Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

HJB Equations for Ergodic Control Problems for Constrained Diffusions in Polyhedral Domains.

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***Abstract*— Recently in [10] an ergodic control problem for a class of diffusion processes, constrained to take values in a polyhedral cone, was considered. The main result of that paper was that under appropriate conditions on the model, there is a Markov control for which the infimum of the cost function is attained. In the current work we characterize the value of the ergodic control problem via a suitable Hamilton-Jacobi-Bellman (HJB) equation. The theory of existence and uniqueness of classical solutions, for PDEs in domains with corners and reflection fields which are oblique, discontinuous and multi-valued on corners, is not available. We show that the natural HJB equation for the ergodic control problem admits a unique continuous viscosity solution which enables us to characterize the value function of the control problem. The existence of a solution to this HJB equation is established via the classical vanishing discount argument. The key step is proving the pre-compactness of the family of suitably re-normalized discounted value functions. In this regard we use a recent technique, introduced in [4], of using the Athreya-Ney-Nummelin pseudo-atom construction for obtaining a coupling of a pair of embedded, discrete time, controlled Markov chains.**

I. INTRODUCTION

This paper is a condensed version of Borkar and Budhiraja[8]. The reader is referred to [8] for proofs of all the results stated here. In a recent work [10] an ergodic control problem for a class of constrained diffusion processes, in polyhedral cones, was studied. Such controlled constrained diffusion processes arise in the heavy traffic analysis of single class open queuing networks with state dependent arrival and service rates with control in the marginal service rates (cf. [17]). The domain $G \subset I\!R^k$, which is the state space of the controlled Markov process, is given as an intersection of N half spaces $G_i; i = 1, \cdots N$. Associated with each G_i is a vector d_i which defines the "direction of constraint" in the relative interior of ∂G_i. At a point $x \in \partial G$ where several faces meet, there is more than one possible direction of constraint, in fact the set of permissible directions is a cone denoted by $d(x)$. Roughly speaking, the constrained version of a given unrestricted trajectory in $I\!R^k$ is obtained by pushing back the trajectory, whenever it is about to exit the domain, in one of the permissible directions of constraint using the minimal force required to keep the trajectory inside the domain. Precise definitions will be given in Section 2. The constraining mechanism is described via the notion of a Skorohod problem. Under appropriate conditions on $(d_i)_{i=1}^N$ it follows from the results in [14] that one can define the "Skorohod map", denoted as $\Gamma(\cdot)$ which takes an unrestricted trajectory $\psi(\cdot)$ and maps it to a trajectory $\phi(\cdot) \doteq \Gamma(\psi)(\cdot)$ such that $\phi(t) \in G$ for all $t \in (0, \infty)$.

The controlled constrained diffusion process that we will study is obtained as a solution to the equation:

$$\begin{aligned} X(t) &= \Gamma\left(X(0) + \int_0^{\cdot} b(X(s), u(s))ds \right. \\ &+ \left. \int_0^{\cdot} \sigma(X(s))dW(s) \right)(t); t \geq 0, \qquad \text{(I.1)} \end{aligned}$$

where $W(\cdot)$ is a standard Wiener process, $b : G \times U \to I\!R^k$; $\sigma : G \to I\!R^{k\times k}$ are suitable coefficients, U is a given control set and $u(\cdot)$ is a U valued "admissible" control process. The cost of interest is the ergodic cost criterion:

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T k(X(s), u(s))ds, \qquad \text{(I.2)}$$

where the limit above is taken almost surely and $k : G \times U \to I\!R$ is a suitable map.

It was shown in [10] that, under appropriate conditions on the model, (Conditions II.2, II.3, II.4) there is a Markov control for which the infimum of the cost in (I.2) is attained. In the current work our goal is the characterization of the value function of the control problem via a suitable Hamilton-Jacobi-Bellman (HJB) equation. For unconstrained diffusions this problem has been extensively studied and we refer the reader to [6] for a detailed account. For the controlled Markov processes in the present work, the problem is quite challenging since the domain in which the process is constrained to lie is not smooth (because of the corners where the faces meet) and the reflection field is oblique, discontinuous and multi-valued at the boundary points which lie on more than one face. The theory of existence and uniqueness of classical solutions for PDEs in such domains is not available. However, using the fundamental ideas of Crandall and Lions [13], [18], Dupuis and Ishii [15] have developed existence and uniqueness theory of

Research of Vivek Borkar was supported in part by a grant for 'Nonlinear Studies' from the Indian Space Research Organization and the Defense Research and Development Organization, Government of India, administered through the Indian Institute of Science.
Research of Amarjit Budhiraja supported in part by ARO (Grant W911NF-04-1-0230).

