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A Mathematical Framework for Robust Control Over Uncertain Communication Channels

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***Abstract*— In this paper, a mathematical framework for studying robust control over uncertain communication channels is introduced. The theory is developed by 1) Generalizing the classical information theoretic measures to the robust analogous, which are subject to uncertainty in the source and the communication channel, 2) Deriving a lower bound for the robust rate distortion, and 3) Finding a necessary condition on the communication blocks subject to uncertainty for reliable communication up to distortion level D_v. By invoking this mathematical framework, necessary conditions for uniform asymptotic observability and stabilizability are derived for the following uncertain plants controlled over uncertain communication channels. 1) A probabilistic uncertain plant defined via a relative entropy constraint and 2) A frequency domain uncertain plant defined via an H^∞ constraint.**

I. INTRODUCTION

One of the issues that has begun to emerge in a number of applications, such as sensor networking, large scale teleoperation, and etc., is how to control plants by communicating information reliably, through limited capacity channels, when the subsystems are subject to uncertainty. Typical examples are applications in which a single dynamical system sends feedback information to a distant controller via a communication link with finite capacity. In the absence of uncertainty in the plant and the communication channel, important results are derived in [1]-[5]. The aim of these articles is to find a necessary and sufficient condition for stabilizability, when there are channel capacity and power constraints. For finite dimensional linear discrete time-invariant systems, it is shown that for controlling such systems over communication constraints the transmission data rate (or channel capacity) is lower bounded by the summation of the logarithms of the unstable eigenvalues.

The objective of this paper is to address similar questions, when there is uncertainty in the plant and the communication channel. In particular, to find necessary conditions on the channel capacity which ensure uniform asymptotic observability and stabilizability. The necessary steps in realizing such a study consists of the followings.

1. Generalizing the classical information theoretic measures of Shannon entropy, channel capacity and the rate distortion to the robust analogous, which are subject to uncertainty in the source and the communication channel.
2. Deriving a lower bound for the robust rate distortion in terms of the robust Shannon entropy.
3. Deriving necessary condition on the communication blocks subject to uncertainty in order to ensure reliable communication.

After extending the classical information theoretic measure to the robust analogous, we show that the so called robust transmission rate of the communication channel must be lower bounded by the robust Shannon entropy of the source in order to ensure reliable communication. Subsequently, we find necessary conditions for uniform asymptotic observability and stabilizability of uncertain plants over uncertain communication channels. Our derived results give known results [6] as special case.

In Section II, the precise notion of a robust communication system in the presence of feedback, and the corresponding information theoretic measures which are necessary to analyze this system are introduced. One of the fundamental results is the derivation of a lower bound for the robust rate distortion. Furthermore, a robust version of the Information Transmission theorem is introduced. This theorem provides a necessary condition for reliable communication. In Section III, the mathematical framework developed in Section II, is applied to two different classes of uncertain plant controlled over uncertain communication channels to address necessary conditions for uniform asymptotic observability and stabilizability.

II. ROBUST COMMUNICATION SYSTEMS

A. Communication System

Let $(\Omega, \mathcal{F}(\Omega), P)$ be a complete probability space. Consider the communication subsystems of the control/communication system of Fig. 1. Let $(\mathcal{Y}_S, \mathcal{F}(\mathcal{Y}_S))$ denote the source alphabet set, and $(\tilde{\mathcal{Y}}_R, \mathcal{F}(\tilde{\mathcal{Y}}_R))$ be the source reproduction alphabet set. The channel input and output alphabet sets are $(\mathcal{Z}_I, \mathcal{F}(\mathcal{Z}_I))$ and $(\tilde{\mathcal{Z}}_O, \mathcal{F}(\tilde{\mathcal{Z}}_O))$, respectively. Consider the case when the source and reproduction of the source measurable spaces correspond to the sequences with length T, and the channel input and output measurable spaces correspond to the sequences with length n ($T \leq n$). That is

$$
\begin{aligned}
(\mathcal{Y}, \mathcal{F}(\mathcal{Y})) &= (\mathcal{Y}_{0,T-1}, \mathcal{F}^{\mathcal{Y}}_{0,T-1}) \stackrel{\triangle}{=} \times_{k=0}^{T-1}(\mathcal{Y}_k, \mathcal{F}(\mathcal{Y}_k)),\\
(\tilde{\mathcal{Y}}, \mathcal{F}(\tilde{\mathcal{Y}})) &= (\tilde{\mathcal{Y}}_{0,T-1}, \mathcal{F}^{\tilde{\mathcal{Y}}}_{0,T-1}) \stackrel{\triangle}{=} \times_{k=0}^{T-1}(\tilde{\mathcal{Y}}_k, \mathcal{F}(\tilde{\mathcal{Y}}_k)),\\
T &= 1, 2, ..., \infty.
\end{aligned}
$$

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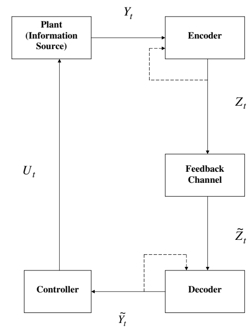

