Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Third-Order Nilpotency, Finite Switchings and Asymptotic Stability

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***Abstract*— We show that a switched system generated by a pair of globally asymptotically stable nonlinear vector fields, which span a third-order nilpotent Lie algebra, is globally asymptotically stable under arbitrary switching. This generalizes a known fact for switched linear systems and provides a partial solution to the open problem posed in [1]. To prove the result, we consider an optimal control problem which consists of finding the "most unstable" trajectory for an associated control system. We use the Agrachev-Gamkrelidze second-order maximum principle to show that there always exists an optimal control that is piecewise constant with no more than four switches. This property is obtained as a special case of a reachability result by piecewise constant controls that is of independent interest. By construction, our criterion also holds for the more general case of differential inclusions.**

Keywords: Switched nonlinear system, global asymptotic stability, Lie bracket, optimal control, maximum principle, differential inclusion.

I. INTRODUCTION

Consider the *differential inclusion* (DI)

$$\dot{\boldsymbol{x}} \in \operatorname{co}\{\boldsymbol{f}_0(\boldsymbol{x}), \boldsymbol{f}_1(\boldsymbol{x})\} \tag{1}$$

where $\boldsymbol{f}_0, \boldsymbol{f}_1 : \mathbb{R}^n \to \mathbb{R}^n$ are two analytic vector fields and co denotes the convex hull. A solution of (1) is an absolutely continuous function $\boldsymbol{x}(\cdot) : \mathbb{R} \to \mathbb{R}^n$ satisfying (1) for (almost) all t. In particular, the set of solutions of (1) includes all the solutions of the *switched system*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_\sigma(\boldsymbol{x}), \tag{2}$$

where $\sigma : [0, \infty) \to \{0, 1\}$ is a piecewise constant function of time, called a *switching signal*. Switched systems have numerous applications and present a subject of extensive ongoing research (see, e.g., [2]).

An important special case, which we will occasionally use for illustration, is when the given vector fields are linear: $\boldsymbol{f}_i(\boldsymbol{x}) = A_i \boldsymbol{x}$, with $A_i \in \mathbb{R}^{n \times n}$. Then (2) becomes the *switched linear system*

$$\dot{\boldsymbol{x}} = A_\sigma \boldsymbol{x}. \tag{3}$$

The DI (1) is called *globally asymptotically stable* (GAS) if there exists a class $\mathcal{KL}$ function[1] β such that for every initial condition $\boldsymbol{x}(0)$ *every* solution of (1) satisfies

$$|\boldsymbol{x}(t)| \le \beta(|\boldsymbol{x}(0)|, t) \qquad \forall\, t \ge 0. \tag{4}$$

This work was supported by the ISF under grant number 199/03.

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[1]Recall that a function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(s, t)$ decreases to 0 as $t \to \infty$ for each fixed $s \ge 0$.

If the DI is GAS, then the switched system (2) is globally asymptotically stable, uniformly over the set of all switching signals, because solutions of (2) are contained in those of (1). We then say that (2) is *globally uniformly asymptotically stable* (GUAS). Determining a necessary and sufficient condition for GUAS is a formidable challenge. In fact, for the special case (3) this is equivalent to solving one of the oldest open problems in the theory of control: the problem of absolute stability (see, e.g., [3]).

It is well known and easy to demonstrate that global asymptotic stability of the individual subsystems $\dot{\boldsymbol{x}} = \boldsymbol{f}_i(\boldsymbol{x})$ is necessary but *not* sufficient for GUAS of the switched system (2). In this paper, we are concerned with the problem of identifying conditions for the individual subsystems—apart from the obviously necessary requirement as to their global asymptotic stability—which guarantee GUAS of (2). This problem has received considerable attention in the literature; see [2, Chapter 2] for some available results.

The difficulty in analyzing the stability of (2) is that the switched system admits an infinite number of solutions for each initial condition. A natural idea is to try to characterize the "worst-case" (that is, the "most unstable") switching law, and then analyze the behavior of the unique trajectory produced by this law. This approach led to many important results in the context of the absolute stability problem [4], [5]. For the particular case of second-order switched linear systems, this approach yields an easily verifiable necessary and sufficient condition for GUAS [6].

Another particularly promising research avenue is to explore the role of commutation relations among the subsystems being switched. We now briefly review available results, starting with the case of the switched linear system (3). The commutator, or *Lie bracket*, is defined as $[A_0, A_1] := A_0 A_1 - A_1 A_0$. We say that the Lie algebra spanned by the pair A_0, A_1 is *kth-order nilpotent* if all iterated Lie brackets containing $k+1$ terms vanish, and there exists a Lie bracket containing k terms that does not vanish. It was shown in [7] that if the Lie algebra is nilpotent (in fact, solvable) of *any* order, then the linear switched system is GUAS (see also [8]).

The nonlinear switched system (2) is much less thoroughly understood. To tackle the global stability question, one can try to inspect commutation relations between the nonlinear vector fields $\boldsymbol{f}_0, \boldsymbol{f}_1$. The *Lie bracket* is now defined as

$$[\boldsymbol{f}_1, \boldsymbol{f}_0](\boldsymbol{x}) := \frac{\partial \boldsymbol{f}_1(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}_0(\boldsymbol{x}) - \frac{\partial \boldsymbol{f}_0(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}_1(\boldsymbol{x}). \tag{5}$$

We say that the Lie algebra spanned by $\boldsymbol{f}_0$ and $\boldsymbol{f}_1$ is kth-order nilpotent if all iterated Lie brackets containing $k+1$ terms vanish and there exists a Lie bracket containing k

terms that does not vanish. Mancilla-Aguilar [9] proved that if the Lie algebra is first-order nilpotent (that is, the two vector fields commute), then global asymptotic stability of the individual subsystems still implies GUAS of the switched system (2).

