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A predictive control strategy for Norm-Bounded LPV linear systems with bounded rates of parameter change

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***Abstract*— A novel predictive control strategy for input-saturated Norm-Bounded LPV discrete-time systems is proposed. The solution is computed by minimizing an upper-bound to the "worst-case" infinite horizon quadratic cost under the constraint of steering the future state evolutions, emanating from the current state, into a feasible and positive invariant set. It will be shown that the "size" of this terminal set depends on the rate of changes of the scheduling parameter which is assumed bounded and measurable.**

I. INTRODUCTION

In this paper, the robust predictive control setup conceived in [3] for input-saturated discrete-time uncertain linear systems subject to norm-bounded model uncertainty, is modified to deal with the case of LPV linear systems described by linear fractional representations (LFR). This generalization is of interest here, beyond genuine LPV systems, in that it allows improvements in the achievable control performance, usually modest under robust control approaches, when nonlinear plants are considered via embedding approaches.

The MPC literature is vast and the robustness issue has been addressed in several aspects (see [8], [9], [12] and references therein). On the contrary, contributions regarding the design of predictive strategies for LPV frameworks are a few and all related to the case of multi-models (polytopic) representations (see [2], [13], [10], [11]).

The contribution of this paper is to export ideas quite well developed for MPC control of LPV systems expressed via polytopic representations to the complementary LFR systems description. Up to our best acknowledge, this class of algorithms is missing in the MPC literature and potentially applicable to class of nonlinear MPC control problems with only modest performance degradations w.r.t. direct nonlinear MPC approaches.

We consider the general case of arbitrary N-steps control horizons and assume the LPV vector measurable at each time instant. Moreover, a bound on its rate of change is also assumed to be known. Based on the above setup, the key point of the proposed procedure consists in refining and, therefore, redefining step-by-step the "tube" of all k-steps state trajectories originating from the actual state under a quadratically stabilizing state-feedback law. The control input is computed on-line by solving a semi-definite programming problem involving linear matrix inequalities and applied to the plant in a receding horizon fashion. The closed-loop stability and feasibility properties of the solution can be proved via standard arguments and are here summarized. A numerical example involving a rational nonlinear plant, which admit an exact LFR description [6] is considered and comparisons with the Norm-Bounded robust approach of [3] shown.

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II. PROBLEM FORMULATION

Consider the following discrete linear LFR system representation of a LPV system [1]

$$\left\{\begin{array}{lcl} x(t+1) & = & \Phi\, x(t) + G\, u(t) + B_p\, p(t) \\ y(t) & = & C\, x(t) \\ q(t) & = & C_q\, x(t) + D_q\, u(t) \\ p(t) & = & \Delta(t)\, q(t) \end{array}\right. \quad (1)$$

with $x \in \mathbb{R}^{n_x}$ denoting the state, $u \in \mathbb{R}^{n_u}$ the control input, $y \in \mathbb{R}^{n_y}$ the output, $p, q \in \mathbb{R}^{n_p}$ additional variables accounting for the LPV scheduling matrix $\Delta(t)$ which satisfies the following additional properties:

- **LPV-1** - $\Delta(t)$ is measurable at each time instant;
- **LPV-2** - The following bound on its rate of change

  $$(\Delta(t+1) - \Delta(t))^T\, (\Delta(t+1) - \Delta(t)) \leq \delta^2\, I, \quad (2)$$

  is assumed with δ known;

The term $\Delta(t)$ may either represent 1) time-varying system parameters (i.e. spring constants, nonlinear resistances etc.), whose dynamics are exogenous and not directly related to the plant dynamics, or 2) state space components, when the LPV plant is obtained by embedding a nonlinear plant. In this case, (2) consists in an explicit constraint on the rate of change of (some components of) the state. It is further assumed that the plant input is subject to the following ellipsoidal constraint

$$u(t) \in \Omega_u, \qquad \Omega_u \triangleq \{u \in \mathbb{R}^{n_u} \ : \ u^T Q_u u \leq \bar{u}\} \quad (3)$$

with $Q_u = Q_u^T > 0$ and $\bar{u} > 0$. The aim is to find a state-feedback input strategy $u(t) = g(x(t))$, which possibly asymptotically stabilizes (1) subject to (3) and ensures a certain level of quadratic performance.

In what follows we will suppose that the family of systems (1) is quadratically stabilizable by a linear state-feedback control law

$$u(t) = Kx(t) \quad (4)$$

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If such a control law exists, the family of closed-loop systems (1) under (4) can be written as

$$x(t+1) = \Phi_K\, x(t) + B_p\, p(t) \tag{5}$$

with $p(t)$ belonging to the following uncertainty set

$$S(t) \triangleq \left\{ p \mid \|p\|_2^2 \leq \|(C_K\, x(t)\|_2^2 \right\} \tag{6}$$

($\Phi_K \triangleq \Phi + G\, K$, $C_K \triangleq C_q + D_q\, K$). Consider next the following quadratic performance index [8]

$$J(x(0),\, u(\cdot)) \triangleq \max_{p(t)\in S_t} \sum_{t=0}^{\infty} \left\{ \|x(t)\|_{R_x}^2 + \|u(t)\|_{R_u}^2 \right\} \tag{7}$$

A computable upper-bound on (7) exists and has the form $J(x(0),\, u(\cdot)) \leq x(0)^T\, P\, x(0)$ with matrix $P = P^T$ to be determined. It is well known that it characterizes the ellipsoidal sets $C(P,\rho) \triangleq \left\{ x \in \mathbb{R}^n \mid x^T\, P\, x \leq \rho \right\}$ which are positively invariant regions for the closed-system (5).