Fig. 1. Block diagram of control/communication system

$$
\begin{aligned}
(\mathcal{Z},\mathcal{F}(\mathcal{Z})) &= (\mathcal{Z}_{0,n-1},\mathcal{F}^{\mathcal{Z}}_{0,n-1}) \stackrel{\triangle}{=} \times_{k=0}^{n-1}(\mathcal{Z}_k,\mathcal{F}(\mathcal{Z}_k)),\\
(\tilde{\mathcal{Z}},\mathcal{F}(\tilde{\mathcal{Z}})) &= (\tilde{\mathcal{Z}}_{0,n-1},\mathcal{F}^{\tilde{\mathcal{Z}}}_{0,n-1}) \stackrel{\triangle}{=} \times_{k=0}^{n-1}(\tilde{\mathcal{Z}}_k,\mathcal{F}(\tilde{\mathcal{Z}}_k)),\\
n &= 1,2,...,\infty, \qquad (1)
\end{aligned}
$$

where $(\mathcal{Y}_k,\mathcal{F}(\mathcal{Y}_k))$, $(\tilde{\mathcal{Y}}_k,\mathcal{F}(\tilde{\mathcal{Y}}_k))$, $(\mathcal{Z}_k,\mathcal{F}(\mathcal{Z}_k))$, and $(\tilde{\mathcal{Z}}_k,\mathcal{F}(\tilde{\mathcal{Z}}_k))$ are exemplars of the source, reproduction, channel input and output alphabet sets respectively.

Let $Y^T \stackrel{\triangle}{=} (Y_0,Y_1,...,Y_{T-1}) \in \mathcal{Y}_{0,T-1}$, and $\tilde{Y}^T \stackrel{\triangle}{=} (\tilde{Y}_0,\tilde{Y}_1,...,\tilde{Y}_{T-1}) \in \tilde{Y}_{0,T-1}$, be a sequences with length T of source and reproduction of the source respectively, and $Z^n \stackrel{\triangle}{=} (Z_0,Z_1,...,Z_{n-1}) \in \mathcal{Z}_{0,n-1}$ and $\tilde{Z}^n \stackrel{\triangle}{=} (\tilde{Z}_0,\tilde{Z}_1,...,\tilde{Z}_{n-1}) \in \tilde{\mathcal{Z}}_{0,n-1}$ be sequences with length n of the channel input and output, respectively. Denote the set of Probability Density Functions (PDF's) on $\mathcal{Y}_{0,T-1}$ by $\mathcal{D} \stackrel{\triangle}{=} \{f_{Y^T}(y^T) : \mathcal{Y}_{0,T-1} \rightarrow \Re_+ : f_{Y^T}$ is $\mathcal{B}(\mathcal{Y}_{0,T-1})$ Borel measurable, $f_{Y^T}(y^T) \geq 0, \forall y^T \in \mathcal{Y}_{0,T-1}, \int_{\mathcal{Y}_{0,T-1}} f_{Y^T}(y^T)dy^T = 1\}$. Let also for $(\mathcal{Y}_S,\mathcal{F}(\mathcal{Y}_S)) = (\Re^d,\mathcal{B}(\Re^d))$, $\mathcal{P} = \{S_Y(e^{jw}) : \mathcal{C} \rightarrow \Re^{d\times d}; \det S_Y(e^{jw}) \geq 0, \forall e^{jw} \in \mathcal{C}, w \in [-\pi,\pi]\}$ denote the set of Power Spectral Density (PSD's).

Information Source. The information source $Y^T \in \mathcal{Y}_{0,T-1}$ induces a PDF $f_{Y^T}(y^T)$ on $(\mathcal{Y}_{0,T-1},\mathcal{F}^{\mathcal{Y}}_{0,T-1})$. In general, the source is uncertain, that is, $f_{Y^T}(y^T)$ is unknown but belongs to the source uncertainty set $f_{Y^T}(y^T) \in \mathcal{D}_{SU} \subseteq \mathcal{D}$.

Communication Channel. The communication channel is identified by a sequence of PDF's $\left\{f_{\tilde{Z}_k|Z^k,\tilde{Z}^{k-1}}\right\}_{k=0}^{n-1}$. Since the channel, in general, is uncertain, the joint PDF $f_{\tilde{Z^n}|Z^n}$ of the channel is uncertain and belongs to the channel uncertainty set $f_{\tilde{Z^n}|Z^n} \in \mathcal{D}_{CU} \subseteq \mathcal{D}$.

Encoder. An encoder is identified by a sequence of PDF's that is describing the probabilistic relationship between the current output of the encoder and the previous outputs of the encoder and inputs to the encoder, up to current time. That is, it is identified by a sequence of PDF's $\left\{f_{Z_k|Y^k,Z^{k-1}}\right\}_{k=0}^{n-1}$.

Decoder. A decoder is identified by a sequence of PDF's that is describing the probabilistic relationship between the current output of the decoder and the previous outputs of the decoder and inputs to the decoder, up to current time. That is, it is identified by a sequence of PDF's $\left\{f_{\tilde{Y}_k|\tilde{Z}^k,\tilde{Y}^{k-1}}\right\}_{k=0}^{n-1}$.

Please note that deterministic encoders and decoders can be modeled as sequences of PDF's that are Dirac functions.

Here, we assume conditional independence between the outputs of the information source, encoder, channel and decoder. That is, $\mathcal{Y} \rightarrow \mathcal{Z} \rightarrow \widetilde{\mathcal{Z}} \rightarrow \widetilde{\mathcal{Y}}$ forms a Markov chain.