Until very recently, all attempts to formulate global asymptotic stability criteria valid beyond the commuting nonlinear case have been unsuccessful. This is due to the fact that the methods employed to obtain the corresponding results for switched linear systems do not seem to apply. These issues are explained in [1], where this is presented as an open problem which seems to require an entirely different approach.

It is a well-known fact that Lie brackets play an essential role in the Maximum Principle (MP) (see, e.g. [10][11]), which is an important tool in analyzing the "worst-case" switching law [4][5]. This suggests that the two approaches for stability analysis described above are actually related. Indeed, using a combination of both approaches, it was recently proved that if the Lie algebra is second-order nilpotent then there always exists a "worst-case" switching law with no more than two switches [12]. Then GUAS of the switched system can be deduced from global asymptotic stability of the individual subsystems. This approach provided the first stability criterion for switched nonlinear systems that involves Lie brackets of the individual vector fields but does not require that these vector fields commute.

In this paper, we extend this approach to the case where the Lie algebra spanned by $\boldsymbol{f}_0$ and $\boldsymbol{f}_1$ is third-order nilpotent, that is, $[\boldsymbol{f}_i, [\boldsymbol{f}_j, [\boldsymbol{f}_k, \boldsymbol{f}_l]]](\boldsymbol{x}) = 0, \forall i, j, k, l \in \{0, 1\}$. Our main result (Theorem 1) is that this property implies that the DI (1) is GAS. In fact, this follows as a special case of a more general "reachability with nice controls" result (Theorem 5) which is of independent interest.

An analysis of the arguments in [12] shows that this result cannot be derived using the (classical) MP. Hence, we apply a new approach based on a powerful second-order MP developed by Agrachev and Gamkrelidze [13] (see also [14]).

II. OPTIMAL CONTROL APPROACH

Our starting point is to rewrite the differential inclusion (1) as the control system with drift

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u, \qquad u \in \mathcal{U} \tag{6}$$

where $\boldsymbol{f}(\boldsymbol{x}) := \boldsymbol{f}_0(\boldsymbol{x})$, $\boldsymbol{g}(\boldsymbol{x}) := \boldsymbol{f}_1(\boldsymbol{x}) - \boldsymbol{f}_0(\boldsymbol{x})$ and $\mathcal{U}$ is the set of measurable functions $u(\cdot) : \mathbb{R} \to [0, 1]$.

It is obvious that every solution of (6) is also a solution of (1). It follows from Filippov's Selection Lemma (see, e.g., [15, Theorem 2.3.13]) that every solution of (1) is also a solution of (6). Thus, the set of solutions of the control system (6) coincides with the set of solutions of the differential inclusion (1).

Note that trajectories of the switched system (2) correspond to *piecewise constant* controls taking values in $\{0, 1\}$. In particular, setting $u \equiv 0$ [$u \equiv 1$] in (6) yields $\dot{\boldsymbol{x}} = \boldsymbol{f}_0(\boldsymbol{x})$ [$\dot{\boldsymbol{x}} = \boldsymbol{f}_1(\boldsymbol{x})$]. We also remark that the switched linear system (3) is associated in this way with the bilinear control system $\dot{\boldsymbol{x}} = A_0\boldsymbol{x} + (A_1 - A_0)\boldsymbol{x}u$.

Fix an arbitrary point $\boldsymbol{p} \in \mathbb{R}^n$, and let $\boldsymbol{x}(\cdot; \boldsymbol{p}, u)$ denote the solution of the system (6) with initial condition $\boldsymbol{x}(0) = \boldsymbol{p}$ corresponding to a control $u \in \mathcal{U}$. Since the right-hand side of (6) is bounded on every bounded ball in $\mathbb{R}^n$, there exists a largest time $T_{\max} \in (0, \infty]$ (that depends on $|\boldsymbol{p}|$) such that $\boldsymbol{x}(\cdot; \boldsymbol{p}, u)$ is well defined for all $u \in \mathcal{U}$ and all $t \in [0, T_{\max})$. Picking a positive final time $t_f < T_{\max}$, we define

$$J(t_f, \boldsymbol{p}, u) := |\boldsymbol{x}(t_f; \boldsymbol{p}, u)|^2,$$

and pose the following optimal control problem.

Problem 1: Find a control $u \in \mathcal{U}$ that maximizes J along the solutions of (6).

It follows from [16, §7, Theorem 3] that this problem is well posed, i.e., an optimal control u^* does exist.

The intuitive interpretation of Problem 1 is clear: find a control that "pushes" the corresponding trajectory $\boldsymbol{x}^*$ as far away from the origin as possible (from a given initial condition in a given amount of time). If we can show that $\boldsymbol{x}^*$ satisfies the bound (4), then the same will be true for any other trajectory of the DI, and stability of the DI—as well as the switched system—will be established.

