)^T Pf(x)$ to obtain a globally asymptotically stabilizing state feedback. This is worked out in the second part of this section, starting with the following theorem:

Theorem 4: If there exists a polynomial matrix function M and a positive definite matrix Q such that

$$y^T f_x(x)(Q+G(x)M(x))y \quad < \quad 0, \quad (13)$$

for all nonzero x,y. Then $u = K(x)f(x)$ with $K(x) = M(x)Q^{-1}$ is a globally asymptotically stabilizing state feedback for the control system (1).

Proof. The change of coordinates $z = Qy$ leads to

$$z^T Q^{-1} f_x(x)(I+G(x)M(x)Q^{-1})z \quad < \quad 0.$$

With $z = f(x)$ and $K(x) = M(x)Q^{-1}$ one obtains

$$f(x)^T(Q^{-1}f_x(x)+Q^{-1}f_x(x)G(x)K(x))f(x) \quad < \quad 0,$$

which is nothing else then the derivative of the Lyapunov function $V(x) = \frac{1}{2}f(x)^T Q^{-1} f(x)$ w.r.t. the control system (1). Note that a necessary condition that (13) can be satisfied, is that the Jacobian f_x must have full rank. Hence the mapping $y = f(x)$ is one-to-one and since f is polynomial, $V(x) = \frac{1}{2}f(x)^T Q^{-1} f(x)$ is positive definite and radially unbounded. □

Moreover, the idea of Krasovskii can be generalized by the following type of Lyapunov function:

$$V(x) = w(x)^T Pw(x), \quad (14)$$

where $s(x) = W(x)x$ and $W = (W_{ij})$ is a quadratic, lower triangular, polynomial matrix function with all diagonal elements are 1, i.e.,

$$w(x) = W(x)x = \begin{bmatrix} 1 & 0 & \dots & 0 \\ w_{11}(x) & 1 & 0 & 0 \\ \vdots & & \ddots & 0 \\ w_{n1}(x) & w_{n2}(x) & \dots & 1 \end{bmatrix} x. \quad (15)$$

The idea behind this Ansatz is twofold. First, if the control engineer has some inside in the structure of the control system (1), then it can be reasonable and useful to incooperate this structural inside into the design strategy by specifying W. Of course, it is needless to say that the right choice of W may be hard to find even in the case some inside in the structure of the control system is given. However, the design allows to combine analytic reasoning with efficient computating. In Section IV, an example is given for the well-known class of systems in strict feedback form for which also backstepping [19] can be applied and in which also a lower triangular (backstepping) structure of the control system (1) appears. Using the Ansatz above and the fact that the inverse of W is polynomial, since $det(W) = 1$, the following state feedback design method is proposed:

Theorem 5: Let W be a given polynomial matrix function as defined in (15). If there exists a polynomial matrix function M and a positive definite matrix Q such that

$$y^T w_x(x)(A(x)W(x)^{-1}Q+G(x)M(x))y \quad < \quad 0, \quad (16)$$

for all nonzero x,y. Then $u = K(x)x$ with $K(x) = M(x)Q^{-1}W(x)$ is a globally asymptotically stabilizing state feedback for the control system (1).

Proof. The change of coordinates $y = Q^{-1}W(x)z$, leads to

$$z^T W^T(x)Q^{-1}w_x(x)(A(x)W(x)^{-1}Q \quad + \\ G(x)M(x))Q^{-1}W(x)z \quad < \quad 0.$$

With $z = x$ and $K(x) = M(x)Q^{-1}W(x)$ one obtains

$$x^T W^T(x)Q^{-1}w_x(x)(A(x)x + G(x)K(x))x \quad < \quad 0,$$

which is nothing else then the derivative of the Lyapunov function $V(x) = \frac{1}{2}w(x)^T Q^{-1}w(x)$ w.r.t. the control system (1). □

Summarizing, Theorem 2, 4, and 5 give different variations of state feedback design strategies based on nonquadratic Lyapunov functions. Of course, the nonquadratic Lyapunov functions are motivated by quadratic Lyapunov functions. However, the problems with general nonquadratic Lyapunov functions like no guaranteed stability or computational complexity motivates the use such type of Lyapunov functions as considered here.