Often, it is necessary to impose certain limitation on the input to the channel (such as average channel input power). These kinds of limitation are introduced by assuming the PDF corresponding to the channel input belongs to a smaller class $f_{Z^n} \in \mathcal{D}_{CI} \subseteq \mathcal{D}$.

B. Robust Information Theoretic Measures

In this Section, we address the following questions. First, we introduce robust analog of the Shannon entropy of the source, channel capacity and the rate distortion. Second, we find a lower bound for the robust rate distortion and finally a robust version of the Information Transmission theorem is introduced.

1) Robust Definition of Information Theoretic Measures: The robust definition of information theoretic measures are given using the mutual information between two Random Variables (R.V.'s). For recalling the definition of the mutual information, see [7] or [8]. We proceed by defining the robust Shannon entropy of the Source.

Definition 2.1: The robust Shannon entropy is defined as follows.

i) Probabilistic Model. Let Y be a R.V., and $f_Y(y) \in \mathcal{D}_{SU} \subseteq \mathcal{D}$ the corresponding PDF, where $\mathcal{D}_{SU}$ is the set of all densities induced by the R.V., Y.

A) Robust Shannon Entropy. The robust Shannon entropy associated with the family $\mathcal{D}_{SU}$ is defined by

$$
H_{robust}(f^*_Y) \stackrel{\triangle}{=} \sup_{f_Y \in \mathcal{D}_{SU}} H_S(f_Y), \qquad (2)
$$

where $H_S(.)$ denote the Shannon entropy [7].

B) Robust Shannon Entropy Rate. The robust Shannon entropy rate associated with the family $\mathcal{D}_{SU}$ of the joint PDF, Y^T is defined by

$$
\begin{aligned}
\mathcal{H}_{robust}(\mathcal{Y}) &\stackrel{\triangle}{=} \lim_{T\rightarrow\infty} \frac{1}{T} H_{robust}(f^*_{Y^T}),\\
H_{robust}(f^*_{Y^T}) &= \sup_{f_{Y^T} \in \mathcal{D}_{SU}} H_S(f_{Y^T}) \qquad (3)
\end{aligned}
$$

provided the limit exists.

ii) Frequency Domain Model. Let $\{Y_t; t \in \mathbf{N}_+\}$, $\mathbf{N}_+ \stackrel{\triangle}{=} \{0,1,2,...\}$, $Y_t : (\Omega,\mathcal{F}(\Omega)) \rightarrow (\mathcal{Y}_S,\mathcal{F}(\mathcal{Y}_S))$, $(\mathcal{Y}_S,\mathcal{F}(\mathcal{Y}_S)) = (\Re^d,\mathcal{B}(\Re^d))$ be a Wide Sense Stationary (WSS) Gaussian process and $S_Y(e^{jw}) \in \mathcal{P}_{SU} \subseteq \mathcal{P}$, where $\mathcal{P}_{SU}$ is the set of all Power Spectral Densities (PSD's) induced by the random process $\{Y_t; t \in \mathbf{N}_+\}$.

The robust Shannon entropy rate associated with the family $\mathcal{P}_{SU}$ is defined by

$$
\mathcal{H}_{robust}(\mathcal{Y}) \stackrel{\triangle}{=} \sup_{S_Y \in \mathcal{P}_{SU}} \mathcal{H}_S(S_Y), \qquad (4)
$$

where $\mathcal{H}_S(S_Y)$ is the Shannon entropy rate of the WSS Gaussian random process $\{Y_t; t \in \mathbf{N}_+\}$.
Next, we define the robust channel capacity in the presence of feedback.

Definition 2.2: (Robust Channel Capacity in the Presence of Feedback) When the channel is uncertain, the robust channel capacity in the presence of feedback is defined by

$$C_{robust} = \lim_{n \to \infty} \frac{1}{n} C_{n,robust} \tag{5}$$

$$\triangleq \lim_{n \to \infty} \frac{1}{n} \sup_{P_{Z^n} \in \mathcal{D}_{CI}} \inf_{P_{\tilde{Z}^n|Z^n} \in \mathcal{D}_{CU}} I(Z^n; \tilde{Z}^n),$$

where $I(.;.)$ denote the mutual information.
Next we proceed by defining the robust rate distortion. This is a measure of the minimum rate under which an end to end transmission with distortion, up to distortion level D_v is possible.

Definition 2.3: (Robust Rate Distortion) Let $\mathcal{D}_{DC} = \{q_{\tilde{Y}^T|Y^T}; \int_{\mathcal{Y}_{0,T-1} \times \tilde{\mathcal{Y}}_{0,T-1}} \rho_T(y^T, \tilde{y}^T) q_{\tilde{Y}^T|Y^T}(\tilde{y}^T) f_{Y^T}(y^T) .dy^T d\tilde{y}^T \leq D_v\}$ be the set of distortion constraints, in which $q_{\tilde{Y}^T|Y^T}$ is a PDF representing the probabilistic relationship between Y^T and $\tilde{Y}^T$, $D_v \geq 0$ is the distortion level, and $\rho_T : \mathcal{Y}_{0,T-1} \times \tilde{\mathcal{Y}}_{0,T-1} \to [0, \infty)$ is the distortion measure. The robust rate distortion is defined by

$$R_{robust}(D_v) = \lim_{T \to \infty} \frac{1}{T} R_{T,robust}(D_v) \triangleq \tag{6}$$

$$\lim_{T \to \infty} \frac{1}{T} \inf_{q_{\tilde{Y}^T|Y^T} \in \mathcal{D}_{DC}} \sup_{f_{Y^T} \in \mathcal{D}_{SU}} I(Y^T; \tilde{Y}^T).$$

2) Lower Bound for Robust Rate Distortion: Since the explicit expression for the robust rate distortion is difficult to obtain, it is desirable to have a lower bound which is easily computed. Moreover, these lower bound will be used in the next Section to address uniform asymptotic observability and stabilizability of control/communication system of Fig. 1.