III. MAIN RESULT

For $\tau, k > 0$ we let $\mathcal{PC}(\tau, k) \subset \mathcal{U}$ denote the set of piecewise constant controls that have no more than k switches over the interval $[0, \tau]$. We are now ready to state our main result.

Theorem 1: Consider Problem 1 with $\boldsymbol{x}(0) = \boldsymbol{p}$ and $t_f < T_{\max}(\boldsymbol{p})$. If $\{\boldsymbol{f}_0, \boldsymbol{f}_1\}_{LA}$ is third-order nilpotent then there exists an optimal control u^* that satisfies $u^* \in \mathcal{PC}(t_f, 4)$.

Note that there is rich literature on conditions guaranteeing that optimal controls are, in some sense, regular;[2] see e.g. [17][18] and the references therein. However, many of these results are local. That is, they guarantee that there exists a (sufficiently small) $T > 0$ such that the restriction of the optimal control to the interval $[0, T]$ is regular. In contrast, our result is global, as the bound on the number of switches *does not depend* on t_f.

The next result will allow us to apply Theorem 1 to the stability analysis of (1). Loosely speaking, it states that to obtain instability in a DI, composed of GAS subsystems, we must never stop switching.

Proposition 1: Suppose that: (1) for any *fixed* $c \in [0, 1]$ the system $\dot{\boldsymbol{x}} = \boldsymbol{h}_c(\boldsymbol{x})$ is GAS; and (2) there exists a finite integer d such that for *any* $t_f < T_{max}$ there exist an optimal control $u^* \in \mathcal{PC}(t_f, d)$. Then the DI (1) is GAS.

proof. See [12, §4].

Combining Theorem 1 and Proposition 1 yields:

Corollary 1: Suppose that: (1) for any *fixed* $c \in [0, 1]$ the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + c\boldsymbol{g}(\boldsymbol{x})$ is GAS; and (2) $\{\boldsymbol{f}_0, \boldsymbol{f}_1\}_{LA}$ is third-order nilpotent. Then the DI (1) is globally asymptotically stable, and in particular the switched system (2) is GUAS.

[2] e.g., a finite combination of bang-bang and singular arcs.

The remainder of this paper is devoted to the proof of Theorem 1. We first consider the case where u^* is bang-bang and then consider the singular case.

IV. THE BANG-BANG CASE

The next result follows from applying the MP to Problem 1 We use the notation $D\boldsymbol{f} := \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$.

Theorem 2: Let u^* be an optimal control for Problem 1. Define the *costate* $\boldsymbol{\lambda} : [0, t_f] \to \mathbb{R}^n$ by

$$\dot{\boldsymbol{\lambda}} = -\left(Df(\boldsymbol{x}^*) + u^* D\boldsymbol{g}(\boldsymbol{x}^*)\right)^T \boldsymbol{\lambda}, \qquad \lambda(t_f) = \boldsymbol{x}^*(t_f), \tag{7}$$

and let $\varphi(t) := \boldsymbol{\lambda}^T(t)\boldsymbol{g}(\boldsymbol{x}^*(t))$. Then

$$u^*(t) = \begin{cases} 1 & \text{if } \varphi(t) > 0 \\ 0 & \text{if } \varphi(t) < 0. \end{cases} \tag{8}$$

In this section, we assume that $\varphi(t) = 0$ holds only for isolated points, so any optimal control u^* must be bang-bang.

A. First-order analysis

The next result will be used in the sequel. For easy reference, we state it formally.

Fact 1: If $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field, and

$$\psi(t) := \boldsymbol{\lambda}^T(t)\boldsymbol{h}(\boldsymbol{x}(t)), \tag{9}$$

then

$$\dot{\psi}(t) = \boldsymbol{\lambda}^T(t)([\boldsymbol{h}, \boldsymbol{f}] + u(t)[\boldsymbol{h}, \boldsymbol{g}])(\boldsymbol{x}(t)). \tag{10}$$

Proof: Follows immediately from differentiating the (absolutely continuous) function ψ and using (6), (7) and (5). ■

The next result relates $\varphi(t)$ to the Lie algebra spanned by $\boldsymbol{f}$ and $\boldsymbol{g}$.

Proposition 2: Suppose that the Lie algebra spanned by the vector fields $\boldsymbol{f}$ and $\boldsymbol{g}$ is kth-order nilpotent for some integer $k > 0$. If $I \subseteq [0, t_f]$ is an interval such that $u^*(t) \equiv c$ for all $t \in I$, then the restriction of $\varphi(t)$ to I is a polynomial in t with degree $< k$.

Proof: It follows from the definition of φ and Fact 1 that $\dot{\varphi} = \boldsymbol{\lambda}^T[\boldsymbol{g}, \boldsymbol{f}](\boldsymbol{x})$. Differentiating again and again, using Fact 1, we deduce that $\varphi^{(k)}$ contains iterated Lie brackets of $\boldsymbol{f}$ and $\boldsymbol{g}$ with $k+1$ terms, so $\varphi^{(k)} \equiv 0$ on I. Using the absolute continuity of φ (and its derivatives) on I we conclude that the restriction of $\varphi(t)$ to I is a polynomial in t with degree $< k$. ■

Proposition 2 implies that if u^* is piecewise constant, then the corresponding $\varphi(t)$ is *piecewise polynomial* in t, and every polynomial has a degree $< k$. This result is closely related to the fact that control systems with a nilpotent Lie algebra can be represented using a differential equation with polynomial vector fields [19].