III. Observer Design

The following problem is considered in this section:

State Observer Problem Given a control system of the form

$$\begin{aligned} \dot{x} &= f(x) + \phi(y,u), \\ y &= Cx \end{aligned} \tag{17}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, and $y \in \mathbb{R}^q$ the output. f is a polynomial vector field with $f(0) = 0$ and C is a constant matrix of appropriate dimension. Finally ϕ is an general (nonpolynomial) function. Furthermore, given a Luenberger-type observer

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + \phi(y,u) + L(y-\hat{y},y)(\hat{y}-y) \\ \hat{y} &= C\hat{x}. \end{aligned} \tag{18}$$

Find an observer gain $L = L(y-\hat{y},y)$, such that the observer error $e = \hat{x} - x$ is asymptotically stable. Note that in contrast to state feedback design, almost no attention has been paid to state observer design for the class of polynomial control systems, semidefinite programming and the sum of squares decomposition. A standard approach to obtain a globally asymptotically stabilizing observer gain is to stabilize the error dynamics

$$\dot{e} \quad = \quad f(x+e) - f(x) + L(Ce,y)Ce \tag{19}$$

by the observer gain $L = L(Ce,y)$. Basically, the problem boils down to find an observer gain $L = L(Ce,y)$ and (observer) Lyapunov function $V = V(e)$ such that

$$\nabla V(e)(f(x+e) - f(x) + L(Ce,y)Ce) < 0 \tag{20}$$

for all x and all nonzero e. This is definitely a hard problem. However, almost all Luenberger-type observer design strategies are based on stabilizing the error dynamics. This is often achieved under particular assumptions and often by using a quadratic Lyapunov function V. Instead of introducing a lot of assumptions, one can directly try to solve inequality (20) using semidefinite programming and the sum of squares decomposition. But this is not possible, since in inequality (20), the unknown V,L does not appear in a linear fashion. Therefore, semidefinite programming and the sum of squares decomposition cannot be applied directly. However, similar to the state feedback problem, one can try to find a formulation which is as general as possible but affine in the unknowns. This is worked out in the next theorem:

Theorem 6: If there exists a polynomial matrix function M, an (observer) Lyapunov function V and a positive definite matrix function $Q = Q(Ce)$ such that

$$\begin{aligned} &\nabla V(e) = e^T Q(Ce) \\ &e^T Q(Ce)(f(x+e) - f(x)) + e^T M(Ce,y)Ce < 0 \end{aligned} \tag{21}$$

for all x and all nonzero e. Then $L(Ce,y) = Q(Ce)^{-1}M(Ce,y)$ is a globally asymptotically stabilizing observer gain for the observer error (19).

Proof. The proof follows from $Q(Ce)L(Ce,y) = M(Ce,y)$ and by the Lyapunov inequality

$$\nabla V(e)(f(x+e) - f(x) + L(Ce,y)Ce) < 0. \square$$

Notice that in contrast to the state feedback design, an inversion of $Q(Ce)^{-1}$ is not necessary. Therefore, from this point of view, the observer design is easier than the the state feedback design and allows “more” nonquadratic Lyapunov function candidates. However, a general Lyapunov function V seems to be not feasible since one has to ensure that L contains only measurable quantities. This is the reason why Q depends only on the output error $\hat{y} - y$ and L depends on $\hat{y} - y$ and y. Moreover, in some sense, the inequality in (21) in Theorem 6 is not very satisfactory, since it is hard to say when or for which class it is possible to find an observer gain. Notice, that up to now, no observability assumptions are made. However, complementary to an analytic design strategy, the spirit of a computer-aided design strategy is to allow a more general setup but instead one cannot say in advance that the computer will give a (satisfactory) solution until to put the problem on the computer and to run the algorithms. Nevertheless, it is needless to say that it is very important to have a kind of ”lower bound for success”. The next theorem is such a kind of result.

Theorem 7: Let the control system (17) be of the form

$$\begin{aligned} \dot{x} &= Ax + G\Phi(x,y) + \phi(y,u), \\ y &= Cx. \end{aligned} \tag{22}$$

Assume that Φ is Lipschitz in x, i.e.,

$$\|\Phi(x,y) - \Phi(\hat{x},y)\| \leq \gamma(y)\|x - \hat{x}\|. \tag{23}$$

and assume that the linear control system

$$\begin{aligned} \dot{\xi} &= (A^T - L_0 C)\xi + Gv \\ \eta &= C\xi \end{aligned} \tag{24}$$

can be (strictly) passified via a constant matrix L_0 and a (storage) Lyapunov function $S = \xi^T Q \xi$, i.e., there exist positive definite matrices Q, P and a matrix L_0 such that: $Q(A^T - L_0 C) + (A^T - L_0 C)^T Q = -P < 0$ and $QG = C^T$. Then, inequality (21) in Theorem 6 has a solution.