Lemma 2.4: (Lower Bound for Robust Rate Distortion) [9]. Let $(\mathcal{Y}_S, \mathcal{F}(\mathcal{Y}_S)) = (\Re^d, \mathcal{B}(\Re^d))$, $(\tilde{\mathcal{Y}}_R, \mathcal{F}(\tilde{\mathcal{Y}}_R)) = (\Re^d, \mathcal{B}(\Re^d))$ and $f_{Y^T}(y^T) \in \mathcal{D}_{SU}$, $(y^T \in \mathcal{Y}_{0,T-1})$ denote the joint PDF corresponding to a sequence of R.V.'s with length T produced by the source. Consider a single letter distortion measure of the form $\rho_T(y^T; \tilde{y}^T) = \frac{1}{T}\sum_{i=0}^{T-1} \rho(y_i; \tilde{y}_i)$, where $\rho(y_i; \tilde{y}_i) = \rho(y_i - \tilde{y}_i) : \Re^d \to [0, \infty)$ is Borel measurable. Then a lower bound for $\frac{1}{T} R_{T,robust}(D_v)$ is given by

$$\begin{aligned} \frac{1}{T} R_{T,robust}(D_v) \quad &\geq \quad \sup_{f_{Y^T} \in \mathcal{D}_{SU}} \frac{1}{T} H_S(f_{Y^T}) \\ &\qquad - \max_{g \in G_D} H_S(g), \end{aligned} \tag{7}$$

where G_D is defined by

$$\begin{aligned} G_D \quad = \quad &\{g : \Re^d \to [0, \infty); \int_{\Re^d} g(y) dy = 1, \\ &\int_{\Re^d} \rho(y) g(y) dy \leq D_v\}. \end{aligned} \tag{8}$$

Moreover, if ρ is such that $\int e^{s\rho(y)} dy < \infty$ $(s < 0)$, the maximum over $g \in G_D$ is attained at $g^*(y)$, satisfying the following two conditions

$$\begin{aligned} g^*(y) &= \frac{e^{s\rho(y)}}{\int_{\Re^d} e^{s\rho(y)} dy} \\ \int_{\Re^d} \rho(y) g^*(y) dy &= D_v. \end{aligned} \tag{9}$$

It is also shown that if $\bar{R}(D_v) \triangleq \lim_{T \to \infty} \frac{1}{T} R_T(D_v)$ and $\mathcal{H}_S(\mathcal{Y}) \triangleq \lim_{T \to \infty} \frac{1}{T} H_S(f_{Y^T})$ exist, then the Shannon lower bound is tight [9]. That is, $\lim_{D_v \to 0} \Big(\bar{R}(D_v) - (\mathcal{H}_S(\mathcal{Y}) - H_S(g^*))\Big) = 0$.

3) Robust Information Transmission Theorem: In this Section by invoking the data processing inequality, we derive a robust version of the Information Transmission theorem. This theorem provides a necessary condition for end to end transmission up to a distortion level D_v (e.g. $E\rho_T(Y^T; \tilde{Y}^T) \leq D_v$), when there is uncertainty on the source as well as communication channel. In the next Section, this theorem will be used to relate the robust channel capacity required for uniform asymptotic observability and stabilizability to the robust rate distortion.

Theorem 2.5: (Robust Information Transmission Theorem) [9] A necessary condition for reproducing the source output Y^T up to distortion level D_v by $\widetilde{Y}^T$ at the output of the decoder for n-times channel use ($T \leq n$), when there is uncertainty on the source and communication channel, is

$$C_{n,robust} \geq R_{T,robust}(D_v). \tag{10}$$

III. Necessary Conditions for Uniform Asymptotic Observability and Stabilizability

In this Section by invoking the mathematical framework developed in the previous Section, general necessary conditions for uniform asymptotic observability and stabilizability are derived. Then the obtained results are applied to two different classes of uncertain plant. Throughout this Section, we assume that the control law at time t, $U_t = \mu(t, \tilde{Y}_0, \ldots, \tilde{Y}_t)$, is a non-anticipative functional of the decoder output up to time t. The encoder law at time t, $Z_t = \mathcal{E}(t, Y_0, Y_1, ..., Y_t, Z_0, Z_1, ..., Z_{t-1})$, is a non-anticipative functional of the information source output up to time t and the previous output of the encoder up to time $t-1$. Finally, the decoder law at time t, $\tilde{Y}_t = \mathcal{A}(t, \tilde{Z}_0, \tilde{Z}_1, ..., \tilde{Z}_t, \tilde{Y}_0, \tilde{Y}_1, ..., \tilde{Y}_{t-1})$, is a non-anticipative functional of the channel output up to time t and the previous output of the decoder up to time $t-1$.

