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On the Number of Integrators Needed for Dynamic Observer Error Linearization via Integrators

Kyung Tak Yu, Nam H. Jo and Jin H. Seo

Abstract— **In this paper, an illustrative example is given, which shows that the number of integrators needed for the dynamic observer error linearization using integrators can not be bounded by a function of the dimension of the system and the number of outputs in contrast to dynamic feedback linearization results. Some necessary conditions for dynamic observer error linearization are also provided.**

I. Introduction

Recently, a dynamic observer error linearization approach which is in some sense dual to dynamic feedback linearization [1], [2] was proposed in [3]. Sufficient conditions for a nonlinear multi-output system to be dynamically observer error linearizable by adding integrators to the outputs of the original system were provided. Although the results expand the class of nonlinear systems to which the observer error linearization approach is applicable, there still remains the problem of determining the bound of the number of the integrators needed for the dynamic observer error linearization. To solve the problem, we investigate the number of integrators needed for the dynamic observer error linearization via integrators in this paper.

An upper bound on the number of integrators needed for dynamic feedback linearization using integrators was provided in [4] and [5]. This bound can be determined by the dimension of the system and the number of inputs. In contrast to dynamic feedback linearization results, it will be shown by means of an example that the number of integrators needed for the dynamic observer error linearization using integrators can not be bounded by a function of the number of states and outputs. Under the assumption of the equal observability indices, some necessary conditions are provided for a system to be dynamic observer error linearizable.

The paper is organized as follows. Some preliminary results and notations are given in Section II. Necessary conditions for dynamic observer error linearization are given in Section III. An illustrative example is given in Section IV, which shows that the number of integrators needed for the dynamic observer error linearization using integrators can not be bounded. Finally, some conclusions follow in Section V.

II. Preliminaries

Throughout the paper, O_n denotes a $1\times n$ matrix of which all elements are zeros. O denotes a matrix of a suitable dimension of which all elements are zeros. I_n denotes an $n\times n$ identity matrix. Some basic notations of the differential geometry used in this paper can be found in [6] and [7].

A. Observer Error linearization

Consider a multi-output nonlinear system described by

$$\begin{aligned}\dot{\xi} &= f(\xi), \quad \xi \in \mathbb{R}^n, \\ y &= h(\xi) = \left[h_1(\xi), \cdots, h_s(\xi)\right]^T, y \in \mathbb{R}^s\end{aligned} \tag{2.1}$$

where $\xi = [\xi_1, \cdots, \xi_n]^T$ is a state with an initial condition $\xi(0) = \xi_0$, $y = [y_1, \cdots, y_s]^T$ is an output, $h_1(\cdot), \cdots, h_s(\cdot)$ are smooth functions on $\mathbb{R}^n$ and $f(\cdot)$ is a smooth vector field on $\mathbb{R}^n$. The system (2.1) is said to be (locally) observer error linearizable if there exist a neighborhood $U_0 \subset \mathbb{R}^n$ of the initial state ξ_0 and a state coordinate change $z = T(\xi)$ that transforms it into the nonlinear observer canonical form in $\mathbb{R}^n$:

$$\begin{aligned}\dot{z} &= A_o z + \gamma(y), \quad z \in \mathbb{R}^n, \\ y &= C_o z, \quad y \in \mathbb{R}^s\end{aligned} \tag{2.2}$$

with

$$A_o = \mathrm{diag}(A_1, \cdots, A_s), C_o = \mathrm{diag}(C_1, \cdots, C_s),$$

where

$$A_i = \begin{bmatrix} O & I_{k_i-1} \\ O & O \end{bmatrix}_{k_i\times k_i}, \quad C_i = \begin{bmatrix} 1 & O_{k_i-1}\end{bmatrix}_{1\times k_i}$$

for $i = 1, \cdots, s$. Necessary and sufficient conditions for the system (2.1) to be locally observer error linearizable were obtained in [8] and [9].