In the presence of input constraints $u \in \Omega_u$, the above setup keeps holding true provided that the pair (P, K) is chosen so that $x(0) \in C(P,\rho)$ with $K\, C(P,\rho) \subset \Omega_u$. A receding-horizon procedure for computing, at each time instant t on the basis of the current state $x(t)$, the pair (P, K) minimizing the upper-bound $x(0)^T P x(0)$ to the cost (7) and ensuring quadratic stability and constraints fulfilment from t onward has been proposed in [8]. Such procedure will be used off-line in our framework to derive, given the initial state $x(0)$, a couple (P, K) compatible with the invariance state-trajectory and input constraints requirements (see [3] for details).

We will discuss now how the LPV hypotheses **LPV-1**, and **LPV-2** affect the state predictions. To this end, we denote as $\hat{v}_k(t) \triangleq \hat{v}(t+k|t)$ the k-steps ahead predictions of a generic system variable based on all information available at time t. Consider the following input strategy

$$u(\cdot|t) = \begin{cases} K\, \hat{x}_k(t) + c_k(t), & k = 0, 1, \ldots, N-1, \\ K\, \hat{x}_k(t) & k \geq N, \end{cases} \tag{8}$$

It adds N additional free control moves c_k to the control action (4). Based on (8), the k-steps ahead state predictions for $x(t)$ and $p(t)$, say $\hat{x}_k(t) \triangleq \hat{x}(t+k|t)$ and $\hat{p}_k(t) \triangleq \hat{p}(t+k|t)$, can be computed. First, one has that

$$\hat{p}_0(t) = \Delta(t)\,(C_K\, x(t) + D_q\, c_0(t))$$

and

$$\begin{aligned} \hat{x}_1(t) \;&=\; \Phi_K\, x(t) + G\, c_0(t) + B_p\, p_0(t) = \\ &\quad\; \hat{\Phi}_K(t)\, x(t) + \hat{G}(t)\, x(t) \end{aligned} \tag{9}$$

are vectors exactly computable from

$$(\hat{\Phi}_K(t), \hat{G}(t)) \triangleq (\Phi_K + B_p\, \Delta(t)\, C_K,\; G + B_p\, \Delta(t)\, D_q)\,,$$

the latter pair representing the current realization of the plant family (1). On the contrary, when the prediction step exceeds one, the state and the p predictions are set-valued and related each other by

$$\hat{x}_{k+1}(t) = \Phi_K\, \hat{x}_k(t) + G\, c_k(t) + B_p\, \hat{p}_k(t) \tag{10}$$

Now, by denoting as $\hat{\Delta}_k(t)$ a whatever k-steps ahead prediction of $\Delta(t)$, one has that following identity

$$\hat{\Delta}_k(t) = \Delta(t) + \sum_{i=1}^{k} \left(\hat{\Delta}_i(t) - \hat{\Delta}_{i-1}(t) \right)$$

trivially follows when $\hat{\Delta}_0(t) = \Delta(t)$. Moreover, the matrix predictions $\hat{\Delta}_i(t)$ have to satisfy

$$\left(\hat{\Delta}_i(t) - \hat{\Delta}_{i-1}(t) \right)^T \left(\hat{\Delta}_i(t) - \hat{\Delta}_{i-1}(t) \right) \leq \delta^2\, I, \tag{11}$$

$i = 1, \ldots, k$ because of **LPV-2**. Then, the p-predictions can be rewritten

$$\begin{aligned} \hat{p}_k(t) \;&= \hat{\Delta}_k(t)\hat{q}_k(t) \\ &= \Delta(t)\hat{q}_k(t) + \sum_{i=1}^{k} \left(\hat{\Delta}_i(t) - \hat{\Delta}_{i-1}(t) \right) \hat{q}_k(t) \\ &= \bar{p}_k(t) + \sum_{i=1}^{k} \tilde{p}_{k,i}(t), \end{aligned} \tag{12}$$

where $\bar{p}_k(t) \triangleq \Delta(t) q_k(t)$ is the component of $\hat{p}_k(t)$ which is not affected by uncertainties on the rate of change which are all collected in

$$\tilde{p}_{k,i}(t) \triangleq \left(\hat{\Delta}_i(t) - \hat{\Delta}_{i-1}(t) \right) \hat{q}_k(t),\; i = 1, \ldots, k \tag{13}$$

It can easily shown that $\tilde{p}_{k,i}(t) \in S_k^\delta(t),\; i = 1, .., k$, where

$$S_k^\delta(t) \triangleq \left\{ \tilde{p} \mid \|\tilde{p}\|_2^2 \leq \max_{\hat{x}_k(t)} \delta^2\, \|C_K \hat{x}_k(t) + D_q c_k(t)\|_2^2 \right\}, \tag{14}$$

Notice that, although $\tilde{p}_{k,i}(t)$ is also quantifiable as $\tilde{p}_{k,i}(t) = \left(\hat{\Delta}_k(t) - \Delta(t) \right) \hat{q}_k(t)$, we prefer to consider it as the sum of k independent contributions because this leads to less conservative solutions. Back-substituting $\hat{p}_k(t)$ in (10) we obtain

$$\hat{x}_{k+1}(t) = \hat{\Phi}_K(t) \hat{x}_k(t) + \hat{G}(t) c_k(t) + B_p\; \sum_{i=1}^{k} \tilde{p}_{k,i}(t) \tag{15}$$