Proposition 3: If the Lie algebra spanned by $\boldsymbol{f}$ and $\boldsymbol{g}$ is third-order nilpotent, and $\varphi(t) = 0$ holds only on isolated points, then u^* is bang-bang and either: (1) u^* is periodic; or (2) u^* contains no more than two switches for all $t_f < T_{max}$.

Proof: It is sufficient to prove that any bang-bang control u^* with more than two switches is periodic. Thus, suppose that u^* has switches at times $\tau_1 < \tau_2 < \tau_3$. We assume, without loss of generality, that $\tau_1 > 0$, $\tau_3 < t_f$, and that $u^*(t) = 0$ for $t \in [0, \tau_1)$.

Differentiating φ and applying Fact 1 yields

$$\dot{\varphi}(t) = a(t), \qquad \ddot{\varphi}(t) = b + u(t)c, \tag{11}$$

where

$$\begin{aligned} a(t) &= \boldsymbol{\lambda}^T(t)[\boldsymbol{g}, \boldsymbol{f}](\boldsymbol{x}^*(t)) \\ b &= \boldsymbol{\lambda}^T(t)[[\boldsymbol{g}, \boldsymbol{f}], \boldsymbol{f}](\boldsymbol{x}^*(t)) \\ c &= \boldsymbol{\lambda}^T(t)[[\boldsymbol{g}, \boldsymbol{f}], \boldsymbol{g}](\boldsymbol{x}^*(t)) \end{aligned} \tag{12}$$

(the fact that b and c do not depend on t is an immediate consequence of Fact 1 and the absolute continuity of $\boldsymbol{x}^*(t)$ and $\boldsymbol{\lambda}(t)$). Note that (12) implies that $\dot{\varphi}$ is absolutely continuous.

Let $\varphi_i(t)$ denote the restriction of the absolutely continuous function φ on the interval $[\tau_i, \tau_{i+1})$. Combining Proposition 2 with the fact that φ must vanish on the switching points yields $\varphi_i = p_i(t - \tau_i)(t - \tau_{i+1})$, for some $p_i \in \mathbb{R}$, so $\dot{\varphi}_i(t) = p_i(2t - \tau_i - \tau_{i+1})$. This implies that $\dot{\varphi}_i(\tau_{i+1}) = -\dot{\varphi}_i(\tau_i)$, so $a(\tau_3) = -a(\tau_2) = a(\tau_1)$.

It now follows from (11) that $\varphi^{(j)}(\tau_1) = \varphi^{(j)}(\tau_3)$ for $j = 0, 1, 2$. Since φ is composed of second-order polynomials this implies that φ is periodic. Eq. (8) implies that u^* is periodic. ■

The analysis based on the classical, first-order, MP provides considerable information on u^*. However, the fact that u^* might be periodic implies that the number of switches can increase as a function of t_f.

B. Second-order analysis

In this section, we apply a second-order MP [13] to prove that in the non-singular case, any bang-bang control with more than three switches is not optimal. To make this paper self-contained, we provide an independent proof for our particular case. Due to space limitations, and to make the proof more transparent, from here on we consider the special case of linear vector fields: $\boldsymbol{f}_0(\boldsymbol{x}) = A\boldsymbol{x}$ and $\boldsymbol{f}_1(\boldsymbol{x}) = B\boldsymbol{x}$. The generalization to the nonlinear case is not difficult, as all the tools that we use hold for nonlinear vector fields as well.

Assume that there exits an optimal control u^* with exactly four switches $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < t_f$. Without loss of generality, we assume that $u^*(t) = 0$ on $t \in [0, \tau_1)$. For notational convenience, we define $\tau_0 := 0$ and $\tau_5 := t_f$ (note, however, that these are not switching points). Then $\boldsymbol{x}^*(t_f) = \exp(Aq_5)\exp(Bq_4)\exp(Aq_3)\exp(Bq_2)\exp(Aq_1)\boldsymbol{p}$, where $q_i := \tau_i - \tau_{i-1}$.

We define a set $\mathcal{S}^5 \subset \mathbb{R}^5$ by

$$\mathcal{S}^5 := \{\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_4)^T : \sum_{i=0}^{4} \alpha_i = 0\}. \tag{13}$$

For $\boldsymbol{\alpha} \in \mathcal{S}^5$ and $s > 0$ we define a new control $\tilde{u}(t; \boldsymbol{\alpha}, s)$ by perturbing the switching times of u^* to $\tilde{\tau}_i := \tau_i + s\sum_{k=0}^{i-1}\alpha_i$, $i = 1, 2, \ldots, 5$. In other words, $\tilde{u}(t) = 0$ for $t \in [0, \tilde{\tau}_1)$, $\tilde{u}(t) = 1$ for $t \in [\tilde{\tau}_1, \tilde{\tau}_2)$, and so on. Note that (13) implies that $\tilde{\tau}_5 = \tau_5 = t_f$, that is, the final time is not changed.

It is clear that for any $\boldsymbol{\alpha}$, there exists a sufficiently small $s_0 > 0$ such that $\tilde{u}(s, \boldsymbol{\alpha}) \in \mathcal{U}$ for all $s \in [0, s_0]$. In other words, $\tilde{u}(s, \boldsymbol{\alpha})$ is an admissible control for all sufficiently small s. The corresponding trajectory satisfies

$$\tilde{\boldsymbol{x}}(t_f; s, \boldsymbol{\alpha}) = \exp(A\tilde{q}_5)\exp(B\tilde{q}_4)\exp(A\tilde{q}_3)\exp(B\tilde{q}_2)\exp(A\tilde{q}_1)\boldsymbol{p}, \tag{14}$$

where $\tilde{q}_i := \tilde{\tau}_i - \tilde{\tau}_{i-1} = q_i + s\alpha_{i-1}$.