0-7803-9568-9/05/$20.00 ©2005 IEEE

viscosity solutions for fully nonlinear second order elliptic PDEs on such domains. In this work we will show that the value of the ergodic control problem introduced above can be characterized via the unique viscosity solution of an appropriate HJB equation. The usual approach to the HJB equation for the ergodic control is via the "vanishing discount method" (cf. [12], [9], [6], [3]). In this approach one first studies the value function $V_\alpha(x)$ of the discounted control problem:

$$V_\alpha(x) = \inf_u I\!\!E \left(\int_0^\infty e^{-\alpha s} k(X^x(s), u(s)) ds \right), \qquad (I.3)$$

where $\alpha \in (0, \infty)$, the infimum is taken over all admissible controls u and $X^x(\cdot)$ is the solution of (I.1) with $X(0) \equiv x$. For $f \in C_b^2(G)$ let $Lf : G \times U \to I\!\!R$ be defined as follows. For$(x, u) \in G \times U$,

$$(Lf)(x,u) \doteq \frac{1}{2} \sum_{i,j=1}^k a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^k b_i(x,u) \frac{\partial f}{\partial x_i}(x), \qquad (I.4)$$

where $a_{ij}(x) \doteq \sigma(x)\sigma^T(x)$. Using results of [15] we will show that the value function $V_\alpha(x)$ is the unique viscosity solution (see Definition III.1) of the following HJB equation.

$$\inf_{u \in U} \left(L\psi(x,u) + k(x,u) - \alpha\psi(x) \right) = 0, \;\; x \in G$$
$$\langle \nabla\psi(x), d_i \rangle = 0, \;\; x \in \partial G; \;\; i \in \mathrm{In}(x), \qquad (I.5)$$

where $\mathrm{In}(x) \doteq \{i \in \{1, 2, \cdots N\} : x \in \partial G_i\}$(as a convention we set, for $x \in G^o$, $I_n(x) = \emptyset$). We remark that the work [15] considers the case where the domain is bounded, however by a slight modification the techniques there can be used to cover the case in the present work.

In order to study the HJB equation of the ergodic control problem, we need to take the limit as $\alpha \to 0$. The key step in this program is to show that the family:

$$\{\overline{V}_\alpha(\cdot) \doteq V_\alpha(\cdot) - V_\alpha(0), \;\; \alpha \in (0, \infty)\} \qquad (I.6)$$

is pre-compact in $C(G)$. The classical derivation (see Theorem VI.3.1 of [6]) of such a result makes use of certain gradient estimates on $V_\alpha(x)$, uniform in α, which we are unable to prove for the model considered in the present work. Another approach based on viscosity solutions, taken in [3], proves the above pre-compactness by making some strong stability assumptions on the model (a restoring force towards bounded sets that grows without bound as $|x| \to \infty$) which are not natural for the constrained diffusion models that arise from the heavy traffic analysis of queuing networks. In the present work we prove the pre-compactness of the family in (I.6) by using the Athreya-Ney-Nummelin pseudo-atom construction which was recently introduced in the context of partially observed ergodic control problems in [4]. Using this construction, the pre-compactness of the family of (re-normalized) discounted value functions for a partially observed control problem was proved in [7]. One of the key requirements for the coupling methods used in the above cited works to work, is the existence of a suitable Lyapunov function for the underlying controlled Markov processes. For the processes considered in the present work, the existence of such a Lyapunov function was proved in [1]. Using this Lyapunov function one can show that a Foster type drift criterion is satisfied for an appropriate embedded discrete time controlled Markov chain. This, along with the pseudo-atom construction enables us to show that the coupling time, for two embedded controlled Markov chains driven by the same Markov control and independent noise processes but two different initial conditions, has finite moments. The above step is the main ingredient to the proof of the pre-compactness of (I.6).