A more compact expression of the state predictions $\hat{x}_{k+1}(t)$ in terms of initial state $x(t)$, input moves $c_i(t)$, $i = 0, 1, \ldots, k$ and uncertain terms $\tilde{p}_{i,j}(t)$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, i$ is obtained by defining the following matrices and vectors for $k = 0, 1, 2, \ldots, N-2$

$$\bar{\Phi}_{k+1}(t) \triangleq \hat{\Phi}_K^{k+1}(t),$$

$$\bar{G}_k(t) \triangleq [\hat{\Phi}_K^k(t) \hat{G}(t)\; \hat{\Phi}_K^{k-1}(t) \hat{G}(t)\; \cdots$$
$$\hat{\Phi}_K(t) \hat{G}(t)\; \hat{G}(t)] \in \mathbb{R}^{n_x \times (k+1) n_u}$$

$$\bar{B}_k(t) \triangleq [\hat{\Phi}_K^k(t) B_p\, \tilde{I}_1,\; \hat{\Phi}^{k-1}(t) B_p\, \tilde{I}_2,\; \cdots$$
$$\hat{\Phi}_K(t) B_p \tilde{I}_{k-2},\; B_p \tilde{I}_{k-1}] \in \mathbb{R}^{n_x \times \frac{k(k+1)}{2} n_p}$$

$\tilde{I}_l \in \mathbb{R}^{n_p \times l n_p}$, $l = 1, \ldots, k$ are row block matrices whose elements are the identity matrix I_{n_p},

$$\underline{c}_k(t) \triangleq [c_0(t)^T\; c_1(t)^T\; \cdots\; c_k(t)^T]^T \in \mathbb{R}^{(k+1) n_u},$$

$$\underline{\tilde{p}}_k(t) \triangleq [\tilde{p}_{1,1}^T(t),\, \tilde{p}_{2,1}^T(t),\, \tilde{p}_{2,2}^T(t),\, \ldots$$

$\tilde{p}_{k,1}^T(t),\ \tilde{p}_{k,2}^T(t),\ \ldots \tilde{p}_{k,k}^T(t)]^T \in \mathbb{R}^{\frac{k(k+1)}{2}n_p}$,

Thus, convex set-valued state predictions $\hat{x}_{k+1}(t)$ can be rewritten as

$$\begin{aligned}\hat{x}_{k+1}(t) &= \bar{\Phi}_{k+1}(t)\, x(t) + \bar{G}_k(t)\, \underline{c}_k(t) + \\ &\quad \bar{B}_k(t)\, \underline{\tilde{p}}_k(t),\ k \geq 1\end{aligned} \tag{16}$$

subject to $\tilde{p}_{i,j}(t) \in S_i^\delta(t),\ i = 1, \ldots, k;\ j = 1, \ldots, i$.

According to the strategy (8) and given the state predictions $\hat{x}_{k+1}(t)$, from (9) and (16) a convenient upper-bound to the cost (7) is given by the following quadratic index $V \triangleq V(x(t), P, c_k(t))$,

$$\begin{aligned}V \triangleq\ & \|x(t)\|_{R_x}^2 + \sum_{k=1}^{N-1}\left(\max_{\hat{x}_k(t)} \|\hat{x}_k(t)\|_{R_x}^2 + \|c_{k-1}(t)\|_{R_u}^2\right) \\ & + \max_{\hat{x}_N(t)} \|\hat{x}_N(t)\|_P^2 + \|c_{N-1}(t)\|_{R_u}^2,\end{aligned} \tag{17}$$

to be minimized w.r.t. $c_k(t)$, $k = 0, \ldots N-1$. In (17) $R_x > 0$, $R_u \geq 0$ are symmetric state and input weighting matrices and $P \geq 0$. Then, at each time instant t, our solution will consist of computing

$$c_k^*(t) \triangleq \arg\min_{c_k(t)} V(x(t), P, c_k(t)) \tag{18}$$

subject to

$$K\hat{x}_k(t) + c_k(t) \subset \Omega_u, \quad k = 0, 1, ..., N-1 \tag{19}$$

$$\hat{x}_N(t) \subset C(P, \rho), \ \ Kz \in \Omega_u,\ \forall z \in C(P, \rho) \tag{20}$$

where $C(P, \rho)$ is a robust invariant set under a quadratically stabilizing feedback gain K that is a solution of the off-line LMI conditions whose details can be found in [3].

III. LMI FORMULATION OF COST UPPER-BOUNDS, INPUT, STATE AND SET-INVARIANCE CONSTRAINTS

In this section we will determine LMI conditions which guarantee an upper-bound to the quadratic cost (17) and ensure satisfaction of the prescribed input, state and quadratic set-invariance constraints. As a consequence, the derivations hereafter will be properly shortened. For notational simplicity we can consider w.l.o.g. the generic time t instant equal to zero and denote $c_k = c_k(0)$, $\hat{x}_k = \hat{x}_k(0)$, $x = x(0)$ for $k = 0, \ldots, N-1$, $i = 0, 1, \ldots, k$, and $S_k^\delta = S_k^\delta(0)$, $\tilde{p}_{k,i} = \tilde{p}_{k,i}(0)$, for $k = 1, \ldots, N-1$, $i = 0, 1, \ldots, k$.