A. Necessary Conditions for Uniform Asymptotic Observability and Stabilizability in Probability and r-Mean

In this Section, we find general necessary conditions for uniform observability and stabilizability in probability and r-mean.
Consider the control/communication system of Fig. 1. Let $(\mathcal{Y}_S, \mathcal{F}(\mathcal{Y}_S)) = (\Re^d, \mathcal{B}(\Re^d))$ and $(\tilde{\mathcal{Y}}_R, \mathcal{F}(\tilde{Y}_R)) =$

$(\Re^d, \mathcal{B}(\Re^d))$. That is, $Y_t \in \Re^d$, where Y_t is the observation from the uncertain plant obtained by sensors at time t. The objective is to find a necessary condition for uniform asymptotic observability and stabilizability in probability and r-mean defined as follows.

Definition 3.1: (Uniform Asymptotic Observability in Probability and r- Mean). Consider the control/communication system of Fig. 1.

Uniform Asymptotic Observability in Probability. The uncertain plant is uniform asymptotic observable in probability over uncertain communication channel if there exists an encoder and decoder such that

$$\lim_{t\to+\infty} \sup_{f_{Y^t}\in\mathcal{D}_{SU}} \frac{1}{t}\sum_{k=0}^{t-1} E\rho(Y_k,\tilde{Y}_k) \leq D_v, \tag{11}$$

where $f_{Y^t}(y^t)$ is the joint PDF of Y^t produced by the uncertain plant, $D_v \geq 0$ is arbitrary small and $\rho(Y_k,\tilde{Y}_k)$ is defined by

$$\rho(Y_k,\tilde{Y}_k) \triangleq \begin{cases} 1 & \text{if } ||Y_k-\tilde{Y}_k|| > \delta, \\ 0 & \text{if } ||Y_k-\tilde{Y}_k|| \leq \delta, \end{cases} \tag{12}$$

in which $||.||$ is Euclidian norm, that is, $||y-\tilde{y}|| \triangleq \Big((y-\tilde{y})^{tr}(y-\tilde{y})\Big)^{\frac{1}{2}}$ and $\delta \geq 0$ is fixed.

Uniform Asymptotic Observability in r-Mean. The uncertain plant is uniform asymptotic observable in r-mean over uncertain communication channel if there exists an encoder and decoder such that (11) is satisfied for a given fixed $D_v \geq 0$ and $\rho(Y_k,\tilde{Y}_k) = ||Y_k-\tilde{Y}_k||^r, r>0$.

Next, assume there is a linear relationship between the observed signal, Y_t, and the state variable, X_t, of the uncertain plant. That is, $Y_t = CX_t + \Upsilon_t$, where Υ_t, in general, is subject to uncertainty and it is a function of time, control signal and measurement noises. Under this assumption, the uniform asymptotic stabilizability in probability and r-mean is defined as follow.

Definition 3.2: (Uniform Asymptotic Stabilizability in Probability and r- Mean). Consider the control/communication system of Fig. 1.

Uniform Asymptotic Stabilizability in Probability. The uncertain plant is uniform asymptotic stabilizable in probability over the uncertain communication channel if there exists an encoder, decoder, and controller such that

$$\lim_{t\to\infty} \sup_{f_{Y^t}\in\mathcal{D}_{SU}} \frac{1}{t}\sum_{k=0}^{t-1} E\rho(X_k,0) \leq D_v, \tag{13}$$

where $D_v \geq 0$ is arbitrary small and $\rho(X_k,0)$ is defined by

$$\rho(X_k,0) \triangleq \begin{cases} 1 & \text{if } ||X_k-0||_{C^{tr}C} > \delta, \\ 0 & \text{if } ||X_k-0||_{C^{tr}C} \leq \delta, \end{cases} \tag{14}$$

in which $||x-0||_{C^{tr}C} \triangleq \Big(x^{tr}C^{tr}Cx\Big)^{\frac{1}{2}}$.

Uniform Asymptotic Stabilizability in r-Mean. The uncertain plant is uniform asymptotic stabilizable in r-mean if there exists an encoder, decoder and controller such that (13) is satisfied for a given $D_v \geq 0$ and $\rho(X_k,0) = ||X_k-0||^r_{C^{tr}C}, r>0$.

Next, using Lemma 2.4 and Theorem 2.5, the main result of this Section is presented in the following theorem.

Theorem 3.3: [9]. i) For uniform asymptotic observability and stabilizability in probability, a necessary condition on the robust channel capacity is

$$\mathcal{C}_{robust} \quad \geq \quad \mathcal{H}_{robust}(\mathcal{Y}) - \frac{1}{2}\log[(2\pi e)^d \det\Gamma_g], \tag{15}$$

where $\mathcal{H}_{robust}(\mathcal{Y})$ is the robust Shannon entropy rate of the observed process and Γ_g is the covariance matrix of the Gaussian distribution $g^*(y) \sim N(0,\Gamma_g), (y \in \Re^d)$ which satisfies

$$\int_{||y||>\delta} g^*(y)dy = D_v, \tag{16}$$

in which $D_v \geq 0$ is arbitrary small.

ii) A necessary condition for r-mean uniform asymptotic observability and stabilizability is

$$\mathcal{C}_{robust} \geq \mathcal{H}_{robust}(\mathcal{Y}) - \frac{d}{r} + \log(\frac{r}{dV_d\Gamma(\frac{d}{r})}(\frac{d}{rD_v})^{\frac{d}{r}}), \tag{17}$$

where $\Gamma(.)$ is the gamma function and V_d is the volume of the unit sphere (e.g., $V_d = Vol(S_d)$; $S_d \triangleq \{x \in \Re^d; ||x|| \leq 1\}$).