Once the pre-compactness is proved, one can take the limit of $(\alpha V_\alpha(0), \overline{V}_\alpha(\cdot))$, along a subsequence, as $\alpha \to 0$. Then by stability (under perturbations) properties of viscosity solutions it follows that the limit, denoted as $(\rho, V(\cdot))$ is a viscosity solution of the HJB equation for the ergodic control problem (see (V.15)). The rest of the work involves showing that this equation admits a unique solution and that ρ is the infimum, over all admissible controls, of the cost function in (I.2).

II. Preliminaries and Background Results.

Let $G \subset I\!\!R^k$ be a polyhedral cone with the vertex at origin given as the intersection of half spaces G_i, $i = 1, \cdots, N$. Each half space G_i is associated with a unit vector n_i via the relation $G_i = \{x \in I\!\!R^k : \langle x, n_i \rangle \geq 0\}$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $I\!\!R^k$. Denote the boundary of a set $B \subset I\!\!R^k$ by ∂B. We will denote the set $\{x \in \partial G : \langle x, n_i \rangle = 0\}$ by F_i. For $x \in \partial G$, define the set, $n(x)$, of inward normals to G at x by $n(x) \doteq \{r : |r| = 1, \quad \langle r, x - y \rangle \leq 0, \quad \forall y \in G\}$. With each face F_i we associate a unit vector d_i such that $\langle d_i, n_i \rangle > 0$. This vector defines the *direction of constraint* associated with the face F_i. For $x \in \partial G$ define $d(x) \doteq \left\{ d \in I\!\!R^k : d = \sum_{i \in \mathrm{In}(x)} \alpha_i d_i; \alpha_i \geq 0; \;\; ||d|| = 1 \right\}$. We will denote the collection of all subsets of $\{1, \cdots N\}$ by Λ.

Let $D([0,\infty) : I\!\!R^k)$ denote the set of functions mapping $[0, \infty)$ to $I\!\!R^k$ that are right continuous and have limits from the left. We endow $D([0,\infty) : I\!\!R^k)$ with the usual Skorohod topology. Let $D_G([0,\infty) : I\!\!R^k) \doteq \{\psi \in D([0,\infty) : I\!\!R^k) : \psi(0) \in G\}$. For $\eta \in D([0,\infty) : I\!\!R^k)$ let $|\eta|(T)$ denote the total variation of η on $[0, T]$ with respect to the Euclidean norm on $I\!\!R^k$.

Definition II.1: Let $\psi \in D_G([0,\infty) : I\!\!R^k)$ be given. Then $(\phi, \eta) \in D([0,\infty) : I\!\!R^k) \times D([0,\infty) : I\!\!R^k)$ solves the Skorohod problem (SP) for ψ with respect to G and d if and only if $\phi(0) = \psi(0)$, and for all $t \in [0, \infty)$ (1) $\phi(t) = \psi(t) + \eta(t)$; (2) $\phi(t) \in G$; (3) $|\eta|(t) < \infty$; (4) $|\eta|(t) = \int_{[0,t]} I_{\{\phi(s) \in \partial G\}} d|\eta|(s)$; (5) There exists (Borel) measurable $\gamma : [0, \infty) \to I\!\!R^k$ such that $\gamma(t) \in d(\phi(t))$ ($d|\eta|$-almost everywhere) and $\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s)$.

On the domain $D \subset D_G([0,\infty) : I\!\!R^k)$ on which there is a unique solution to the Skorohod problem we define the Skorohod map (SM) Γ as $\Gamma(\psi) \doteq \phi$, if $(\phi, \psi - \phi)$ is the unique solution of the Skorohod problem posed by ψ. The following is the key assumption made in [10] on the data defining the Skorohod problem.