Given a sequence of non-negative reals $J_0, ..., J_{N-1}$ such that,

$$V(x(t), P, c_k(t)) \leq x^T R_x\, x + \sum_{k=0}^{N-1} J_k \tag{21}$$

for arbitrary P, K and c_k, $k = 0, .., N-1$, the following constrained inequalities must be satisfied:

$$\hat{x}_1^T R_x\, \hat{x}_1 + c_0^T R_u\, c_0 \leq J_0 \tag{22}$$

$$\max_{\substack{\tilde{p}_{i,j} \in S_i^\delta \\ i=1,....,k, \\ j=1,...,i}} \hat{x}_k^T R_x\, \hat{x}_k + c_{k-1}^T R_u\, c_{k-1} \leq J_{k-1},\ k \geq 2, \tag{23}$$

$$\max_{\substack{\tilde{p}_{i,j} \in S_i^\delta \\ i=1,....,N-1, \\ j=1,...,i}} \hat{x}_N^T P\, \hat{x}_N + c_{N-1}^T R_u\, c_{N-1} \leq J_{N-1} \tag{24}$$

Due to the fact that $\hat{x}_1$ is a given vector, via Schur Complements it can be shown that all triplets (x, c_0, J_0) for which (22) holds true make the following LMI

$$\Sigma_0 \triangleq \begin{bmatrix} J_0 & \begin{pmatrix} x^T & c_0^T \end{pmatrix} \\ * & E_0^{-1} \end{bmatrix} \tag{25}$$

positive definite, where

$$E_0 \triangleq \begin{bmatrix} \hat{\Phi}_K^T\, R_x\, \hat{\Phi}_K & \hat{\Phi}_K^T\, R_x\, \hat{G} \\ * & \hat{G}^T\, R_x\, \hat{G} + R_u \end{bmatrix}$$

In order to derive LMI conditions related to the satisfaction of (23) consider, for $k \geq 1$

$$\hat{x}_{k+1} = \bar{\Phi}_{k+1}\, x + \bar{G}_k\, \underline{c}_k + \bar{B}_k\, \underline{\tilde{p}}_k, \tag{26}$$

with $\forall \tilde{p}_{i,j} \in S_i^\delta$, $i = 1, \ldots, k$, $j = 1, \ldots, i$. Then, (23) is satisfied by a generic $(x, \underline{c}_k, J_k)$ provided that

$$\begin{aligned} J_k - \begin{bmatrix} x \\ \underline{c}_k \end{bmatrix}^T E_k \begin{bmatrix} x \\ \underline{c}_k \end{bmatrix} \\ -2\begin{bmatrix} x \\ \underline{c}_k \end{bmatrix}^T D_k \underline{\tilde{p}}_k - \underline{\tilde{p}}_k^T \bar{B}_k^T\, R_x\, \bar{B}_k \underline{\tilde{p}}_k > 0 \end{aligned} \tag{27}$$

holds true when

$$\begin{aligned} -\underline{\tilde{p}}_k^T \tilde{O}_{i,j} \underline{\tilde{p}}_k + 2\begin{bmatrix} x \\ \underline{c}_k \end{bmatrix}^T \tilde{H}_i^T \underline{\tilde{p}}_k + \begin{bmatrix} x \\ \underline{c}_k \end{bmatrix}^T \tilde{F}_i \begin{bmatrix} x \\ \underline{c}_k \end{bmatrix} \geq 0, \\ i = 1, ..., k, j = 1, ..., i \end{aligned} \tag{28}$$

holds true as well. In (28), D_k^T, $E_k = E_k^T \geq 0$ are matrices defined by

$$D_k^T \triangleq \begin{bmatrix} \bar{\Phi}_{k+1}^T \\ \bar{G}_k^T \end{bmatrix} R_x \bar{B}_k^T, \tag{29}$$

$$E_k \triangleq \begin{bmatrix} \bar{\Phi}_{k+1}^T R_x \bar{\Phi}_{k+1} & \bar{\Phi}_{k+1}^T R_x \bar{G}_k \\ * & \bar{G}_k^T R_x \bar{G}_k + \begin{bmatrix} 0 & 0 \\ 0 & R_u \end{bmatrix} \end{bmatrix} \tag{30}$$

with the square matrix R_u in the $(2, 2)$-entry of E_k added to the last n_u rows and columns of the sub-matrix $\bar{G}_k^T R_x \bar{G}_k$. Moreover, the matrices $\tilde{H}_i$, $\tilde{G}_i = \tilde{G}_i^T$ and $\tilde{F}_i = \tilde{F}_i^T \geq 0$ are defined as

$$\tilde{H}_i^T \triangleq \begin{bmatrix} H_i^T & 0 \\ * & 0 \end{bmatrix}, \quad \tilde{G}_i \triangleq \begin{bmatrix} G_i & 0 \\ * & 0 \end{bmatrix}, \quad \tilde{F}_i \triangleq \begin{bmatrix} F_i & 0 \\ * & 0 \end{bmatrix}$$

with

$$H_1^T \triangleq 0_{(n_x+n_u)\times n_p}, \quad G_1 \triangleq I_{n_p} \tag{31}$$