Remark 3.4: We have the following remarks regarding the above theorem.

i) The robust Shannon entropy rate is a function of the control signal.

ii) For the case $d=1$, condition (16) is reduced to

$$2\Phi(-\frac{\delta}{\sqrt{\Gamma_g}}) = D_v, \tag{18}$$

where $\Phi(t) \triangleq \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}du$. Using a table for this integral, we notice that for an arbitrary small D_v, $\Gamma_g = \frac{\delta^2}{16}$ should be used in (15).

iii) Finally, it is pointed out that the necessary conditions derived in Theorem 3.3, are practically important because they give flexibility to the designer to relate the observability and stabilizability error to the minimum capacity necessary for observability and stabilizability.

B. Uncertain Plants Defined via the Relative Entropy and H^∞ Constraints

In this Section, we apply Theorem 3.3 to the following uncertain plants. 1) The probabilistic uncertain plant defined via relative entropy constraint and 2) The frequency domain uncertain plant defined via H^∞ constraint.

We start by defining the probabilistic uncertain plant defined via relative entropy constraint. Let $Y^T \triangleq (Y_0, Y_1, ..., Y_{T-1})$, $Y_k : (\Omega, \mathcal{F}(\Omega)) \to (\Re^d, \mathcal{B}(\Re^d))$, $k = 0,1,...,T-1$ be a sequence of R.V.'s with length T of observation process of uncertain plant and $f_{Y^T}(y^T) \in \mathcal{D}$ denote the joint PDF of Y^T. Let g_{Y^T} be the joint PDF of Y^T produced by the following state space form.

$$(\Omega, \mathcal{F}(\Omega), \{\mathcal{F}\}_{t\geq 0}, P): \quad \begin{cases} X_{t+1} &= AX_t + BW_t + NU_t, \quad X_0 = X, \\ Y_t &= CX_t + DV_t + MU_t. \end{cases} \tag{19}$$

where $t \in \mathbf{N}_+$, $X_t \in \Re^n$ is the unobserved (state) process, $Y_t \in \Re^d$ is the observed process, $U_t \in \Re^o$ is the control signal, $W_t \in \Re^m$, $V_t \in \Re^l$, in which $\{W_t; t \in \mathbf{N}_+\}$ is Independent Identically Distributed (i.i.d.)$\sim N(0, I_{m\times m})$, $\{V_t; t \in \mathbf{N}_+\}$, is i.i.d. $\sim N(0, I_{l\times l})$, $X_0 \sim N(\bar{x}_0, \bar{V}_0)$, $\{W_t, V_t, X_0; t \in \mathbf{N}_+\}$ are mutually independent and $D \neq 0$. Here, it is assumed that (C, A) is detectable and $(A, (BB^{tr})^{\frac{1}{2}})$ is stabilizable.

Definition 3.5: (Probabilistic Uncertain Plant Defined via a Relative Entropy Constraint). The probabilistic uncertain plant is the one that its joint PDF, $f_{Y^T}(y^T) \in \mathcal{D}$, belongs to the following relative entropy uncertainty set.

$$\begin{aligned} \mathcal{D}_{SU}(g_{Y^T}) &\stackrel{\triangle}{=} \{f_{Y^T} \in \mathcal{D}; H(f_{Y^T}|g_{Y^T}) \leq TR_c\}, \\ &\quad R_c \in (0, \infty), \end{aligned} \tag{20}$$

where $H(.|.)$ denotes the relative entropy between two density functions [7].

Next, we consider Gaussian uncertain plants, in which the uncertainty is described via the H^∞ norm linear space model. Define $\beta(1) \stackrel{\triangle}{=} \{z; z \in \mathcal{C}, |z| \leq 1\}$ and let H^∞ denote the space of scalar, bounded, analytic functions of $z \in \beta(1)$. When this space is endowed with the norm $||H||_\infty \stackrel{\triangle}{=} \sup_{-\pi\leq w\leq\pi} |H(e^{jw})|, (z = e^{jw})$, $H \in H^\infty$, then $(H^\infty, ||.||_\infty)$ is a Banach space.

