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A Numerical Algorithm for Solving the Absolute Stability Problem in $\mathbb{R}^3$

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Abstract— **The problem of absolute stability is one of the oldest open problems in the theory of control. For low-order systems, the most general results were obtained by Pyatnitskiy and Rapoport. They derived an implicit characterization of the "most destabilizing" nonlinearity using the maximum principle. In this paper, we show that their approach yields a simple and efficient numerical scheme for solving the problem in the case of third-order systems. This allows the determination of the critical value where stability is lost in a tractable and accurate fashion. This value is important in many practical applications and we believe that it can also be used to develop a deeper theoretical understanding of this interesting problem.**

Keywords: Switched linear systems, global asymptotic stability under arbitrary switching, switched controllers, differential inclusions.

I. INTRODUCTION

The absolute stability problem, posed by Lure and his colleagues in the 1940s, is one of the oldest open problems in the theory of stability. The problem is to determine a critical value $k^* \in \mathbb{R}$ for which an nth-order system, composed of a linear and a nonlinear part, loses its stability.

Attempts for solving this problem led to numerous important results in the mathematical theories of stability and control, including: Popov's criterion; the circle criterion; the positive-real lemma [1]; and the theory of integral quadratic constraints [2]. However, all these approaches lead to sufficient, but *not* necessary and sufficient stability conditions. An analysis of the computational complexity of closely related problems can be found in [3].

More general results were obtained in the pioneering work of Pyatnitskiy and Rapoport [4], [5] who developed a variational approach to tackle the absolute stability problem. For the special cases $n = 2$ and $n = 3$ this approach led to a deep qualitative understanding of the problem. More recently, a dynamic programming approach [6] provided the first complete solution for the case $n = 2$ [7] (see also [8] and the closely related work [9]).

Recently, the problem is attracting renewed interest because determining k^* is equivalent to providing a necessary and sufficient condition for stability of switched linear systems. Switched systems have numerous applications and represent a subject of extensive ongoing research (see, e.g., [10], [11], [12]).

In this paper, we derive a simple and efficient algorithm for estimating k^* for the case $n = 3$. The main advantage of this algorithm is that it can be used to estimate k^* to arbitrary precision in a computationally tractable manner.

This accurate estimate of k^* can be used for several purposes. First, k^* provides a necessary and sufficient stability condition for switched linear systems under arbitrary switching. Second, it follows from a simple dual argument that it also provides a necessary and sufficient stability condition for a controller based on switching between two linear systems. Third, it can be used to estimate how conservative are various analytic conditions that provide a sufficient, but not necessary and sufficient condition, for stability of switched linear systems. Fourth, it can help examine the behavior of different numerical algorithms that can also be used to tackle this problem. These include, for example, algorithms that calculate polyhedral Lyapunov functions for switched linear systems (see, e.g., [13], [14]). Finally, an accurate estimate of k^* may serve as the first step in developing a better theoretical understanding of the absolute stability problem and, in particular, of the interesting behavior that takes place at the edge of stability.

The remainder of this paper is organized as follows. In Section II, we review the absolute stability problem. In Section III, we present a simple algorithm for calculating an accurate estimate of the critical value k^* for the case $n = 3$. Two illustrative examples for showing the effectiveness of the algorithm are presented in section IV. The final section concludes.

II. THE ABSOLUTE STABILITY PROBLEM

Consider the system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + \mathbf{b}\phi(t, y(t)) \\ y(t) &= \mathbf{c}^T\mathbf{x}(t) \end{aligned} \qquad (1)$$

where $\mathbf{b}, \mathbf{c}, \mathbf{x}(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$ is an asymptotically stable matrix, and ϕ belongs to S_k, the set of scalar time-varying functions in the sector $[0, k]$, i.e., $\phi(t, 0) = 0$ and $0 \le z\phi(t, z) \le kz^2$ for all $t \ge 0$. Note that we can view (1) as the feedback connection of a linear system and a single nonlinear function from S_k. Note also that, by definition, S_0 contains only the function $\phi \equiv 0$, so clearly (1) is asymptotically stable for any $\phi \in S_0$.

Problem 1 (Absolute stability [15, Ch. 5] [16]) *Find the value*

$$k^* := \inf\{k > 0 : \exists \phi^* \in S_k \text{ such that } (1) \text{ is not asymptotically stable}\}.$$

In other words, for $k \in [0, k^*)$, (1) is asymptotically stable for *any* $\phi \in S_k$. The problem is difficult because S_k

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contains an infinite number of functions and, therefore, a solution must actually entail the characterization of the "most destabilizing" nonlinearity ϕ^*.