and, for $i = 2, ...k$,

$$H_i^T \triangleq \delta^2 \begin{bmatrix} \bar{\Phi}_{i-1}^T C_K^T \\ \bar{G}_{i-2}^T C_K^T \\ D_q^T \end{bmatrix} [C_K \bar{B}_{i-2} \ \ 0], \tag{32}$$

$$F_i \triangleq \delta^2 \begin{bmatrix} \bar{\Phi}_i^T C_K^T \\ \bar{G}_{i-1}^T C_K^T \\ D_q^T \end{bmatrix} [C_K \bar{\Phi}_i\ C_K \bar{G}_{i-1}\ D_q] \tag{33}$$

$$\tilde{O}_{i,j} \triangleq \operatorname{diag}\left([-\delta^2 \bar{B}_{i-2}^T C_K^T C_K \bar{B}_{i-2},\ 0, ..., I_{n_p}, ..., 0]\right), \tag{34}$$

where the identity block-matrix is placed in the rows/columns corresponding to the positions that the

vector $\tilde{p}_{i,j}$, $i=1,\dots,k$, $j=1,\dots,i$ has in the stacked-vector $\underline{\tilde{p}}_k$. Via the S-procedure [14], it can be shown that the implication **St. 1** - (27) holds true for all $\underline{\tilde{p}}_k$ satisfying (28)
is satisfied if there exist $\frac{k(k+1)}{2}$ scalars

$$\tau_{i,j}^k \geq 0,\ i=1,\dots,k,\ j=1,\dots,k$$

such that the following matrix

$$\begin{bmatrix} -\bar{B}_k^T R_x \bar{B}_k + \sum_{i,j}\tau_{i,j}^k \tilde{O}_{i,j} & -\left(D_k+\sum_{i,j}\tau_{i,j}^k \tilde{H}_i\right)\begin{bmatrix} x \\ \underline{c}_k \end{bmatrix} \\ * & J_k - \begin{bmatrix} x \\ \underline{c}_k \end{bmatrix}^T \left(E_k+\sum_{i,j}\tau_{i,j}^k \tilde{F}_i\right)\begin{bmatrix} x \\ \underline{c}_k \end{bmatrix} \end{bmatrix} \tag{35}$$

is positive definite. Via Schur complements, the inequality (35) is satisfied if the following LMI

$$\Sigma_k \triangleq \begin{bmatrix} J_k & -[x^T\ \underline{c}_k^T]L_k^T \\ * & I \end{bmatrix} \tag{36}$$

is positive definite, where the matrix L_k is the Choleski [7] factor of

$$\left(E_k+\textstyle\sum_{i,j}\tau_{i,j}^k\tilde{F}_i\right)+\left(D_k+\sum_{i,j}\tau_{i,j}^k\tilde{H}_i\right)^T \left(-\bar{B}_k^T R_x\bar{B}_k+\sum_{i,j}\tau_{i,j}^k\tilde{O}_{i,j}\right)^{-1}\left(D_k+\sum_{i,j}\tau_{i,j}^k\tilde{H}_i\right) \tag{37}$$

Optimal values of the scalars $\tau_{i,j}^k$ are computed by solving the following GEVP problem (see [3] for details).
The constrained inequality related to the terminal state $\hat{x}_N$, (24) can be converted in a LMI similar as (36). The derivation procedure is identical, except that P is put in place of R_x inside (27), (29) and (30), for $k=N-1$ and $\frac{(N-1)(N-2)}{2}$ scalars $\tau_{i,j}^{N-1}$ must be computed by solving a GEVP problem. It results that (24) is satisfied by all $(x, \underline{c}_{N-1}, J_{N-1})$ such that the LMI

$$\Sigma_{N-1} \triangleq \begin{bmatrix} J_{N-1} & -[x^T\ \underline{c}_{N-1}^T]L_{N-1}^T \\ * & I \end{bmatrix} \tag{38}$$

is positive definite, where where the L_{N-1} is the Choleski factor of

$$\left(E_{N-1}+\textstyle\sum_{i,j}\tau_{i,j}^{N-1}\tilde{F}_i\right)+\left(D_{N-1}+\sum_{i,j}\tau_{i,j}^{N-1}\tilde{H}_i\right)^T \left(-\bar{B}_{N-1}^T P\bar{B}_{N-1}+\sum_{i,j}\tau_{i,j}^{N-1}\tilde{O}_{i,j}\right)^{-1}\left(D_{N-1}+\sum_{i,j}\tau_{i,j}^{N-1}\tilde{H}_i\right)$$

A. Terminal State Invariance

The state invariance condition (20) consists, for a given pair (P,ρ), of imposing that all N-steps ahead state predictions

$$\hat{x}_N = \bar{\Phi}_N x + \bar{G}_{N-1}\underline{c}_{N-1} + \bar{B}_{N-1}\underline{\tilde{p}}_{N-1},$$

$$\forall \tilde{p}_{i,j} \in S_i^\delta,\quad i=1,...,N-1,\ j=1,...,i,$$

belong to the ellipsoidal set $C(P,\rho)$. The derivation procedure uses the same matrices of Σ_{N-1}. We obtain that (20) is satisfied if the following LMI constraint

$$\Sigma_N \triangleq \begin{bmatrix} \rho & -[x^T\ \underline{c}_{N-1}^T]L_N^T \\ * & I \end{bmatrix} \tag{39}$$

is positive definite. The matrix L_N is the Choleski Factor of

$$\left(E_N+\textstyle\sum_{i,j}\tau_{i,j}^N\tilde{F}_i\right)+\left(D_{N-1}+\sum_{i,j}\tau_{i,j}^N\tilde{H}_i\right)^T \left(-\bar{B}_{N-1}^T P\bar{B}_{N-1}+\sum_{i,j}\tau_{i,j}^N\tilde{O}_{i,j}\right)^{-1}\left(D_{N-1}+\sum_{i,j}\tau_{i,j}^N\tilde{H}_i\right)$$

where E_N is obtained from E_{N-1} (eq. (30) with P in place of R_x) and the term R_u is not present. Again, a set of $\frac{(N-1)(N-2)}{2}$ scalars τ_{ij}^N, $i=1,\dots,N-1$,$j=1,\dots,i$ must be computed off-line.