Definition 3.6: (Frequency Domain Uncertainty Plant Defined via an H^∞ Constraint). The uncertain plant is obtained by passing the control signals through an uncertain stable linear filter $H_U(z)$ and a stationary Gaussian random process $X_t : (\Omega, \mathcal{F}(\Omega)) \to (\Re, \mathcal{B}(\Re)), t \in \mathbf{N}_+$, with known power spectral density $S_X(e^{jw}) : \mathcal{C} \to [0, \infty)$, through an uncertain stable linear filter $\tilde{H}(z)$, defined by (see Fig. 2)

$$\begin{aligned} &\tilde{H} \in \mathcal{H}_{ad} \stackrel{\triangle}{=} \Big\{\tilde{H} \in H^\infty; \tilde{H}(z) = H(z) + \Delta(z)W(z), \\ &\tilde{H}(z), H(z), \Delta(z), W(z) \in H^\infty, \text{where} \\ &\quad H(z), W(z) \text{ are fixed, } \Delta(z) \text{ is unknown and} \\ &||\Delta||_\infty \leq 1\Big\}. \end{aligned} \tag{21}$$

Here, $Y_t \in \Re$, $H(z)$ is the nominal source transfer function based on previous experience or belief, and $\Delta(z)W(z)$ represents the uncertain part of the source. Clearly, this additive uncertainty model implies $|\tilde{H}(e^{jw}) - H(e^{jw})| \leq |W(e^{jw})|$, $\forall w \in [-\pi, \pi]$, and thus the size of uncertainty is controlled by the fixed transfer function $W(z)$.

Since $\{X_t; t \in \mathbf{N}_+\}$ is stationary and $\tilde{H}(z)$ is stable, $S_Y(e^{jw}) = |\tilde{H}(e^{jw})|^2 S_X(e^{jw})$. Consequently, the set of all PSD's of such uncertain plants is given by

$$\begin{aligned} \mathcal{P}_{SU} \stackrel{\triangle}{=} &\{S_Y(e^{jw}) \in \mathcal{P}; S_Y(e^{jw}) = \\ &|H(e^{jw}) + \Delta(e^{jw})W(e^{jw})|^2 S_X(e^{jw}), \\ &||\Delta||_\infty \leq 1\}. \end{aligned} \tag{22}$$

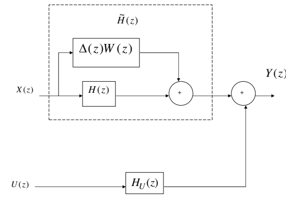

Fig. 2. Uncertain plant defined via H^∞ constraint

We repeat ([10], Proposition 3.3 and the result of Section III.B) here.

Proposition 3.7: [10]. The robust Shannon entropy rate of the uncontrolled (e.g., $\{U_t = 0; t \in \mathbf{N}_+\}$) uncertain plant corresponding to the uncontrolled version of the nominal plant (19) via the relative entropy uncertainty set (20) is

$$\begin{aligned} \mathcal{H}_{robust}(\mathcal{Y}) &= \frac{d}{2}\log(\frac{1+s^*}{s^*}) + \mathcal{H}_S(\mathcal{Y}), \\ \mathcal{H}_S(\mathcal{Y}) &\stackrel{\triangle}{=} \frac{d}{2}\log(2\pi e) + \frac{1}{2}\log\det\Lambda_\infty, \end{aligned} \tag{23}$$

where $s^* > 0$ is the unique solution of $R_c = -\frac{d}{2}\log(\frac{1+s^*}{s^*}) + \frac{d}{2s^*}$ and Λ_∞ is given by

$$\begin{aligned} \Lambda_\infty &= CV_\infty C^{tr} + DD^{tr}, \\ V_\infty &= AV_\infty A^{tr} - AV_\infty C^{tr}[CV_\infty C^{tr} + DD^{tr}]^{-1} \\ &\quad .CV_\infty A^{tr} + BB^{tr}. \end{aligned} \tag{24}$$

Remark 3.8: We have the following observations regarding the robust Shannon entropy rate found in Proposition 3.7.

i) The robust Shannon entropy rate is equal to the Shannon entropy rate if either $R_c \to 0$, or $s^* \to \infty$, as it is expected since the case, $R_c \to 0$ or $s^* \to \infty$, corresponds to the case when there is a single source.

ii) Consider the scalar version of (19), with $n = 1$ and $d = 1$. Then (24) can be solved explicitly and then substituted into (23) to obtain the following results.

A) When $B \neq 0$,

$$\begin{aligned} \mathcal{H}_{robust}(\mathcal{Y}) \geq &\frac{1}{2}\log(\frac{1+s^*}{s^*}) + \frac{1}{2}\log(2\pi e D^2) \\ &+ \max\{0, \log|A|\}. \end{aligned} \tag{25}$$

B) When B is arbitrary small ($B \cong 0$),

$$\begin{aligned} \mathcal{H}_{robust}(\mathcal{Y}) = &\frac{1}{2}\log(\frac{1+s^*}{s^*}) + \frac{1}{2}\log(2\pi e D^2) \\ &+ \max\{0, \log|A|\}. \end{aligned} \tag{26}$$

Notice that (26) contains the term $\max\{0, \log|A|\}$. Therefore, the robust Shannon entropy rate is explicitly related to the unstable eigenvalue of the system matrix A. The general case will be treated later, using the Bode integral formula.

Proposition 3.9: [10]. The robust Shannon entropy rate of the uncontrolled (e.g., $U(z) = 0$) uncertain plant defined via H^∞ constraint is

$$\begin{aligned} &\mathcal{H}_{robust}(\mathcal{Y}) = \frac{1}{2}\log(2\pi e) \\ &+\frac{1}{4\pi}\int_{-\pi}^{\pi}\log\Big((|H(e^{jw})| + |W(e^{jw})|)^2 S_X(e^{jw})dw. \end{aligned} \tag{27}$$

Remark 3.10: For the uncertain plant described by the relative entropy constraint, from the chain rule for the Shannon entropy, it follows that the robust Shannon entropy rate of the controlled plant is lower bounded by the Shannon entropy rate of the uncontrolled plant. Also, it is easily shown that the robust Shannon entropy rate of the controlled uncertain plant described by H^{∞} constraint is always lower bounded by the Shannon entropy rate of the uncontrolled uncertain plant.