Proof. It has to be shown that inequality (22) for the control system 24 has a solution, i.e.,

$$\nabla V(e)(Ae + G(\Phi(x+e,y) - \Phi(x,y)) + L(Ce,y)Ce) < 0.$$

To show this, the Lyapunov function V is of the form $V(e) = \frac{1}{2}e^T Qe$ and $L(Ce,y) = L_0 + L_1(y)$. Hence, one obtains

$$\begin{aligned} &eQ(A - L_0 C)e + \\ &eQG(\Phi(x+e,y) - \Phi(x,y)) + eQL_1(y)Ce < 0 \end{aligned}$$

and by using $Q(A^T - L_0 C) + (A^T - L_0 C)^T Q = -P < 0$, $QG = C^T$, one arrives at

$$-\frac{1}{2}ePe + eC(\Phi(x+e,y) - \Phi(x,y)) + eQL_1(y)Ce) < 0.$$

The expression $eC(\Phi(x+e,y) - \Phi(x,y))$ can be upper bounded by using standard arguments for Lipschitz nonlinearities. In particular, the Cauchy-Schwarz inequality $a^T b \leq \|a\|\|b\|$ and the Young inequality $\|a\|\|b\| \leq \varepsilon\|a\|^2 + \frac{1}{\varepsilon}\|b\|^2$, $\varepsilon > 0$ in combination with the Lipschitz assumption $\|\Phi(x,y) - \Phi(x+e,y)\| \leq \gamma(y)\|e\|$ leads to

$$-\frac{1}{2}ePe + \varepsilon e^T e + \frac{\gamma(y)^2}{\varepsilon}e^T C^T Ce + eQL_1(y)Ce < 0.$$

With $L_1(y) = -\frac{\gamma(y)^2}{\varepsilon}Q^{-1}C^T$, one finally arrives at

$$-\frac{1}{2}ePe + \varepsilon e^T e < 0.$$

Hence, by ε sufficiently small, the inequality holds. □

Summarizing, Theorem 6 allows to design state observer with the help of semidefinite programming and the sum of squares decomposition by using nonquadratic Lyapunov functions. Theorem 7 provides a class of control system, where the proposed observer design is guaranteed feasible in case of polynomial nonlinearities, by using a quadratic observer Lyapunov function. Finally, the proposed observer design strategy can be extended to control systems which contains nonpolynomial but Lipschitz nonlinearities by combination with Lipschitz observers [4]. Furthermore, even constraints for example positivity of states and implicit systems of the form $f(x,\dot{x},u) = 0$ can be incooperated in the design by using S-procedure type arguments (cf. e.g. [12]).

IV. Examples

Feedback Design. The following simple example illustrates the Krasovskii-type feedback design strategy present in Theorem 5. In [20], it was shown that the simple control system in backstepping form

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 \\ \dot{x}_2 &= u \end{aligned}$$

does not have a solution for the state dependend Riccati approach. However, using the backstepping idea, it is clear that if x_2 can be driven to the manifold $x_2 + x_1 + x_1^2 = 0$, stability can be easily established. This inside can be used in the Krasovskii-type feedback design strategy by specifying

$$w(x) = W(x)x = \begin{bmatrix} 1 & 0 \\ 1+x_1 & 1 \end{bmatrix} x = \begin{bmatrix} x1 \\ x_1 + x_1^2 + x_2 \end{bmatrix}.$$

Note that w vanishes only if $x_1 + x_1^2 + x_2 = 0$. The solution of inequality (16) was calculated by using the software package SOSTOOLS [14]. The results, where M was chosen to be a polynomial of degree one, are

$$\begin{aligned} V(x) &= 1.7x_1^2 + 2.2x_1^3 + 2.2x_2x_1 + 1.5x_1^4 + 3x_2x_1^2 + 1.5x_2^2 \\ u &= K(x)x = -1.5x_1 - 2.4x_1^2 - 2x_1^3 - 2.4x_2 - 2x_2x_1. \end{aligned}$$

Notice the nonconvex level sets of V (Fig. 1). Finally, remark that the Krasovskii-type feedback design strategy may be especially helpful for control systems of the form $\dot{x} = Ax + f(x) + G(x)u$, where f has backstepping structure. Moreover, notice that one can use in an analogous way an upper triangular W to apply the ideas of forwarding design.