Theorem 1 ([9]): There exist a local diffeomorphism

$$z = T(\xi), \qquad z \in \mathbb{R}^n$$

and U_0, a neighborhood of the initial state ξ_0, transforming the system (2.1) into the nonlinear observer canonical form (2.2) if and only if there exists s-tuple of integers $(k_1, k_2, \cdots, k_s)$ satisfying $k_1 \geq k_2 \geq \cdots \geq k_s > 0$ and $\sum_{i=1}^{s} k_i = n$ such that we have the followings:

1) If we denote (with a possible reordering of the h_i's)

$$Q(\xi) = \{dL_f^{j-1}h_i(\xi) : i = 1, \cdots, s; j = 1, \cdots, k_i\}$$

then

$$\dim \mathrm{span} Q(\xi) = n$$

for each ξ in U_0.

2) If we denote

$$\begin{aligned} Q_j(\xi) = &\{dL_f^{k-1}h_j(\xi) : 1 \leq k \leq k_j - 1\} \\ &\cup \{dL_f^{k-1}h_i(\xi) : 1 \leq i \leq s,\ i \neq j; 1 \leq k \leq k_j\}\end{aligned}$$

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Kyungtak Yu and Jin H. Seo are with ASRI, School of Electrical Engineering and Computer Science, Seoul National Univ., Kwanak P.O.Box 34, Seoul, 151-600, Korea.

Nam H. Jo is with the School of Electrical Engineering, Soongsil University, Seoul, Korea. Corres. author: Kyungtak Yu (yktak90@naver.com)

and

$$B_j(\xi) = Q_j(\xi) \cap Q(\xi)$$

for $j = 1, \cdots, s$ then

$$\text{span}Q_j(\xi) = \text{span}B_j(\xi)$$

for $j = 1, \cdots, s$ for each ξ in U_0.

3) There exist s vector fields $g_1, \cdots, g_s$ satisfying

$$L_{g_i} L_f^{k-1} h_j = \delta_{i,j}\delta_{k,k_i}, \quad 1 \le i, j \le s, \quad 1 \le k \le k_i, \tag{2.3}$$

with $\delta_{i,j} = 0$ for $i \ne j$, $\delta_{i,i} = 1$, such that

$$[ad_f^k g_i, ad_f^\ell g_j] = 0, \quad 1 \le i, j \le s, 0 \le k \le k_i - 1, \quad 0 \le \ell \le k_j - 1.$$

Under the assumption 1 of Theorem 1, $(k_1, k_2, \cdots, k_s)$ are called the observability indices of the system (2.1) at ξ_0.

B. Dynamic Observer Error Linearization

The concept of the dynamic observer error linearization was firstly introduced in [3].

Definition 2: Consider the system described by (2.1). If there exist an auxiliary dynamic system

$$\begin{aligned} \dot{w} &= \alpha(w, y), \quad w \in \mathbb{R}^q, \\ y_e &= H(w, y), \quad y_e \in \mathbb{R}^s \end{aligned} \tag{2.4}$$

and an extended state space transformation

$$z = T(\xi, w), \quad z \in \mathbb{R}^{n+q}$$

defined in V_0, a neighborhood of an extended initial state $(\xi_0, 0) \in \mathbb{R}^{n+q}$ with $T(\xi_0, 0) = 0$ which transforms the extended system

$$\begin{aligned} \dot{x} &\triangleq \begin{pmatrix} \dot{\xi} \\ \dot{w} \end{pmatrix} = F(x) \triangleq \begin{pmatrix} f(\xi) \\ \alpha(w, h(\xi)) \end{pmatrix}, \\ y_e &= H(x) \triangleq H(w, h(\xi)) \end{aligned}$$

into the nonlinear observer canonical form in $\mathbb{R}^{n+q}$, then the system (2.1) is said to be **(locally) dynamically observer error linearizable** via the auxiliary dynamic system (2.4).