Assumption II.2: **(a)** There exists a compact, convex set $B \in I\!R^k$ with $0 \in B^0$, such that if $v(z)$ denotes the set of inward normals to B at $z \in \partial B$, then for $i = 1, 2, \cdots, N$, $z \in \partial B$ and $|\langle z, n_i \rangle| < 1$ implies that $\langle v, d_i \rangle = 0$ for all $v \in v(z)$. **(b)** There exists a map $\pi : I\!R^k \to G$ such that if $y \in G$, then $\pi(y) = y$, and if $y \notin G$, then $\pi(y) \in \partial G$, and $y - \pi(y) = \alpha\gamma$ for some $\alpha \le 0$ and $\gamma \in d(\pi(y))$. **(c)** For every $x \in \partial G$, there is $n \in n(x)$ such that $\langle d, n \rangle > 0$ for all $d \in d(x)$.
Results of Dupuis and Ishii [14] show that under Assumption II.2 the Skorohod map is well defined and Lipschitz continuous on all of $D_G([0,\infty) : I\!R^k)$, in particular, $D = D_G([0,\infty) \ : \ I\!R^k)$. In rest of the paper Condition II.2 will always be taken to hold. We refer the reader to [16] for sufficient conditions and examples for which the above condition holds.

We now introduce the controlled constrained diffusion processes that will be studied in this paper. Throughout this paper we will assume the relaxed control framework, i.e. there is a compact metric space S such that the control set is $U \doteq \mathcal{P}(S)$ (the space of all probability measures on S endowed with the weak convergence topology). All topological spaces in this paper will be endowed with their natural Borel σ- field. For a topological space $\mathcal{K}$, we will denote its Borel- σ field by $\mathcal{B}(\mathcal{K})$. The space of all real, measurable and bounded functions defined on $\mathcal{K}$ will be denoted as $BM(\mathcal{K})$, the subset of $BM(\mathcal{K})$ consisting of continuous functions will be denoted by $C_b(\mathcal{K})$ and the space of all probability measures on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$ will be denoted by $\mathcal{P}(\mathcal{K})$. The space $\mathcal{P}(\mathcal{K})$ will be endowed with the weak convergence topology. For $A \in \mathcal{B}(\mathcal{K})$, $\mathcal{I}_A(\cdot)$ will denote the indicator function of the set A. Also, we will denote by $C_b^2(G)$ the space of real valued, bounded and twice continuously differentiable functions on G. By a filtered probability space: $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ we will mean a probability space $(\Omega, \mathcal{F}, P)$ endowed by a filtration $(\mathcal{F}_t)_{t \ge 0}$ satisfying the usual hypothesis. A pair of stochastic processes $(u(\cdot), W(\cdot))$ defined on some filtered probability space: $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ is said to be an admissible pair if: $W(\cdot)$ is a $\mathcal{F}_t$- standard Wiener process and $u(\cdot)$ is a U valued, measurable, $\{\mathcal{F}_t\}$ adapted process.

We will consider controlled constrained diffusion processes of the form defined in (I.1), where for $(x,u) \in G \times U$, $b(x,u) \doteq \int_S \bar{b}(x,\alpha)u(d\alpha)$ and the coefficients $\sigma : G \to I\!R^{k \times k}$ and $\bar{b} : G \times S \to I\!R^k$ satisfy the following conditions.

Assumption II.3: (i) There exists $r \in (0,\infty)$ such that $\bar{b}$ is a continuous map and for all $x, y \in G$ and $\alpha \in S$

$$||\bar{b}(x,\alpha) - \bar{b}(y,\alpha)|| + ||\sigma(x) - \sigma(y)|| \le r||x-y||$$

and $||\bar{b}(x,\alpha)|| + ||\sigma(x)|| \le r$.
(ii) There exists $c_0 \in (0,\infty)$ such that for all $x \in G$ and $\alpha \in I\!R^k$ $\alpha'(\sigma(x)\sigma'(x))\alpha \ge c_0\alpha'\alpha$.
Under Assumptions II.2 and II.3, it follows via the Lipschitz property of the Skorohod map and the usual fixed point arguments that, (I.1) admits a unique strong solution (cf. Theorem 2.6 [10]). We now present the blanket stability condition, under which existence of an optimal Markov control was established in [10]. Define

$$\mathcal{C} \doteq \left\{ -\sum_{i=1}^{N} \alpha_i d_i : \alpha_i \ge 0; \ i \in \{1, \cdots, N\} \right\}. \qquad \text{(II.7)}$$

The cone $\mathcal{C}$ was used to characterize stability for a certain class of constrained diffusion processes in [11], [2].

Let $\delta \in (0,\infty)$ be fixed. Define the set

$$\mathcal{C}(\delta) \doteq \{v \in \mathcal{C} : \text{dist}(v, \partial C) \ge \delta\}. \qquad \text{(II.8)}$$

The blanket stability condition below, which will be assumed throughout this paper, stipulates the permissible drifts in the underlying diffusion.