Applying the idea of *global linearization* [1], we can rephrase Problem 1 in a more convenient form. Since $\phi \in S_k$, we have $\phi(t,y) = a(t,y)y$, with $0 \le a(t,y) \le k$, so (1) becomes the *differential inclusion*

$$\dot{\mathbf{x}} \in \mathrm{co}\{A, B_k\}\mathbf{x}, \quad B_k := A + k\mathbf{b}\mathbf{c}^T \tag{2}$$

where co denotes convex hull. Note that for $k = 0$, (2) reduces to $\dot{\mathbf{x}} = A\mathbf{x}$ which is of course asymptotically stable. We can now restate Problem 1.

Problem 2 (Absolute stability) *Find the value*

$$k^* := \inf\{k \ge 0 : (2) \textit{ is not asymptotically stable}\}.$$

Let $\hat{k} \in (0,\infty)$ denote the smallest value of k such that B_k is not an asymptotically stable matrix. It is easy to determine $\hat{k}$ using, for example, the Routh-Hurwitz criterion. In what follows, we will always assume that $k^* < \hat{k}$. This is the "interesting case" where instability is obtained by switching between asymptotically stable subsystems.

Note that specifying ϕ^* in (1) is equivalent to specifying the "most unstable trajectory" $\mathbf{x}^*(t)$ of (2). Note also that (2) is the *relaxed* version [17, Ch. 2] of the switched linear system

$$\dot{\mathbf{x}}(t) \in \{A\mathbf{x}(t), B_k\mathbf{x}(t)\}. \tag{3}$$

Stability analysis of switched linear systems is a very active research area (see, e.g., [10]). For our purposes, (2) and (3) are equivalent since it is well-known [18] that (2) is asymptotically stable if and only if (3) is.

Problem 2 is more difficult than the problem of determining whether the switched linear system $\dot{\mathbf{x}} \in \{A\mathbf{x}, B\mathbf{x}\}$ is asymptotically stable (under arbitrary switching) or not. This is so because Problem 2 requires determining the exact value k^* where stability is lost.

Pyatnitskiy and Rapoport [4], [5] introduced the idea of using a variational approach to describe the "most destabilizing" nonlinearity ϕ^*. Applying the *Maximum Principle*, they derived an *implicit* characterization of ϕ^* in terms of a two-point boundary value problem. Using tools from convex analysis and a Poincare-Bendixson-type argument, they derived the following result.

Theorem 1 *[5] Let $n = 3$. Denote $Q(k,t_1,t_2) := \exp(B_k t_2)\exp(At_1)$, and consider the equation*

$$\det(Q(k,t_1,t_2) + I) = 0. \tag{4}$$

This equation admits a solution $t_1, t_2 > 0$ when $k = k^*$, but such a solution does not exist for any $k \in [0, k^*)$.

Geometrically, this implies that there exists $\mathbf{x}^0 \in \mathbb{R}^3$ such that

$$Q(k^*, t_1, t_2)\mathbf{x}^0 = -\mathbf{x}^0$$

so $\mathbf{x}^0$ is an eigenvector of $Q(k^*, t_1, t_2)$ corresponding to the eigenvalue -1. Defining $\mathbf{x}^1 := \exp(At_1)\mathbf{x}^0$ we see that $\exp(B_{k^*}t_2)\mathbf{x}^1 = -\mathbf{x}^0$. In other words, taking $\mathbf{x}^0$ as an initial point, following the trajectory of $\dot{\mathbf{x}} = A\mathbf{x}$ for t_1 sec., and then the trajectory of $\dot{\mathbf{x}} = B_{k^*}\mathbf{x}$ for t_2 sec., we reach $-\mathbf{x}^0$ (see Fig. 1).

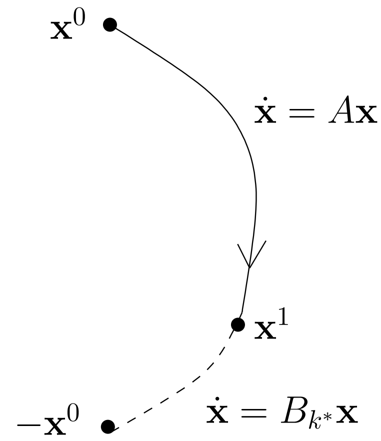

Fig. 1. Schematic interpretation of Theorem 1.

By symmetry, this implies that for $k = k^*$ the system (2) admits a closed, periodic trajectory with four (two) switching points in every period (half period).