It is easy to verify that

$$\begin{aligned}\tilde{\boldsymbol{x}}(t_f; s, \boldsymbol{\alpha}) - \boldsymbol{x}^*(t_f) &= \exp(Aq_5)\exp(Bq_4)\\ &\times \exp(Aq_3)\exp(Bq_2)V(s, \boldsymbol{\alpha})\exp(Aq_1)\boldsymbol{p},\end{aligned} \tag{15}$$

where

$$\begin{aligned}V(s, \boldsymbol{\alpha}) &:= \exp(-Bq_2)\exp(-Aq_3)\exp(-Bq_4)\exp(As\alpha_4)\\ &\times \exp(B\tilde{q}_4)\exp(A\tilde{q}_3)\exp(B\tilde{q}_2)\exp(As\alpha_0) - I.\end{aligned} \tag{16}$$

Note that $V(0, \boldsymbol{\alpha}) = 0$. Using the definition of the costate (7), we get

$$\begin{aligned}&(\boldsymbol{x}^*(t_f))^T(\tilde{\boldsymbol{x}}(t_f; s, \boldsymbol{\alpha}) - \boldsymbol{x}^*(t_f))\\ &= \boldsymbol{\lambda}^T(t_f)(\tilde{\boldsymbol{x}}(t_f; s, \boldsymbol{\alpha}) - \boldsymbol{x}^*(t_f))\\ &= \boldsymbol{\lambda}^T(\tau_1)V(s, \boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1).\end{aligned} \tag{17}$$

This yields the following necessary condition for the optimality of u^*.

Proposition 4: If u^* is optimal then

$$\lim_{s \to 0^+} \frac{1}{s}\boldsymbol{\lambda}^T(\tau_1)V(s, \boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1) \le 0, \qquad \forall \boldsymbol{\alpha} \in \mathcal{S}^5. \tag{18}$$

Proof: Seeking a contradiction, assume that (18) does not hold for some $\boldsymbol{\alpha}^0 \in \mathcal{S}^5$. This implies that there exists $s_0 > 0$ such that $\tilde{u}(s_0, \boldsymbol{\alpha}^0) \in \mathcal{U}$ and $\boldsymbol{\lambda}^T(\tau_1)V(s_0, \boldsymbol{\alpha}^0)\boldsymbol{x}^*(\tau_1) > 0$. Using (17) yields $(\boldsymbol{x}^*(t_f))^T(\tilde{\boldsymbol{x}}(t_f; s_0, \boldsymbol{\alpha}^0) - \boldsymbol{x}^*(t_f)) > 0$. However, this implies that $|\tilde{\boldsymbol{x}}(t_f; s_0, \boldsymbol{\alpha}^0)|^2 > |\boldsymbol{x}^*(t_f)|^2$ and this contradicts the optimality of u^*. ∎

To apply Proposition 4, we will expand $V(s, \boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1)$ as a Taylor series about $s = 0$:

$$V(s, \boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1) = sV_1(\boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1) + s^2V_2(\boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1) + o(s^2),$$

where $o(\epsilon)$ denotes terms that satisfy $\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$.

Theorem 3: Denote:

$$\begin{aligned}H_0 &:= A,\\ H_1 &:= B,\\ H_2 &:= \exp(-Bq_2)A\exp(Bq_2),\\ H_3 &:= \exp(-Bq_2)\exp(-Aq_3)B\exp(Aq_3)\exp(Bq_2),\\ H_4 &:= \exp(-Bq_2)\exp(-Aq_3)\exp(-Bq_4)A\\ &\quad \times \exp(Bq_4)\exp(Aq_3)\exp(Bq_2).\end{aligned}$$

Then

$$V_1(\boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1) = \sum_{i=0}^{4} \alpha_i H_i \boldsymbol{x}^*(\tau_1). \tag{19}$$

Furthermore, for any

$$\boldsymbol{\alpha} \in \mathcal{S}_0^5 := \{\boldsymbol{\alpha} \in \mathcal{S}^5 : \sum_{i=0}^{4} \alpha_i H_i \boldsymbol{x}^*(\tau_1) = 0\}$$

we have

$$V_2(\boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1) = \sum_{i=0}^{4}\sum_{j=i+1}^{4} \alpha_i\alpha_j[H_j, H_i]\boldsymbol{x}^*(\tau_1), \tag{20}$$

and if u^* is optimal then

$$\boldsymbol{\lambda}^T(\tau_1)V_2(\boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1) \le 0, \qquad \forall \boldsymbol{\alpha} \in \mathcal{S}_0^5. \tag{21}$$

Proof: Eqs. (19) and (20) follow directly from the definition of V (16). The proof of (21) is similar to the proof of Proposition 4. ∎

Theorem 3 is the Agrachev-Gamkrelidze MP specialized for our problem. It is important to note that the expression for $V_2(\boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1)$ in (20) is true only for $\boldsymbol{\alpha} \in \mathcal{S}_0^5$, that is, when $V_1(\boldsymbol{\alpha})\boldsymbol{x}^*(\tau_1)$ vanishes. This makes (20) meaningful even in a coordinate-free setting (see [14] for more details).