Assumption II.4: There exist a $\delta \in (0,\infty)$ such that for all $(x,u) \in G \times U$, $b(x,u) \in \mathcal{C}(\delta)$.
Using these stability properties, the following result on the existence of an optimal Markov control was proved in [10].

Theorem II.5: There exists a Markov control $\bar{v}(\cdot)$ such that if for some $\mu \in \mathcal{P}(G)$, $\overline{X}(\cdot)$ is the corresponding process solving (I.1) (with $u(s)$ there replaced by $\bar{v}(X(s))$), on some filtered probability space, with the probability law of $\overline{X}(0)$ being μ then:

$$\begin{aligned} &\limsup_{T\to\infty} \frac{1}{T} \int_0^T k(\overline{X}(s), \bar{v}(\overline{X}(s)))ds \\ &= \inf \quad \text{ess } \inf \limsup_{T\to\infty} \frac{1}{T} \int_0^T k(X(s), u(s))ds, \end{aligned} \qquad \text{(II.9)}$$

a.s., where the outside infimum on the right side above is taken over all controlled processes $X(\cdot)$ with an arbitrary initial distribution and solving (I.1) over some filtered probability space with some admissible pair $(W(\cdot), u(\cdot))$.

III. The Discounted Cost Problem.

One of the important goals in optimal control theory is to derive the Hamilton-Jacobi-Bellman equation for the value function and characterize the value function as the unique solution (in an appropriate class) of the PDE. The classical approach to the HJB equation for the ergodic control is by the "vanishing discount method" (cf. [12], [9], [6], [3]). In this approach the first step is to study the value function $V_\alpha(x)$ of the discounted control problem defined via (I.3). In this section we will characterize the value function $V_\alpha(x)$ via a suitable HJB equation.

With an abuse of notation we will write for $\alpha \in S$, $(Lf)(x, \delta_{\{\alpha\}})$, merely as $(Lf)(x,\alpha)$, where $\delta_{\{\alpha\}}$ denotes the probability measure concentrated at the point α. Thus with this notation, for $(x,u) \in G \times U$, $(Lf)(x,u) = \int_S (Lf)(x,\alpha)u(d\alpha)$. For $i = 1, 2, \cdots N$ and $f \in C_b^2(G)$ let $D_i f : G \to I\!R$ be defined as: $(D_i f)(x) \doteq \langle d_i, \nabla f(x) \rangle, \quad x \in G$. The natural HJB equation associated with the control problem (I.3) is the one given in (I.5). Theory of classical solutions for such a PDE is not available, and therefore we will consider solutions in the viscosity sense[13], [18], [15]. We now describe what we mean by a viscosity solution of (I.5). Denote by S^k the space of $k \times k$ real symmetric matrices. Let $F_\alpha : G \times I\!R \times I\!R^k \times S^k \to I\!R$ be defined as

$$F_\alpha(x,r,p,M) \doteq -\frac{1}{2}\mathrm{Tr}(aM) + \alpha r - \inf_{u \in U}\{\langle b(x,u), p\rangle + k(x,u)\}. \quad \text{(III.1)}$$

Also, define $F_{\alpha,*}$ and F^*_α as maps from $G \times I\!R \times I\!R^k \times S^k$ to $I\!R$, as follows.

$$F_{\alpha,*}(x,r,p,M) = F_\alpha(x,r,p,M) \wedge \min\{-\langle d_i, p\rangle; i \in \mathrm{In}(x)\} \quad \text{(III.2)}$$

$$F^*_\alpha(x,r,p,M) = F_\alpha(x,r,p,M) \vee \max\{-\langle d_i, p\rangle; i \in \mathrm{In}(x)\}, \quad \text{(III.3)}$$

where minimum [maximum] over an empty set is, by convention, ∞[resp. $-\infty$].

Definition III.1: We say that $\phi \in C_b(G)$ is a viscosity solution of the equation (I.5) if and only if, for all $x_0 \in G$ and for all $\psi \in C^2(G)$, such that x_0 is a strict maximum [minimum] point of $\phi - \psi$,

$$F_{\alpha,*}(x_0, \phi(x_0), (D\psi)(x_0), (D^2\psi)(x_0)) \leq \text{ [resp. } \geq] \; 0. \quad \text{(III.4)}$$

We now have the following result.