The intuition underlying Theorem 1 can be explained as follows. Consider the "most unstable" solution $\mathbf{x}^*(t)$ of (2) (this can be defined rigorously as the solution of a certain variational problem). If $k < k^*$ then, by definition, *all* trajectories converge to the origin and, therefore, so does $\mathbf{x}^*$. In particular, there cannot exist a solution $t_1, t_2 > 0$ for (4). For $k > k^*$, $\mathbf{x}^*$ diverges. Between these two extremes, that is when $k = k^*$, there exists an initial point $\mathbf{x}^0$ such that $\mathbf{x}^*(t)$, with $\mathbf{x}^*(0) = \mathbf{x}^0$, is a closed and periodic trajectory.

III. The Algorithm

Solving Eq. (4) is quite difficult since it is a nonlinear equation with three unknowns k, t_1, and t_2. Nevertheless, it does suggest a simple numerical algorithm for estimating k^*. Namely, implement three nested loops: the first on k, the second on t_1 and the innermost one on t_2. Inside the nested loops calculate $\det(Q(k,t_1,t_2) + I)$, and find the first value of k where this value changes its sign. However, this implementation yields numerical instabilities. Instead, we use the geometric interpretation of Theorem 1 in terms of eigenvalues of the matrix Q. This yields the algorithm depicted in Fig. 2.

Note that it is possible that $\hat{k} = \infty$ (that is, B_k is Hurwitz for all $k > 0$). In this case, we simply replace $\hat{k}$ by some large value.

To implement our numerical algorithm we need upper bounds for the values t_1, t_2 in (4). Suitable values for the bounds $\hat{t}_1, \hat{t}_2$ can be computed explicitly using the following approach.

For a matrix $D \in \mathbb{R}^{n\times n}$, let $\|D\| := (\lambda_{\max}\{D^T D\})^{1/2}$, that is, $\|D\|$ is the square root of the largest eigenvalue of the symmetric matrix $D^T D$.

For $k = 0 : \Delta k : \hat{k}$
    For $t_1 = 0 : \Delta t_1 : \hat{t}_1$
        For $t_2 = 0 : \Delta t_2 : \hat{t}_2$
            Calculate the eigenvalues λ_i of $Q(k, t_1, t_2)$
            For $i = 1 : 1 : 3$
                If (λ_i is real) and ($\lambda_i \leq -1$) then
                  return(k, t_1, t_2)
                EndIf
            EndFor
        EndFor
    EndFor
EndFor

Fig. 2. The Algorithm.

Recall that we assume that $k^* < \hat{k}$, so for the range of k values we are interested in, B_k is Hurwitz. Thus, the Lyapunov equation $PB_k + B_k^T P = -I$ admits a symmetric and positive-definite solution P_k.

Proposition 1 *Denote* $M_k := \lambda_{\max}\{P_k\}$, $m_k := \lambda_{\min}\{P_k\}$, $\hat{t}_1 := M_0 \log(\frac{M_0 M_k}{m_0 m_k})$ *and* $\hat{t}_2 := M_k \log(\frac{M_0 M_k}{m_0 m_k})$. *If* t_1, t_2 *satisfy* (4)*, then*

$$t_1 \leq \hat{t}_1 \text{ and } t_2 \leq \hat{t}_2.$$

Proof: See the Appendix.

If the algorithm returns three values k, t_1, t_2 then k is an estimate of k^*. Using these values we can easily estimate $\mathbf{x}^0$ by computing the eigenvector of Q corresponding to the eigenvalue $\lambda \approx -1$.

The running time and accuracy depend, of course, on the bounds and the size of the iteration steps Δk, Δt_1, and Δt_2. We applied a simple refining scheme, starting from large initial values and using the returned values k, t_1, t_2 to narrow the range of the loops as well as reduce the step sizes. Careful experimentation with finer iteration steps for k, t_1, t_2 leads to estimates of the real values with arbitrary accuracy.

We applied this algorithm to many specific examples, and it provides accurate estimates of k^* in a stable fashion. The running time of a MATLAB implementation on a standard PC is a few minutes.

IV. Examples

We demonstrate the results using two examples. These will also allow us to compare the results with those derived using other algorithms and analytic conditions.