Combining Proposition 4 and (19) yields

$$\boldsymbol{\lambda}^T(\tau_1)(\sum_{i=0}^{4} \alpha_i H_i)\boldsymbol{x}^*(\tau_1) \le 0, \qquad \forall \boldsymbol{\alpha} \in \mathcal{S}^5, \tag{22}$$

and noting that if $\boldsymbol{\alpha} \in \mathcal{S}^5$ then $-\boldsymbol{\alpha} \in \mathcal{S}^5$ immediately yields the following.

Proposition 5: If the bang-bang control u^* is optimal then

$$\boldsymbol{\lambda}^T(\tau_1)(\sum_{i=0}^{4} \alpha_i H_i)\boldsymbol{x}^*(\tau_1) = 0, \qquad \forall \boldsymbol{\alpha} \in \mathcal{S}^5. \tag{23}$$

To proceed, we use the nilpotency assumption, combined with the celebrated Baker-Campbell-Hausdorff (BCH) formula, to simplify the H_is and their Lie brackets.

Theorem 4: (BCH)

$$\exp(At)B\exp(-At) = B + [A, B]t + [A, [A, B]]\frac{t^2}{2!} + \dots.$$

Note that this result can be considered as a particular case of the *pullback formula* (see, e.g., [20, Appendix I]), which provides a similar expansion for the case of nonlinear vector fields.

Denote $C := [B, A]$, $D := [B, [B, A]]$, and $E := [A, [B, A]]$. Using Theorem 4, and the fact that Lie brackets involving four (or more) terms of A and B vanish, we get

$$\begin{aligned}H_2 &= A - q_2C + \frac{1}{2}q_2^2D\\ H_3 &= B + q_3C - q_2q_3D - \frac{1}{2}q_3^2E\\ H_4 &= H_2 - q_4C + q_4(q_2 + \frac{1}{2}q_4)D + q_3q_4E.\end{aligned} \tag{24}$$

Proposition 6: Let $\boldsymbol{\alpha}^0 := (q_3, 2q_2, 0, -2q_2, -q_3)^T \in \mathcal{S}^5$. Then $\boldsymbol{\alpha}^0 \in \mathcal{S}_0^5$ and $\boldsymbol{\lambda}^T(\tau_1)V_2(\boldsymbol{\alpha}^0)\boldsymbol{x}^*(\tau_1) > 0$.

Proof: It follows from the proof of Proposition 3 that $q_4 = q_2$. Using this and (24), we get $\sum_{i=0}^{4} \alpha_i^0 H_i = 0$, so $\boldsymbol{\alpha}_0 \in \mathcal{S}_0^5$.

Eqs. (20) and (24) yield

$$\begin{aligned}V_2(\boldsymbol{\alpha}^0)\boldsymbol{x}^*(\tau_1) &= \sum_{i=0}^{4}\sum_{j=i+1}^{4} \alpha_i^0\alpha_j^0[H_j, H_i]\boldsymbol{x}^*(\tau_1)\\ &= 2q_2q_3(q_3E + 2q_2D)\boldsymbol{x}^*(\tau_1).\end{aligned} \tag{25}$$

To simplify this expression, let $\boldsymbol{\alpha}^1 := (q_3, q_2, -q_3, -q_2, 0)^T \in \mathcal{S}^5$, then $\sum_{i=0}^4 \alpha_i^1 H_i = \frac{1}{2} q_2 q_3 (q_2 D + q_3 E)$. Proposition 5 yields

$$\boldsymbol{\lambda}^T(\tau_1)(q_2 D + q_3 E)\boldsymbol{x}^*(\tau_1) = 0$$

and substituting this in (25), we get

$$\boldsymbol{\lambda}^T(\tau_1) V_2(\boldsymbol{\alpha}^0)\boldsymbol{x}^*(\tau_1) = 2q_2^2 q_3 \boldsymbol{\lambda}^T(\tau_1) D\boldsymbol{x}^*(\tau_1). \qquad (26)$$

Arguing as in the proof of Proposition 3, we find that $\varphi_1(t) = p_1(t-\tau_1)(t-\tau_2)$. Using the fact that $u^*(t) = 1$ for $t \in [\tau_1, \tau_2)$, and our non-singularity assumption, we see that $p_1 < 0$, so $\ddot{\varphi}_1(t) = 2p_1 < 0$.

On the other hand, $\ddot{\varphi}_1(t) = -\boldsymbol{\lambda}^T(\tau_1) D\boldsymbol{x}^*(\tau_1)$, Thus, $\boldsymbol{\lambda}^T(\tau_1) D\boldsymbol{x}^*(\tau_1) > 0$, and this completes the proof. ■

Combining Proposition 6 and Theorem 3, we conclude that in the non-singular case, any bang-bang control with exactly four switches is *not* optimal. It is easy to see that this implies that in this case u^* is bang-bang with no more than three switches for *all* $t_f < T_{max}$.

V. THE SINGULAR CASE

We now consider the possibility that there exists an interval of time $I \subseteq [0, t_f]$ such that $\varphi(t) \equiv 0$ on I. It follows from (11) that $a(t) = b + u(t)c = 0$ on I. If $c \neq 0$, then this equation uniquely determines u^* along the singular arc. If $c = 0$, then the MP provides no information on u^*. We now consider these cases in detail.