Theorem III.2: The value function V_α defined via (I.3) is the unique viscosity solution of (I.5).

IV. THE VANISHING DISCOUNT LIMIT.

In this section we will show that if V_α is given via (I.3) and $\overline{V}_\alpha(x) \doteq V_\alpha(x) - V_\alpha(0);\; x \in G$, then the family $\{\overline{V}_\alpha; \quad \alpha \in (0,\infty)\}$ is pre-compact in $C(G)$. We begin, following [4], by an embedding of the continuous time control problem in a discrete time control problem. Define $\mathcal{U} \doteq \{\theta : [0,1] \to U : \theta \quad \text{is a measurable map } \}$. We endow $\mathcal{U}$ with the coarsest topology under which , for every $e \in L^2[0,1]$ and $f \in C_b(S)$, the map $\Psi : \mathcal{U} \to I\!R$ defined as $\Psi(u) \doteq \int_0^1 e(t) \int_S f(\theta) u_t(d\theta) dt$, is continuous. Let $\hat{\Phi} \subset \mathcal{P}(C([0,1] : I\!R^k) \times \mathcal{U})$ be the class of all probability measures which correspond to the probability law of some admissible pair $(u(t), W(t))_{0 \leq t \leq 1}$. It follows from [6], Chapter 1, that $\hat{\Phi}$ is a compact metric space.

Let $\hat{\phi} \in \hat{\Phi}$ and let $(u(\cdot), W(\cdot))$ be the corresponding admissible pair on a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$. Define

$$\hat{k}_\alpha(x, \hat{\phi}) \doteq I\!E\left(\int_0^1 e^{-\alpha s} k(X^x(s), u(s)) ds\right), \quad \text{(IV.5)}$$

where $X^x(\cdot)$ is given as a solution of (I.1) with $X(0) \equiv x$. Then setting $\hat{\alpha} \doteq e^{-\alpha}$ we have that

$$I\!E\left(\int_0^\infty e^{-\alpha t} k(X^x(t), u(t)) dt\right) = \sum_{n=0}^\infty \hat{\alpha}^n I\!E\left(\hat{k}_\alpha(X^x_n, \phi_n)\right), \quad \text{(IV.6)}$$

where for $n \in I\!N_0$, ϕ_n is the conditional law of $(u(n+s), W(n+s) - W(n))_{0 \leq s \leq 1}$ given $\mathcal{F}_n$ and $X^x_n \doteq X^x(n)$. Note that ϕ_n is a sequence of $\mathcal{F}_n$ measurable, $\hat{\Phi}$ valued random variables. We will call the sequence $\{\phi_n\}$ as the admissible control sequence corresponding to the admissible pair $(u(\cdot), W(\cdot))$. Also, note that $\{X^x_n\}$ is a controlled Markov chain with control set $\hat{\Phi}$ and (controlled) transition probability kernel $\hat{p}(x_1, \phi, dy_1)$, on G, given as follows

$$\int_G f(y)\hat{p}(x,\phi,dy) \doteq I\!E(f(\xi)), \quad f \in BM(G), \quad \phi \in \hat{\Phi}, \quad \text{(IV.7)}$$

where $\xi \doteq X(1)$ and $X(\cdot)$ is given via (I.1) with $X(0) \equiv x$ and the control pair $(u(t), W(t))_{0 \leq t \leq 1}$ having the probability law ϕ.

We now introduce a Lyapunov function for the controlled Markov chain $\{X^x_n\}$. This Lyapunov function was constructed in [1].

Theorem IV.3: There exist $c_0, \ell_0, M_0 \in (0,\infty)$ and a function $F : G \to I\!R$ such that it is twice continuously differentiable on $G \setminus \{0\}$ and for any admissible pair $(u(\cdot), W(\cdot))$ on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, $x \in G$ and $X^x(\cdot)$ given by (I.1), we have that

$$I\!E(F(X^x_{n+1}) \mid \mathcal{F}_n) - F(X^x_n) \leq -c_0 1_{X^x_n \in B^c} + M_0 1_{X^x_n \in B}, \quad \text{(IV.8)}$$

where $B \doteq \{x \in G \mid |x| \leq \ell_0\}$ and $X^x_n \doteq X^x(n)$.