Example 1 Consider Problem 2 with

$$A = \begin{pmatrix} -1.5 & -3 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{b} = [1\ 0\ 0]^T, \mathbf{c} = [0\ {-1}\ {-1}]^T.$$

Our algorithm yields

$$k^* = 3.82695, \text{ and}$$

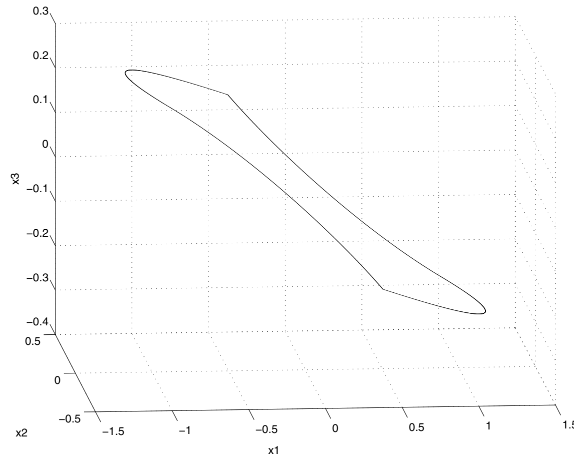

Fig. 3. Periodic trajectory of switched system in Example 1.

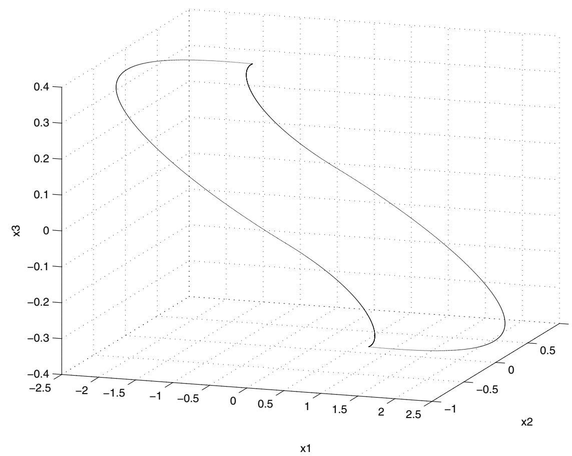

Fig. 4. Periodic trajectory of switched system in Example 2.

$$t_1 = 0.874,\ t_2 = 0.696\ ,\ \mathbf{x}^0 = [0.9422\ 0.2381\ {-0.2357}]^T.$$

Fig. 3 depicts the trajectory corresponding to this case. Note that we obtain a closed trajectory as expected.

It is interesting to compare our result with previous results derived for this problem. The circle criterion gives us a conservative lower bound $k^* \geq 2.27$. The polyhedral Lyapunov function (PLF) method in [13] yields $k^* \approx 3.00$ with 40 layers (6402 vertices), while the PLF method in [14] yields $k^* \approx 3.82$ with 250 layers (250002 vertices). This latter result is a good estimate of k^*, however, the computational time required to compute such a highly complex PLF is significant [14]. Our algorithm yields an estimate of k^*, to 5 digits accuracy, in a few minutes.

Example 2 Consider again Problem 2 with

$$A = \begin{pmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{b} = [1\ 0\ 0]^T, \mathbf{c} = [0\ 0\ {-24}]^T.$$

Our algorithm yields

$$k^* = 1.739, \text{ and}$$

$t_1 = 0.65,\ t_2 = 0.74\ ,\ \mathbf{x}^0 = [0.4362\ -0.8999\ -0.0010]^T$.

Fig. 4 depicts the trajectory corresponding to this case.

The circle criterion gives us the lower bound $k^* \geq 1.166$. The method in [13] yields $k^* \approx 1.50$ with 40 layers (6402 vertices), while the PLF method in [14] yields $k^* \approx 1.72$ with 250 layers (250002 vertices).

An important point regarding the comparison of the algorithm proposed with previous methods should be made: techniques such as the circle criterion and the methods based on PLFs, provide us with a value of $k < k^*$ for which absolute stability is *assured*. This is because the estimate of k^* is accompanied with a Lyapunov function that proves stability. In contrast, our algorithm yields a value k for which loss of stability has been detected. This estimate can be made to approach the critical value k^* with arbitrary accuracy.

V. Conclusions

The absolute stability problem, posed in the 1940s, is still an open problem. It is strongly related with the stability analysis of switched linear systems, a topic that has attracted considerable research interest in the last decade.

The work of Pyatniskiy and Rapoport has provided a deep qualitative understanding of the low-order special cases $n = 2$ and $n = 3$. However, even for the case $n = 2$ a complete solution of the problem was presented only very recently [7].