Next, since each lower bound for (15) or (17) represent a necessary condition for uniform asymptotic observability and stabilizability, from Theorem 3.3, Propositions 3.7, 3.9 and Remark 3.10, we have the following corollary as necessary conditions for observability and stabilizability of uncertain plants described by the relative entropy and H^{∞} constraints.

Corollary 3.11: A necessary condition for uniform asymptotic observability and stabilizability in probability and r-mean of uncertain plants described by the relative entropy or H^{∞} constraint is given by (15) and (17) respectively, in which $\mathcal{H}_{robust}(\mathcal{Y})$ is given by Proposition 3.7 for uncertain plants described by relative entropy constraint, or by Proposition 3.9 for uncertain plants described by H^{∞} constraint.

Please note that the necessary conditions given in Corollary 3.11 are independent of the control signals, so we do not need to be aware of the control signals to present the necessary conditions, unlike [2].

Next, we apply the results of Theorem 3.3 (by invoking the Bode integral formula [11]) to the control/communication system of Fig. 1, in which the channel is Additive White Gaussian Noise (AWGN) channel, and the encoder and decoder are linear time-invariant with transfer functions equal to 1. We shall recover the results derived in [6] as special case.

Corollary 3.12: Consider the probabilistic uncertain plant described via a relative entropy constraint, in which the corresponding nominal plant is the special case of the nominal plant (19) with $X_t \in \Re^n$, $W_t \in \Re^m$, $Y_t \in \Re, U_t \in \Re$, $V_t \in \Re^l$, and $M = 0$. Assume this system is controlled over a linear time-invariant single-input single-output discrete time, additive Gaussian noise stable channel. That is, in compact notation, $\tilde{Y}(z) = H_c(z)Y(z) + W_c(z)$, where $H_c(z)$ (the channel transfer function) has poles inside unit circle and $W_c(z)$ is the frequency response of the channel noise $\{W_c(t); t \in \mathbf{N}_+\}$ which is AWGN process with mean zero and variance $\sigma^2_{W_c}$ and it is mutually independent of $\{X_0, W_t, V_t; t \in \mathbf{N}_+\}$. Assume the controller is stable linear time-invariant. That is, the controller transfer function $K_c(z)$ has poles inside unit circle. Moreover, assume the open loop transfer function $L(z) = P(z)K_c(z)H_c(z)$, $P(z) = C(zI - A)^{-1}N$ is strictly proper transfer function.

An application of Bode integral formula [11] implies that for uniform asymptotic stabilizability in r-mean, the required channel capacity must satisfy

$$\mathcal{C}^{cap} \geq \frac{1}{2}\log(\frac{1+s^*}{s^*}) + \sum_{\{i;|\lambda_i(A)|\geq 1\}} \log|\lambda_i(A)|$$

$$+\frac{1}{4\pi}\int_{-\pi}^{\pi} \log(|F(e^{jw})|^2 + DD^{tr} + |G(e^{jw})|^2\sigma^2_{W_c})dw$$

$$+\frac{1}{2}\log(2\pi e) - \Delta, \qquad (28)$$

where $\mathcal{C}^{cap}$ denote the AWGN channel capacity, $s^* > 0$ is given in Proposition 3.7, $F(e^{jw}) = C(e^{jw}I - A)^{-1}B$, $|F(e^{jw})|^2 \stackrel{\triangle}{=} F(e^{jw})F^{tr}(e^{-jw})$, $G(e^{jw}) = P(e^{jw})K_c(e^{jw})$ and $|G(e^{jw})|^2 = G(e^{jw})G^{tr}(e^{jw})$. Moreover, Δ could be equal to $\Delta = \frac{1}{2}\log\frac{\pi e\delta^2}{8}$, ($\delta > 0$ is large enough) or $\Delta = \frac{1}{r} - \log\left(\frac{r}{2\Gamma(\frac{1}{r})}(\frac{1}{rD_v})^{\frac{1}{r}}\right)$.

Remark 3.13: Condition (28) gives as special case the result of [6] in which a digital noiseless channel (with rate $\mathcal{R}$, e.g., $\mathcal{C}^{cap} = \mathcal{R}$) is used. This follows from the results of Corollary 3.12, condition (28), for $r = 2$ and D_v large, by letting the quantization parameter Δy found in [6] takes the value $\Delta y = \exp\{\Delta\}$, and setting, $R_c \to 0$, $D = 1$ and $B = 0$ for the plant and $H_c(z) = 1$ and $\sigma_{W_c} = 0$ for the channel, which implies that when these values are substituted into (28), then

$$\mathcal{R} \geq \frac{1}{2}\log(2\pi e) + \sum_{\{i;|\lambda_i(A)|\geq 1\}} \log|\lambda_i(A)| - \log\Delta y. \quad (29)$$

Clearly, (29) is precisely the condition derived in [6].

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