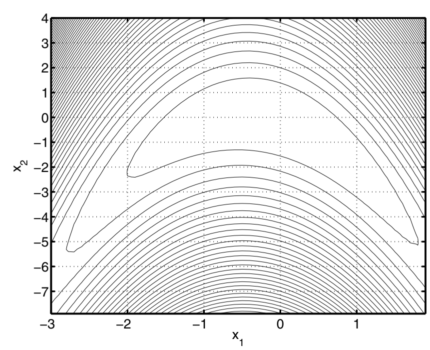

Fig. 1. Nonconvex level sets of the Lyapunov Function V.

Observer Design. A polynomial predator-prey model of HIV-1 is given by [2]:

$$\begin{aligned} \dot{x} &= a(x_0 - x) - bxz \\ \dot{y} &= c(y_0 - y) + dyz \\ \dot{z} &= z(ex - fy), \end{aligned}$$

with the (normalized) parameters $a = 0.25$, $b = 0.01$, $c = 0.25$, $d = 0.002$, $e = 0.01$, $f = 0.006$, $x_0 = 1000$, $y_0 = 550$ and where x, y is the CD4 and CD8 lymphocyte population and z is the HIV-1 viral load. It is assumed that the virual load z is measurable. Notice, that the linearization of the model is not observable. However, a nonlinear observer gain via solving inequality (21) was found using the software package SOSTOOLS [14]. The results, where M was chosen to be a polynomial of degree one and Q constant, are:

$$V(e) = e^T Q e = e^T \begin{bmatrix} 1.06 & 0.09 & -0.46 \\ 0.09 & 0.33 & -0.13 \\ -046 & -0.13 & 0.47 \end{bmatrix} e$$

$$L(z) = Q^{-1} M(z) = \begin{bmatrix} -1.88 - 0.034z \\ -0.94 + 0.069z \\ -5.25 + 0.013z \end{bmatrix}.$$

Simulation results for the initial condition $[x_0\ y_0\ z_0] = [1000\quad 550\quad 0.0003]$, $[\hat{x}_0\ \hat{y}_0\ \hat{z}_0] = [1300\quad 165\quad 0.00003]$ are shown below in Fig. 2 and 3.

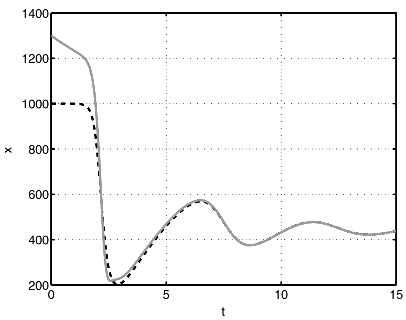

Fig. 2. x: System (solid), Observer (dashed).

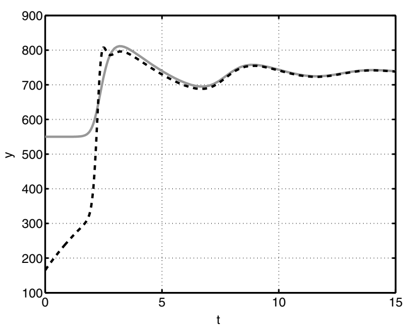

Fig. 3. y: System (solid), Observer (dashed).

V. SUMMARY

In the present paper, new state feedback and observer design strategies based on semidefinite programming and the sum of squares decomposition are proposed for the class of polynomial control systems. The proposed design strategies uses nonquadratic Lyapunov functions without jeopardizing global stability. In particular, homogeneous Lyapunov functions and generalized Krasovskii-type Lyapunov functions are studied for state feedback design and Lyapunov functions which are nonquadratic with respect to the control system output are studied for observer design. One may argue that in some sense a kind of quadratic structure allways appears in the proposed Lyapunov functions. This is true, however, one has to keep in mind the many so called analytic design strategies use often quadratic Lyapunov functions or the assumptions on the control system structure are strong enough to construct Lyapunov functions easily. In contrast to that, the advantages of the proposed design strategies are that no special requirements on the system structure are imposed and that a computer-aided design is possible which leads to global stability results. Finally, it should be pointed out that the results derived here may be of special interest for an incremental stability analysis [10], [13], with applications, for example, to synchronization problems. In particular, the conservatism introducted in the present state feedback design in order to get affine dissipation inequalities is in an incremental stability analysis "necessary", since the notion of incremental stability is a stronger notion than the ordinary notion of stability [10], [13].

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