Several researchers developed numerical schemes, based on the construction of PLFs, for solving the problem [14], [13]. These algorithms work reasonably well for the case $n = 2$, but the case $n = 3$ requires considerable computation times. This is caused by the fact that the Lyapunov function that proves stability of the switched system as k approaches k^* is transcendental [7]. Therefore, any method based on polyhedral Lyapunov functions tends to "explode" as k approaches k^*.

In this paper, we have presented a new simple and efficient algorithm for solving the absolute stability problem in $\mathbb{R}^3$, i.e. determining the critical value k^* for which stability is lost. The algorithm is based on the results of Pyatnitskiy and Rapoport [4], [5], who proved that for $k = k^*$ the system admits a periodic trajectory with two switching points in every half period.

We believe that our algorithm is an efficient computational technique for estimating k^*, with arbitrary accuracy, in the case $n = 3$. We have shown that our technique yields the most accurate results in the literature and is computationally much more efficient than other numerical techniques available.

Appendix
Proof of Proposition 1

We require the following results.

Fact 1 *If λ is an eigenvalue of a matrix A, then $|\lambda| \leq \|A\|$.*

Proof: Let $\mathbf{v}$ be the eigenvector corresponding to λ, i.e. $A\mathbf{v} = \lambda\mathbf{v}$. Let X be the square matrix $X := [\mathbf{v}, \mathbf{v}, \ldots, \mathbf{v}]$. Using the properties of (any) matrix norm, we get

$$|\lambda|\,\|X\| = \|\lambda X\| = \|AX\| \leq \|A\|\|X\|,$$

hence $|\lambda| \leq \|A\|$. ■

Fact 2 *Suppose that $D \in \mathbb{R}^{n\times n}$ is Hurwitz. Let P denote the symmetric and positive-definite solution of $PD + D^T P = -I$. Denote $M := \lambda_{\max}\{P\}$ and $m := \lambda_{\min}\{P\}$, so $M \geq m > 0$. Then*

$$\|\exp(D\,t)\| \leq \sqrt{\frac{M}{m}}\exp(-t/(2M)), \qquad \forall t \geq 0.$$

Proof: Define $V(\mathbf{y}) := \mathbf{y}^T P\mathbf{y}$ and let $\mathbf{x}(t)$ be the solution of $\dot{\mathbf{x}} = D\,\mathbf{x}$, with $\mathbf{x}(0) = \mathbf{x}_0$. Then,

$$\frac{d}{dt}\ln(V(\mathbf{x}(t))) = \frac{\frac{d}{dt}V(\mathbf{x}(t))}{V(\mathbf{x}(t))} = \frac{-|\mathbf{x}(t)|^2}{\mathbf{x}^T(t)P\mathbf{x}(t)} \leq -\frac{1}{M}$$

Hence, $V(\mathbf{x}(t)) \leq V(\mathbf{x}_0)\exp(-t/M)$, so

$$\mathbf{x}^T(t)P\mathbf{x}(t) \leq \mathbf{x}_0^T P\mathbf{x}_0\exp(-t/M) \leq |\mathbf{x}_0|^2 M\exp(-t/M).$$

Thus

$$(\exp(Dt)\mathbf{x}_0)^T P\exp(Dt)\mathbf{x}_0 \leq |\mathbf{x}_0|^2 M\exp(-t/M)$$

so

$$\mathbf{x}_0^T\exp(D^T t)\exp(Dt)\mathbf{x}_0 \leq |\mathbf{x}_0|^2(M/m)\exp(-t/M).$$

This holds for any $\mathbf{x}_0 \in \mathbb{R}^n$ and, therefore,

$$\lambda_{\max}\{\exp(D^T t)\exp(Dt)\} \leq (M/m)\exp(-t/M),$$

and recalling the definition of $\|\cdot\|$ completes the proof. ■

Using these results we can now prove Proposition 1. Suppose that λ is an eigenvalue of the matrix $Q(k, t_1, t_2)$. Fact 1 yields

$$\begin{aligned}|\lambda| &\leq \|Q(k, t_1, t_2)\| \\ &\leq \|\exp(At_1)\|\|\exp(B_k t_2)\|.\end{aligned}$$

Denoting $M_k = \lambda_{\max}\{P_k\}$, $m_k = \lambda_{\min}\{P_k\}$, and using Fact 2 yields

$$|\lambda| \leq \sqrt{\frac{M_0 M_k}{m_0 m_k}}\exp(-t_1/(2M_0))\exp(-t_2/(2M_k)).$$

It is easy to verify that if any of the bounds in Proposition 1 does not hold then $|\lambda| < 1$ and, therefore, t_1, t_2 cannot be the solutions of (4). This completes the proof of Proposition 1. ■

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