We will restrict the auxiliary dynamic system to a chain of integrators. If we add q_i integrators $\dot{w}_{i1} = h_i$ and $\dot{w}_{ij} = w_{i(j-1)}, j = 2, \cdots, q_i$ to the ith output channel of the system (2.1) for $i = 1, \cdots, s$, the extended system is given by

$$\begin{aligned} \dot{\xi} &= f(\xi), \\ \dot{w}_{i1} &= h_i(\xi), \\ \dot{w}_{ij} &= w_{i(j-1)}, j = 2, \cdots, q_i \\ y_{ei} &= H_i(x) \end{aligned} \tag{2.5}$$

for $i = 1, \cdots, s$ and $H_i(x)$ is given by

$$H_i(x) = \begin{cases} h_i(\xi), & \text{if } q_i = 0 \\ w_{iq_i}, & \text{if } q_i \ge 1 \end{cases}, \quad 1 \le i \le s,$$

If we rewrite (2.5) using vector field notations, the extended system can be described by

$$\begin{aligned} \dot{x} = F(x) &= \bar{f}(x) + \sum_{\substack{i=1 \\ q_i \ge 1}}^{s} h_i(\xi) \frac{\partial}{\partial w_{i1}} \\ &+ \sum_{\substack{i=1 \\ q_i \ge 2}}^{s} \sum_{m=2}^{q_i} w_{i(m-1)} \frac{\partial}{\partial w_{im}} \\ y_e &= H(x) = \left[H_1, \cdots, H_s\right]^T \end{aligned}$$

where $x = [\xi, w]^T$, $w^T = (w_{i1} \ \dots \ w_{iq_i};\ 1 \le i \le s,\ q_i \ge 1)$ and $\bar{f}$ represents the vector field $f(\xi)$ on the extended state space, i.e.

$$\bar{f}(x) = \sum_{i=1}^{n} f_i(\xi) \frac{\partial}{\partial \xi_i} + \sum_{\substack{i=1 \\ q_i \ge 1}}^{s} \sum_{m=1}^{q_i} 0 \ \frac{\partial}{\partial w_{im}}.$$

In this case, it is said that we extend the system on the ith channel via q_i integrators. If the system (2.1) which is extended on the ith channel via q_i integrators is dynamically observer error linearizable, it is said that the system (2.1) is dynamically observer error linearizable via integrators with the extension indices $(q_1, \cdots, q_s)$.

We have by simple calculations the following relations which will be used in the next section:

$$d(L_F^{k-1} H_i) = \begin{cases} dw_{i(q_i - k + 1)}, & \text{if } 1 \le k \le q_i \\ d(L_f^{k - q_i - 1} h_i), & \text{if } q_i + 1 \le k \end{cases}. \tag{2.6}$$

III. Necessary conditions for dynamic observer error linearization

In this section, some necessary conditions for dynamic observer error linearization are presented. For the system (2.1) satisfying the condition 1 and 2 of Theorem 1, let

$$D_i \triangleq \{g_i, \cdots, ad_f^{k_i - 1} g_i\}.$$

Recall that D_i is said to commute if for any α and $\beta \in D_i$, $[\alpha, \beta] = 0$. The following necessary condition of dynamically observer error linearizability for the system with all equal observability indices is obtained.

Lemma 3: Suppose that the system (2.1) satisfies the condition 1 of Theorem 1 with $k_1 = k_2 = \cdots = k_s$, and is dynamically observer error linearizable via integrators with extension indices $(q_1, \cdots, q_s)$ satisfying $q_m > 0$ and $q_i = 0, i \ne m$. Then, D_m commutes.

Proof: Let $k_0 \triangleq k_1 = \cdots = k_s$. Since the observability indices are all equal, the condition 2 of Theorem 1 is satisfied and the solution of (2.3) is uniquely determined. Hence, D_m is well defined. Without loss of generality, we can exchange the mth output $h_m(\xi)$ with the first output $h_1(\xi)$. Suppose that D_1 does not commute. Then there exist integers ℓ and $p \in \{1, \cdots, k_0\}$ such that

$$[ad_f^{\ell-1} g_1, ad_f^{p-1} g_1] \ne 0.$$

If we extend the system (2.1) on the first channel via q_1 integrators with $q_1 > 0$, the extended system is given by

$$\begin{aligned} \dot{x} = F(x) &= f(\xi)\frac{\partial}{\partial \xi} + h_1(\xi)\frac{\partial}{\partial w_{11}} \\ &+ \sum_{\substack{m=2 \\ q_1 \geq 2}}^{q_1} w_{1(m-1)}\frac{\partial}{\partial w_{1m}} \\ y_e = H(x) &= \left[H_i, \cdots, H_s\right]^T, \end{aligned} \tag{3.7}$$

where $x = [\xi, w]^T$, $w = [w_{11}, \cdots, w_{1q_1}]^T$ and

$$H_i(x) = \begin{cases} h_i(\xi), & \text{if } i \neq 1 \\ w_{1q_1}, & \text{if } i = 1 \end{cases}.$$