We now introduce a controlled probability transition kernel on $H \doteq G \times G$, with control set $\hat{\Phi} \times \hat{\Phi}$ defined as follows. For $\overline{x} \equiv (x_1, x_2) \in H$ and $\overline{\phi} \equiv (\phi_1, \phi_2) \in \hat{\Phi} \times \hat{\Phi}$, let $\overline{p}(\overline{x}, \overline{\phi}, d\overline{y}) \in \mathcal{P}(H)$ be defined as:

$$\overline{p}(\overline{x}, \overline{\phi}, d\overline{y}) \doteq \hat{p}(x_1, \phi_1, dy_1)\hat{p}(x_2, \phi_2, dy_2). \quad \text{(IV.9)}$$

Let v be a Markov control and for $x_1, x_2 \in G$, let $X^{x_i}(\cdot)$ be given via (I.1) with $u(s)$ replaced by $v(X^{x_i}(s))$, $X^{x_i}(0) \equiv x_i$ and driving Wiener process $W^{(i)}$; $i = 1,2$. The Wiener processes $W^{(1)}$ and $W^{(2)}$ are taken to be independent of each other. Denote $X^{x_i}(n)$ by $X^{x_i}_n$. It is easy to see that if $\overline{x} \doteq (x_1, x_2)$ then $\{\overline{X}^{\overline{x}}_n\} \doteq \{(X^{x_1}_n, X^{x_2}_n)\}$ is a H valued controlled Markov chain, starting at $\overline{x}$, with the controlled probability transition kernel $\overline{p}(\overline{x}, \phi, d\overline{y})$ and the Markov control $\overline{\varrho}_v(\overline{x}) \equiv (\varrho_v(x_1), \varrho_v(x_2))$. Also, as noted earlier, for $i = 1,2$, $X^{x_i}_n$ is a G valued controlled Markov chain, starting at x_i, with the controlled probability transition kernel $\hat{p}(x, \phi, dy)$ and the Markov control ϱ_v.

The Pseudo-Atom Construction. We will now proceed, as in [4], to adapt the Athreya-Ney-Nummelin construction of a pseudo-atom to the current problem. Let $H \doteq G \times G$ and B be as in the statement of Theorem IV.3. Define $B^* \doteq B \times B$ and let $H^* \doteq H \times \{0,1\} = G \times G \times \{0,1\}$.

Let λ denote the Lebesgue measure on G. Define $\nu \in \mathcal{P}(H)$ as

$$\nu(A) \doteq \frac{(\lambda \times \lambda)(A \cap B^*)}{(\lambda(B))^2}. \quad \text{(IV.10)}$$

Using the uniform non-degeneracy of the diffusion coefficient in (I.1), it follows that there exists $0 < \delta^* < 1$ such that for all $\phi \in \hat{\Phi} \times \hat{\Phi}$

$$\overline{p}(x, \phi, A) \geq \delta^* 1_{B^*}(x)\nu(A), \quad \forall\, x \in H, \; A \in \mathcal{B}(H). \quad \text{(IV.11)}$$

For a set $A \in \mathcal{B}(H)$, we let $A_0 \doteq A \times \{0\}$ and $A_1 \doteq A \times \{1\}$. For every $\mu \in \mathcal{P}(G \times G)$ we define a $\mu^* \in \mathcal{P}(H^*)$ as follows.

For $A \in \mathcal{B}(H)$,

$$\begin{aligned} \mu^*(A_0) &\doteq (1-\delta^*)\mu(AB^*) + \mu(A(B^*)^c) \\ \mu^*(A_1) &\doteq \delta^*\mu(AB^*). \end{aligned} \quad \text{(IV.12)}$$

Clearly, $\mu^*(A_0) + \mu^*(A_1) = \mu(A)$ and if $A \subset (B^*)^c$ then $\mu^*(A_0) = \mu(A)$.