B. Input constraints

Let us consider now the quadratic input constraints (3) along the predictions for $k=0,...,N-1$

$$(Kx+c_0)^T Q_u (Kx+c_0) \leq \bar{u} \tag{40}$$

$$\left(K\hat{\Phi}_K x + K\hat{G}c_0 + c_1\right)^T Q_u \left(K\hat{\Phi}_K x + K\hat{G}c_0 + c_1\right) \leq \bar{u} \tag{41}$$

$$(K\hat{x}_k + c_k)^T Q_u (K\hat{x}_k + c_k) \leq \bar{u} \tag{42}$$

$$\forall \tilde{p}_{i,j} \in S_i^\delta, i=1,...,k,\ j=1,...,i$$

Due to the LPV hypothesis, LMIs regarding (40) and (41) are straightforwardly derived:

$$\Upsilon_0 \triangleq \begin{bmatrix} \bar{u} & -(Kx+c_0)^T \\ * & Q_u^{-1} \end{bmatrix} \geq 0 \tag{43}$$

$$\Upsilon_1 \triangleq \begin{bmatrix} \bar{u} & -\left(K\hat{\Phi}_K x + K\hat{G}c_0 + c_1\right)^T \\ * & Q_u^{-1} \end{bmatrix} \geq 0 \tag{44}$$

For the generic prediction step $k \geq 2$, sufficient conditions for the fulfillment of (42) can be obtained by observing that such an inequality can be rewritten as

$$-\underline{\tilde{p}}_{k-1}^T \bar{B}_{k-1}^T K^T Q_u K \bar{B}_{k-1}\underline{\tilde{p}}_{k-1} - 2\begin{bmatrix} x \\ \underline{c}_{k-1} \\ c_k \end{bmatrix}^T M_k^T \underline{\tilde{p}}_{k-1} + \bar{u} - \begin{bmatrix} x \\ \underline{c}_{k-1} \\ c_k \end{bmatrix}^T N_k \begin{bmatrix} x \\ \underline{c}_{k-1} \\ c_k \end{bmatrix} \geq 0 \tag{45}$$

and must be true for all $\underline{\tilde{p}}_{k-1}$, $k=2,\dots,N-1$

$$-\underline{\tilde{p}}_{k-1}^T \hat{O}_{i,j}\underline{\tilde{p}}_{k-1} + 2[x^T\ \underline{c}_k^T]\hat{H}_i^T \underline{\tilde{p}}_{k-1} + [x^T\ \underline{c}_k^T]\hat{F}_i \begin{bmatrix} x \\ \underline{c}_k \end{bmatrix} \geq 0,\quad i=0,...,k-1 \tag{46}$$

where M_k^T and $N_k = N_k^T \geq 0$ are the following matrices

$$M_k^T \triangleq \begin{bmatrix} \bar{\Phi}_k^T K^T \\ \bar{G}_{k-1}^T K^T \\ I \end{bmatrix} Q_u K \bar{B}_{k-1}, \quad N_k \triangleq \begin{bmatrix} \bar{\Phi}_k^T K^T \\ \bar{G}_{k-1}^T K^T \\ I \end{bmatrix} Q_u \begin{bmatrix} K\bar{\Phi}_k & K\bar{G}_{k-1} & I \end{bmatrix} \tag{47}$$

The implication **St. 2** - (42) holds true for all $\underline{\tilde{p}}_{k-1}$ satisfying (28)
is verified if the following LMIs

$$\Upsilon_k \triangleq \begin{bmatrix} \bar{u} & -[x^T\ \underline{c}_k^T]T_k \\ * & I \end{bmatrix} \tag{48}$$

is positive definite, where T_k is the Choleski factor of

$$\left(N_k + \textstyle\sum_{i,j}\theta_{i,j}^k \tilde{F}_i\right) + \left(M_k+\sum_{i,j}\theta_{i,j}^k\tilde{H}_i\right)^T\left(-\bar{B}_{k-1}^T K^T Q_u K \bar{B}_{k-1} + \sum_{i,j}\theta_{i,j}^k \tilde{O}_{i,j}\right)^{-1}\left(M_k+\sum_{i,j}\theta_{i,j}^k\tilde{H}_i\right) \tag{49}$$

The optimal values of the $\frac{k(k-1)}{2}$, θ_{ij}^k, $i=1,\ldots,k-1$, $j=1,\ldots,i$ scalars are computed by solving a GEVP problem.
Remark 1 - State constraints

$$\|C_s\,(\hat{x}_{k+1}-\hat{x}_k)\|_2^2 \le \delta^2 \quad (50)$$

can be derived in a similar way as done for input constraints (details can be found in [4]). Namely, we have the following LMI conditions

$$\Psi_0 \ge 0, \quad (51)$$

$$\Psi_k \ge 0,\ k=1,\ldots,N-1, \quad (52)$$

and a set of positive scalars ς_{ij}^k, $i=1,\ldots,k-1$, $j=1,\ldots,i$ is obtained by means of S-procedure arguments. □