It can be easily seen that the system (3.7) satisfies the condition 1 and 2 of Theorem 1 with the observability indices $(k_0+q_1, k_0, \cdots, k_0)$ at $x_0 = [\xi_0, O_{q_1}]^T$. Using (2.6), the solution $\{\bar{g}_1, \cdots, \bar{g}_s\}$ of (2.3) for (3.7) can be easily computed as

$$\bar{g}_i(\xi) = g_i(\xi)\frac{\partial}{\partial \xi}$$

for each $i = 1, \cdots, s$. We claim that

$$ad_F^k \bar{g}_1(x) = ad_f^k g_1(\xi)\frac{\partial}{\partial \xi} + G_k(\xi)\frac{\partial}{\partial w} \tag{3.8}$$

for any nonnegative integer k, where

$$G_k(\xi)\frac{\partial}{\partial w} = \sum_{j=1}^{q_1} G_{kj}(\xi)\frac{\partial}{\partial w_{1j}}$$

is a smooth vector field we do not care about.

Proof of Claim) The claim is true for $k = 0$. As an induction hypothesis, suppose that the claim is true for some nonnegative integer m. Then, we can compute $ad_F^{m+1}\bar{g}_1$ as follows:

$$\begin{aligned} ad_F^{m+1}\bar{g}_1 &= [F, ad_F^m \bar{g}_1] \\ &= ad_f^{m+1} g_1(\xi)\frac{\partial}{\partial \xi} + f(\xi)\frac{\partial G_m(\xi)}{\partial \xi}\frac{\partial}{\partial w} \\ &- ad_f^m g_1 \frac{\partial h_1}{\partial \xi}\frac{\partial}{\partial w_{11}} - \sum_{j=1}^{q_1-1} G_{mj}(\xi)\frac{\partial}{\partial w_{1(j+1)}}. \end{aligned}$$

If we define $G_{m+1}(\xi)$ by

$$\begin{aligned} G_{m+1}(\xi)\frac{\partial}{\partial w} &= f(\xi)\frac{\partial G_m(\xi)}{\partial \xi}\frac{\partial}{\partial w} - ad_f^m g_1 \frac{\partial h_1}{\partial \xi}\frac{\partial}{\partial w_{11}} \\ &- \sum_{j=1}^{q_1-1} G_{mj}(\xi)\frac{\partial}{\partial w_{1(j+1)}}, \end{aligned}$$

we can see that (3.8) is true for $k = m+1$.
End of Proof of Claim.

Since $[ad_f^{\ell-1} g_1, ad_f^{p-1} g_1] \neq 0$ implies that

$$[ad_F^{\ell-1}\bar{g}_1, ad_F^{p-1}\bar{g}_1] = [ad_f^{\ell-1} g_1, ad_f^{p-1} g_1]\frac{\partial}{\partial \xi} + G_0\frac{\partial}{\partial w} \neq 0,$$

where G_0 is a vector field we do not care about, the extended system is not observer error linearizable. ■

Corollary 1: Suppose that the system (2.1) satisfies the condition 1 of Theorem 1 with $k_1 = k_2 = \cdots = k_s$ and D_i does not commute for each $i = 1, \cdots, s$. Then, the system (2.1) is not dynamically observer error linearizable via integrators.

Proof: Without loss of generality, we may assume that we extend the system (2.1) via integrators with extension indices $\{q_1, \cdots, q_s\}$ satisfying $q_1 \geq q_2 \geq \cdots \geq q_s$ by output reordering. The solution $\{\bar{g}_1, \cdots, \bar{g}_s\}$ of (2.3) for the extended system is uniquely determined and given by

$$\bar{g}_i(\xi) = g_i(\xi)\frac{\partial}{\partial \xi}$$

for each $i = 1, \cdots, s$. Since the remaining part of the proof is similar to the proof of Lemma 3, it is omitted. ■

Corollary 2: Suppose that the system (2.1) is a single-output system, *i.e.* $s = 1$. Then, the system (2.1) is not observer error linearizable if and only if it is not dynamically observer error linearizable via integrators.