A. CASE 1: $c \neq 0$

In this case, $u^*(t) \equiv -b/c$ on I. Of course, this is possible only if $-b/c \in [0, 1]$. Now suppose that u^* contains a combination of singular and bang-bang arcs. Say, $u^*(t) = -b/c$ for $t \in [\tau_i, \tau_{i+1})$ and $\varphi(t) \neq 0$ for $t \in (\tau_{i+1}, \tau_{i+2})$. In this case, φ_{i+1} (that is, the restriction of φ over the interval $[\tau_{i+1}, \tau_{i+2}]$) is a second-order polynomial and, since $\varphi(\tau_{i+1}) = \dot{\varphi}(\tau_{i+1}) = 0$, we get that $\varphi_{i+1}(t) = p(t - \tau_{i+1})^2$, for some $p \in \mathbb{R}$. This implies that $\varphi(t) \neq 0$ for all $t > \tau_{i+1}$, so $\tau_{i+2} = t_f$. Similarly, if there is a bang-bang arc preceding the singular arc, say, $u^*(t) = 0$ for $t \in [\tau_{i-1}, \tau_i)$ then we must have $\tau_{i-1} = 0$. Thus, the most general configuration possible is a bang-bang arc, followed by a singular arc (on which u^* is constant), and another bang-bang arc. Hence, $u^* \in \mathcal{PC}(t_f, 2)$.

It is interesting to note that in the second-order nilpotent case there always exists an optimal control that is bang-bang [12]. It is possible to show that this is not longer true in the third-order nilpotent case.

B. CASE 2: $c = 0$

In this case, (12) yields

$$\boldsymbol{\lambda}^T(t) P(\boldsymbol{x}^*(t)) = \mathbf{0}, \qquad \forall t \in I \qquad (27)$$

where P is the matrix defined by

$$P(\boldsymbol{x}) := (\boldsymbol{g},\ [\boldsymbol{g}, \boldsymbol{f}],\ [[\boldsymbol{g}, \boldsymbol{f}], \boldsymbol{f}],\ [[\boldsymbol{g}, \boldsymbol{f}], \boldsymbol{g}])(\boldsymbol{x}) \qquad (28)$$

The MP provides no information on the optimal control. To overcome this problem, we will use a construction due to Sussmann [21].

Let $\boldsymbol{\Phi}_f(t, \boldsymbol{p})$ denote the solution at time t of the system $\dot{\boldsymbol{z}}(t) = \boldsymbol{f}(\boldsymbol{z}(t))$, with $\boldsymbol{z}(0) = \boldsymbol{p}$. Define $\boldsymbol{y}(t) := \Phi_f(-t, \boldsymbol{x}(t))$ (where $\boldsymbol{x}(t)$ is the solution of (6)). Then $\boldsymbol{y}(0) = \boldsymbol{x}(0)$ and

$$\dot{\boldsymbol{y}}(t) = u(t)\left(D\Phi_f(t, \boldsymbol{y}(t))\right)^{-1} \boldsymbol{g}\left(\Phi_f(t, \boldsymbol{y}(t))\right). \qquad (29)$$

Consider the control system (6). For an admissible control u, let $\boldsymbol{p} := \boldsymbol{x}(0)$ and $\boldsymbol{q} := \boldsymbol{x}(t_f)$. We then say that u steers $\boldsymbol{x}$ from $\boldsymbol{p}$ to $\boldsymbol{q}$ in time t_f. It follows from the definition of $\boldsymbol{y}$ that u steers $\boldsymbol{y}$ from $\boldsymbol{y}(0) = \boldsymbol{p}$ to $\boldsymbol{q}' := \Phi_f(-t_f, \boldsymbol{q})$ in time t_f.

Consider the problem of finding an admissible control that steers $\boldsymbol{y}$ from $\boldsymbol{p}$ to $\boldsymbol{q}'$ in *minimum time*. Such a control exists and we denote it by $\boldsymbol{w}^*$. Then $\boldsymbol{w}^*$ steers $\boldsymbol{y}$ from $\boldsymbol{p}$ to $\boldsymbol{q}'$ in some time $\bar{t} \leq t_f$. Hence, $\boldsymbol{w}^*$ steers $\boldsymbol{x}$ from $\boldsymbol{p}$ to $\boldsymbol{x}(\bar{t}) = \boldsymbol{\Phi}_f(\bar{t}, \boldsymbol{q}') = \boldsymbol{\Phi}_f(\bar{t}, \Phi_f(-t_f, \boldsymbol{q})) = \boldsymbol{\Phi}_f(\bar{t} - t_f, \boldsymbol{q})$ at time $\bar{t}$. This implies that the control

$$v(t) := \begin{cases} w^*(t), & t \in [0, \bar{t}) \\ 0, & t \in [\bar{t}, t_f] \end{cases} \qquad (30)$$

steers $\boldsymbol{x}$ from $\boldsymbol{p}$ to $\boldsymbol{q}$ in time t_f. Summarizing, we can replace any control that steers $\boldsymbol{p}$ to $\boldsymbol{q}$ at time t_f, with another control, in the form (30), that does the same.

The analysis of the time-optimal control problem for (29) turns out to be similar to the analysis performed above for Problem 1. In particular, it is possible to prove the following result.