The following Lemma is justified by previous derivations
Lemma 1 - Let the initial state x and the stabilizing control law K be given. Then, under **LPV-1** and **LPV-2**, all pairs (c_k, J_k), $k=0,...,N-1$, which satisfy the LMI conditions (25), (36), (38), (39), (43), (44), (48), (51) and (52)

$$\Sigma_k \ge 0, \quad k=0,...,N$$

$$\Upsilon_k \ge 0,\ \Psi_k \ge 0, \quad k=0,...,N-1$$

provide upper-bounds to the cost (21) and define feasible prediction trajectories which fulfil the input, state and terminal constraints (3), (50) and (20) for $k=0,..,N-1$
Proof - By collecting all the above discussion. □

IV. Feasibility, Stability and NB-LPV MPC Algorithm

One of the crucial points in any MPC design is to prove that it enjoyes a feasibility property. The usual way to prove it is to show that, if $(c_k^*(t), J_k^*(t)),\ k=0,..,N-1$ denotes the optimal solution at time t, the solution

$$(c_1^*(t), J_1^*(t)), ..., c_{N-1}^*(t), J_{N-1}^*(t)), (0_{n_u}, J_{N-1}^*(t)) \quad (53)$$

is admissible, though non optimal at time $t+1$. In order to prove such a property, it must be noted that, unlike the robust algorithm proposed in [3], the positive scalars $\tau_{ij}^k(t)$, $\theta_{ij}^k(t)$, $\varsigma_{ij}^k(t)$ depend on the actual plant realization $\left(\hat{\Phi}_K(t), \hat{G}(t)\right)$ and must be, in principle, recomputed at each time step. However, the following backsubstitions

$$\begin{cases} \tau_{i-1,j-1}^{k-1}(t+1) \leftarrow \tau_{i,j}^k(t), \\ k=2,...,N-1,\ i=2,...,k,\ j=2,...,i, \\ \\ \theta_{i-1,j-1}^{k-1}(t+1) \leftarrow \theta_{i,j}^k(t) \\ k=2,...,N-2,\ i=2,...,k,\ j=2,...,i, \\ \\ \varsigma_{i-1,j-1}^{k-1}(t+1) \leftarrow \varsigma_{i,j}^k(t), \\ k=2,...,N-1,\ i=2,...,k,\ j=2,...,i, \end{cases} \quad (54)$$

allow one to alleviate the problem in that it can be shown, as done [3], that they are feasible choices. Then, only new computation is required for the terminal ($k=N-1$) and invariant ($k=N$) scalars.

All above developments allows one to write down a computable MPC scheme, hereafter denoted as **NB-LPV**, which consists of the following algorithm.

NB-LPV

0. (Off-line) Given $x(0)$ and $\Delta(0)$ and $\left(\hat{\Phi}(0), \hat{G}(0)\right)$, compute $[Y_{\rm opt}, Q_{\rm opt}, \rho_{\rm opt}]$ by solving the robust off-line procedure of $NB-Frozen$, ([8], p. 1367). Compute $K = Y_{\rm opt}\, Q_{\rm opt}^{-1}$, $P=\rho_{\rm opt}\, Q_{\rm opt}^{-1}$, $\rho=\rho_{opt}$. Compute the scalars

$$\begin{array}{llll} \tau_{i,j}^k(0) & i=1,...,k & j=1,...,i & k=0,...,N-1 \\ \tau_{i,j}^N(0) & i=1,...,N-1 & j=1,...,i & \\ \theta_{i,j}^k(0) & i=1,...,k-1 & j=1,...,i & k=0,...,N-1 \\ \varsigma_{i,j}^k(0) & i=1,...,k & j=1,...,i & k=0,...,N-1 \end{array}$$

1.1 (On-line) At each time instant $t \ge 0$, given $x(t)$ and $\Delta(t)$, solve

$$[J_k^*(t), c_k^*(t)] \triangleq \arg\min_{J_k, c_k} \sum_{k=0}^{N-1} J_k$$
subject to
$$\Sigma_k(t) \ge 0,\ \Upsilon_k(t) \ge 0,\ \Psi_k(t) \ge 0 \quad k=0,1,...,N-1$$
$$\Sigma_N(t) \ge 0$$

1.2 feed the plant with $u(t) = Kx(t) + c_0^*(t)$, compute $\left(\hat{\Phi}(t+1), \hat{G}(t+1)\right)$;
1.3 apply the backsubstitution rule (54) and recompute the scalars, $\tau_{i,j}^{N-1}(t+1)$, $\tau_{i,j}^N(t+1)$, $\theta_{i,j}^{N-1}(t+1)$ and $\varsigma_{i,j}^{N-1}(t+1)$.
1.4 $t=t+1$, and go to step 1.1

The following result answers the matter regarding feasibility and closed-loop stability of the proposed strategy.