Proof: Since the proof is a trivial application of Corollary 1, it is omitted. ■

Remark 4: Corollary 2 is dual to Theorem 2.2 in [1].

IV. Illustrative Example

Consider the following system described by

$$\begin{aligned} &\dot{\xi} = f(\xi) = \xi_{12}\frac{\partial}{\partial \xi_{11}} + \xi_{22}^m\frac{\partial}{\partial \xi_{12}} + \xi_{22}\frac{\partial}{\partial \xi_{21}} + \xi_{21}\xi_{22}\frac{\partial}{\partial \xi_{22}} \\ &y_1 = h_1(\xi) = \xi_{11}, y_2 = h_2(\xi) = \xi_{21}, \end{aligned} \tag{4.9}$$

where $\xi = [\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}]^T$ is a state and m is an integer satisfying $m \geq 2$. The initial condition of (4.9) is assumed to be the origin. The system (4.9) satisfies the condition 1 and 2 of Theorem 1 with the observability indices $k_1 = 2$ and $k_2 = 2$. The solution of (2.3) for (4.9) is given by

$$g_1 = \frac{\partial}{\partial \xi_{12}}, g_2 = \frac{\partial}{\partial \xi_{22}}.$$

We can compute $ad_f g_2$ as

$$ad_f g_2 = -m\xi_{22}^{m-1}\frac{\partial}{\partial \xi_{12}} - \frac{\partial}{\partial \xi_{21}} - \xi_{21}\frac{\partial}{\partial \xi_{22}}.$$

Since

$$[g_2, ad_f g_2] = -m(m-1)\xi_{22}^{m-2}\frac{\partial}{\partial \xi_{22}} \neq 0,$$

the system (4.9) does not satisfy the condition 3 of Theorem 1. Since $D_2 = \{g_2, ad_f g_2\}$ does not commute, the system (4.9) is not dynamically observer error linearizable via integrators with extension indices $(0, q_2)$, $q_2 > 0$.

Let q be a nonnegative integer. If we relabel the states of (4.9) as $x_{1(q+1)} = \xi_{11}, x_{1(q+2)} = \xi_{12}, x_{21} = \xi_{21}$ and $x_{22} = \xi_{22}$ and add q integrators $\dot{x}_{1i} = x_{1(i+1)}$, $i = 1, \cdots, q$ on the first channel, the extended system is given by

$$\begin{aligned} \dot{x} = F(x) &= \sum_{j=1}^{q+1} x_{1(j+1)}\frac{\partial}{\partial x_{1j}} + x_{22}^m\frac{\partial}{\partial x_{1(q+2)}} \\ &+ x_{22}\frac{\partial}{\partial x_{21}} + x_{21}x_{22}\frac{\partial}{\partial x_{22}} \\ y_{e1} &= H_1(x) = x_{11}, y_{e2} = H_2(x) = x_{21}, \end{aligned} \tag{4.10}$$

where $x = [x_{11}, x_{12}, \cdots x_{1(q+2)}, x_{21}, x_{22}]^T$ is the state and y_{e1} and y_{e2} are new outputs. The system (4.10) satisfies the condition 1 and 2 of Theorem 1 with the observability indices $k_1 = q+2$ and $k_2 = 2$. Since the system (4.10) is in observable form, we may use Proposition 3.3 in [8]: x_{22}^m is a polynomial with degree m, the system (4.10) is not observer error linearizable if $q + 2 < m$.