Proposition 7: $w^* \in \mathcal{PC}(\bar{t}, 3)$.

Proof: See the appendix.

Combining this with (30), yields the following reachability result:

Theorem 5: Suppose that the conditions of Theorem 1 hold. If there exist $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^n$ and a control $u \in \mathcal{U}$ that steers $\boldsymbol{x}$ from $\boldsymbol{p}$ to $\boldsymbol{q}$ in some time $t \geq 0$, then there exists an admissible control $v \in \mathcal{PC}(t, 4)$ that also steers $\boldsymbol{x}$ from $\boldsymbol{p}$ to $\boldsymbol{q}$ in time t.

It is obvious that Theorem 1 is a corollary of this result, so this completes the proof of Theorem 1.

Note that Theorem 5 has an important practical application. It implies that any point-to-point control problem is reduced to the problem of determining a (small) set of parameters: four switching times and five control values. Of course, the analysis above shows that in many cases the number of unknown parameters is even smaller.

VI. CONCLUSIONS

We studied an optimal control problem for nonlinear differential inclusions and switched systems. We proved that if the vector fields that generate the differential inclusion span a third-order nilpotent Lie algebra, then a piecewise constant optimal control, with no more than four switches over any interval of time, always exists (Theorem 1). This result is a particular case of a more general reachability result (Theorem 5).

This implies that if, in addition, the vector fields are GAS, then so is the differential inclusion (Corollary 1). This is a promising step toward a solution of the open problem described in [1].

ACKNOWLEDGMENTS

We are grateful to Daniel Liberzon and Andrei Agrachev for helpful comments.

APPENDIX

We denote by $(\mathrm{ad}\, \boldsymbol{f})^r$, $r = 0, 1, \ldots$ the operators defined by $(\mathrm{ad}\, \boldsymbol{f})^0(\boldsymbol{g}) := \boldsymbol{g}$ and $(\mathrm{ad}\, \boldsymbol{f})^r(\boldsymbol{g}) := [\boldsymbol{f}, (\mathrm{ad}\, \boldsymbol{f})^{r-1}(\boldsymbol{g})]$ for $r \geq 1$, where $\boldsymbol{f}$ and $\boldsymbol{g}$ are smooth vector fields on $\mathbb{R}^n$.

It follows from the *pullback formula* (see, e.g., [20, Appendix I]), and our nilpotency assumption that (29) can be written as

$$\dot{\boldsymbol{y}}(t) = u(t) \sum_{v=0}^{2} \frac{(-t)^v}{v!} (\mathrm{ad}\, \boldsymbol{f})^v(\boldsymbol{g})(\boldsymbol{y}(t)). \tag{31}$$

This is a time-varying system, and we apply the standard trick of defining a new state-variable y_{n+1} by $\dot{y}_{n+1} = 1$, with $y_{n+1}(0) = 0$, and augmenting it to the vector $\boldsymbol{y}(t)$. For ease of notation, we denote the augmented vector also by $\boldsymbol{y}(t)$. Then we can write (31) as

$$\dot{\boldsymbol{y}}(t) = \boldsymbol{f}_y(\boldsymbol{y}(t)) + u(t)\boldsymbol{g}_y(\boldsymbol{y}(t)), \quad \boldsymbol{y}(0) = (\boldsymbol{x}^T(0) \;\; 0)^T. \tag{32}$$

It is easy to verify that $\{\boldsymbol{f}_y, \boldsymbol{g}_y\}_{LA}$ is also third-order nilpotent. Thus, the analysis of the time-optimal control for (32) is similar to our analysis of (6) above. In particular, the switching function defined in the MP (for time-optimal controls) is composed of second-order polynomials.

We now present an outline of the analysis for the time-optimal control w^*.

A. The bang-bang case

In the non-singular case, w^* is bang-bang, and it follows from the MP that it either contains two switchings or is periodic.

The second-order MP, as described in [14], is defined for a control yielding a trajectory that is on the boundary of the reachable set. However, if $\boldsymbol{y}^*$ is a minimum time trajectory, then $\boldsymbol{y}^*(\bar{t})$ must lie on the boundary of the reachable set at time $\bar{t}$, so the Agrachev-Gamkrelidze MP can be applied to our problem.

The proof in [14] requires that the adjoint $\boldsymbol{\lambda}$ is unique (up to multiplication by a scalar). This is used to show that if condition (21) does not hold, then $\boldsymbol{y}^*(\bar{t})$ is in the interior of the reachable set at time $\bar{t}$ so w^* cannot be optimal.

However, in the proof of Proposition 6, we found a perturbation $\boldsymbol{\alpha}^0$, that does not depend on $\boldsymbol{\lambda}$. A careful analysis of the proof in [14] shows that in this case the uniqueness of $\boldsymbol{\lambda}$ is not required.

B. The singular case

From the MP (defined for the time-optimal problem) it follows that there exists an adjoint that is non-trivial on the tangent space of the manifold spanned by the system, and satisfies $\lambda_{n+1}(t) \equiv 0$. In the special case of the $\boldsymbol{y}$ system, this implies that the condition (27) cannot hold [21] so $c \neq 0$. Hence, as above, $w^* \in \mathcal{PC}(\bar{t}, 2)$.

Summarizing, the time-optimal control always satisfies $w^* \in \mathcal{PC}(\bar{t}, 3)$. ∎

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