Define $\varrho_v^* : H^* \to \hat{\Phi} \times \hat{\Phi}$ as follows. For $(x_1, x_2, i) \in H^*$, $\varrho_v^*(x_1, x_2, i) \doteq \overline{\varrho}_v(x_1, x_2)$. On a suitable probability space $(\Omega^*, \mathcal{F}^*, P^*)$, define a H^* valued controlled Markov chain: $Z_n \equiv (X_n^*, i_n^*)$, where $X_n^* \equiv (X_n^{1,*}, X_n^{2,*})$, with the control set $\hat{\Phi} \times \hat{\Phi}$ and the Markov control ϱ_v^* so that:

(1). The controlled transition kernel of Z_n is given as follows. For $\overline{z} \equiv (z, i) \in H^*$ and $\overline{\phi} \in \hat{\Phi} \times \hat{\Phi}$

$$q(\overline{z}, \overline{\phi}, d\overline{y}) = \begin{cases} \overline{p}^*(z, \overline{\phi}, dy) \ \text{ if } \overline{z} \in H_0 \setminus B_0^* \\ \frac{1}{1-\delta^*}(\overline{p}^*(z, \overline{\phi}, dy) - \delta^* \nu^*(d\overline{y}) \ \text{ if } \ \overline{z} \in B_0^* \\ \nu^*(d\overline{y}) \ \text{ if } \overline{z} \in H_1. \end{cases} \quad \text{(IV.13)}$$

where $\overline{y} \equiv (y, j) \in H^*$.

(2). The initial distributions are given as follows. For $A \in \mathcal{B}(H)$

$$\begin{aligned} P^*(Z_0 \in A_0) &\doteq \left((1-\delta^*)1_{AB^*}(\overline{x}) + 1_{A(B^*)^c}(\overline{x})\right) \\ P^*(Z_0 \in A_1) &\doteq \delta^* 1_{AB^*}(\overline{x}). \end{aligned}$$

The above construction assures that the probability laws of $\{X_n^*, \varrho_v^*(X_n^*)\}_{n \in I\!N_0}$ and $\{\overline{X}_n^{\overline{x}}, \overline{\varrho}_v(\overline{X}_n^{\overline{x}})\}_{n \in I\!N_0}$ are the same. The following theorem is an important consequence of the drift inequality (IV.8).

Theorem IV.4: Let

$$\tau(x_1, x_2) \doteq \inf\{n \in \mathcal{N}_0 : Z_n \in B_1^*\}.$$

Then there exists a $r \in (1, \infty)$, such that for every $M_0 \in (0, \infty)$,

$$\sup_{x_i \in G; |x_i| \le M_0; i=1,2} \sup I\!E^*(r^{\tau(x_1, x_2)}) < \infty,$$

where the inner supremum is taken over all Markov controls v.

An immediate consequence of the above result is the following Corollary.

Corollary IV.5: The family $\{\overline{V}_\alpha(\cdot) : \ \alpha \in (0,1)\}$ is precompact in $C(G)$.

V. The HJB Equation for the Ergodic Control Problem.

Let V_α and $\overline{V}_\alpha$ be as in the previous section. From (I.3) and the boundedness of k it follows that $\sup_{\alpha \in (0,1)} \alpha V_\alpha(0) < \infty$. This, along with Corollary IV.5 implies that there exists a sequence $\alpha_n \to 0$ as $n \to \infty$ and $\rho \in (0, \infty)$, $V \in C(G)$ such that

$$\lim_{n \to \infty} \alpha_n V_{\alpha_n}(0) = \rho, \quad lim_{n \to \infty} \overline{V}_{\alpha_n} = V. \quad \text{(V.14)}$$

where the second limit is taken uniformly on compact sets of G.

For $\rho^* \in [0, \infty)$, consider the following equation.

$$\begin{aligned} &\inf_{u \in U} (L\psi(x,u) + k(x,u) - \rho^*) = 0, \ \ x \in G \\ &\langle \nabla \psi(x), d_i \rangle = 0, \ \ x \in \partial G; \ \ i \in In(x). \end{aligned} \quad \text{(V.15)}$$

A viscosity solution to the above equation is defined in a similar manner as is defined for (I.5) in Definition III.1. The following is the main result of this paper. Denote the right right side of (II.9) by $\overline{\rho}$.

Theorem V.6: Let (V, ρ) be given via (V.14). Then V is a viscosity solution to (V.15) with $\rho^* = \rho = \overline{\rho}$. Furthermore, (V, ρ) is the unique viscosity solution of (V.15) satisfying $V(0) = 0$.

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