Theorem 1 - Let the NB-LPV scheme have solution at time $t=0$ (Both points 0 and 1.1). Then, it has solution at each future time instant t, satisfies the input constraints and yields an asymptotically (quadratically) stable closed-loop system.
Proof - Feasibility can be proved by means of the previous arguments, closed loop stability can be obtained by following similar ideas used in [3]. □

V. Numerical example: Controlled Van Der Pol Oscillator

A controlled Van Der Pol nonlinear oscillator [5], [6] is taken into consideration as an illustration of the proposed MPC algorithm

$$\begin{cases} \dot{x}_1(t) &=& x_2(t) \\ \dot{x}_2(t) &=& -x_1(t) - (1-x_1^2(t))\,x_2(t) + u(t) \end{cases} \quad (55)$$

This system admits a Linear-Fractional Representation LFR (see [5]) of the form

$$\begin{cases} \dot{x}(t) &=& \Phi\, x(t) + G\, u(t) + B_p\, p(t) \\ q(t) &=& C_q\, x(t) + D_{qu}\, u(t) + D_{qp}\, p(t) \\ p(t) &=& \Delta(t)\, q(t), \|\Delta(t)\|_2 \le \frac{1}{\sigma},\ \sigma > 0 \end{cases} \quad (56)$$

that is more general than (1) for the presence of the matrix D_{qp}. For the problem at hand, we have that $\sigma = 1/0.67$, $\Delta(t) =$ diag $(x_1(t), x_1(t))$ and

$$\Phi = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},\ G = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_p = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C_q = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},\ D_{qu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ D_{qp} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

In many cases the LFR structure (56) can be reduced to the that described by (1) via an invertible variable change $\tilde{\Delta}(t) \triangleq (I - \Delta(t)\, D_{qp})^{-1}\, \Delta(t)$. In fact, if $(I - \Delta(t)\, D_{qp})$ is invertible, by eliminating $q(t)$ from the second and third equation of (56) one has

$$p(t) = (I - \Delta(t)\, D_{qp})^{-1}\, \Delta(t)\, (C_q\, x(t) + D_{qu}\, u(t))$$

However, in this example such a change is not necessary in that $\Delta(t)\, D_{qp} = 0$. The system has been discretized using forward Euler differences with a sampling time $T_c = 0.1$ sec. and $x(0) = \left[\begin{array}{cc} 0.3 & 0.3 \end{array}\right]^T$ used as initial state. The following weighting matrices $R_x = \mathrm{diag}\,(0.01,\, 1)\,, R_u = 1$, and input-constraint radius $\bar{u} = 0.2$ have been chosen.

The proposed NB-LPV algorithm has been compared with the NB-Robust algorithm described in [3] for a control horizon $N = 3$. Two values for the upper-bound δ on the rate of change of $\Delta(t)$ have been considered: $\delta = 0.1$ and $\delta = 0.04$.
Figs. 1 and 2 depict the comparisons for the two values of δ (dashed line graph, $\delta = 0.1$ - continuous line graph, $\delta = 0.04$) and the NB-Robust algorithm (dash-dotted line graph) respectively in terms of control input signal (Fig. 1), regulated state trajectories (Fig. 2) It can be observed that, as δ decreases the command becomes more active, the state component $x_1(t)$ has more sluggish responses whereas $x_2(t)$ features lower undershoots and the cost decreases The behavior of $x_1(t)$ is not surprising because decreasing δ means that "fast" variations of $x_1(t)$ are not allowed.

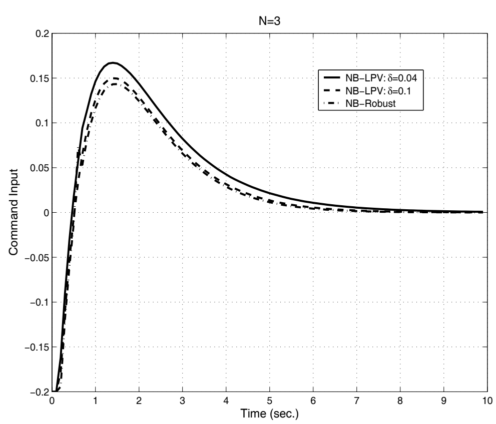

Fig. 1. Command Input

Finally, it has been observed that reducing the input constraint radius from $\bar{u} = 0.2$ to $\bar{u} = 0.15$ makes the NB-Robust algorithm unfeasible whereas the NB-LPV still works, testifying that less conservative results can be achieved by LPV approaches.

VI. Conclusions

In this paper, a LPV predictive control strategy has been presented for discrete-time systems described by LFR representations. The scheduling LPV vector has been assumed measurable and a bound on its rate of change known. A certain level of control performance improvement has been

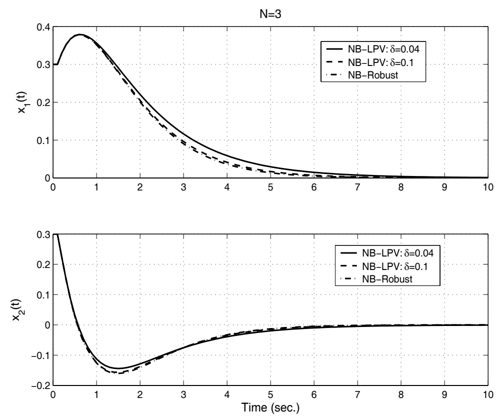

Fig. 2. State Trajectories

observed w.r.t. the robust algorithm proposed in [3]. It is believed that it relies on the use of the actual LPV parameter realization and of its rate of change, which enables to redefine less conservatively the prediction tubes.

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