Since $dL_F^k H_1 = dx_{1(k+1)}$, $k = 0, \cdots, q+1$, $dH_2 = dx_{21}$ and $dL_F H_2 = dx_{22}$, a solution of (2.3) for (4.10) is of the form

$$\bar{g}_1 = \frac{\partial}{\partial x_{1(q+2)}}, \bar{g}_2 = \frac{\partial}{\partial x_{22}} + \sum_{j=1}^{q} \alpha_j(x) \frac{\partial}{\partial x_{1(j+2)}}$$

where $\alpha_j(\cdot) : \mathbb{R}^{q+4} \to \mathbb{R}$ is a smooth function we can choose for each $j = 1, \cdots, q$. To check the condition 3 of Theorem 1, we compute

$$ad_F^k \bar{g}_1 = (-1)^k \frac{\partial}{\partial x_{1(q+2-k)}}$$

for $k = 0, \cdots, q+1$. We need to check the commutativity of $\{\bar{g}_1, ad_F\bar{g}_1, ad_F^2\bar{g}_1, \cdots, ad_F^{q+1}\bar{g}_1, \bar{g}_2, ad_F\bar{g}_2\}$. For the following to hold:

$$[ad_F^k \bar{g}_1, \bar{g}_2] = (-1)^k \sum_{j=1}^{q} \frac{\partial \alpha_j}{\partial x_{1(q+2-k)}} \frac{\partial}{\partial x_{1(j+2)}} = 0$$

for $k = 0, \cdots, q+1$, we have

$$\frac{\partial \alpha_j}{\partial x_{1(q+2-k)}} = 0$$

for $j = 1, \cdots, q; k = 0, \cdots, q+1$. Therefore, $\alpha_j(x)$ should be chosen as a function of x_{21} and x_{22} only, i.e. $\alpha_j(x) = \alpha_j(x_{21}, x_{22})$ for each $j = 1, \cdots, q$. Using this fact, we can compute $ad_F\bar{g}_2$ as follows

$$\begin{aligned} ad_F\bar{g}_2 &= -\alpha_1 \frac{\partial}{\partial x_{12}} \\ &+ \sum_{j=1}^{q-1} \left\{ x_{22} \frac{\partial \alpha_j}{\partial x_{21}} + x_{22}x_{21} \frac{\partial \alpha_j}{\partial x_{22}} - \alpha_{j+1} \right\} \frac{\partial}{\partial x_{1(j+2)}} \\ &+ \left\{ x_{22} \frac{\partial \alpha_q}{\partial x_{21}} + x_{22}x_{21} \frac{\partial \alpha_q}{\partial x_{22}} - m x_{22}^{m-1} \right\} \frac{\partial}{\partial x_{1(q+2)}} \\ &- \frac{\partial}{\partial x_{21}} - x_{21} \frac{\partial}{\partial x_{22}}. \end{aligned}$$

Since $ad_F\bar{g}_2$ is a vector field that depends on x_{21} and x_{22} only, we have $[\bar{g}_1, ad_F\bar{g}_2] = [ad_F\bar{g}_1, ad_F\bar{g}_2] = [ad_F^2\bar{g}_1, ad_F\bar{g}_2] = \cdots = [ad_F^{q+1}\bar{g}_1, ad_F\bar{g}_2] = 0$. Next, we compute

$$\begin{aligned} &[\bar{g}_2, ad_F\bar{g}_2] \\ &= -\frac{\partial \alpha_1}{\partial x_{22}} \frac{\partial}{\partial x_{12}} + \sum_{j=1}^{q-1} \left\{ 2\frac{\partial \alpha_j}{\partial x_{21}} + 2x_{21}\frac{\partial \alpha_j}{\partial x_{22}} + x_{22}\frac{\partial^2 \alpha_j}{\partial x_{22}\partial x_{21}} \right. \\ &\left. + x_{21}x_{22}\frac{\partial^2 \alpha_j}{\partial x_{22}^2} - \frac{\partial \alpha_{j+1}}{\partial x_{22}} \right\} \frac{\partial}{\partial x_{1(j+2)}} \\ &+ \left\{ 2\frac{\partial \alpha_q}{\partial x_{21}} + 2x_{21}\frac{\partial \alpha_q}{\partial x_{22}} + x_{22}\frac{\partial^2 \alpha_q}{\partial x_{22}\partial x_{21}} \right. \\ &\left. + x_{21}x_{22}\frac{\partial^2 \alpha_q}{\partial x_{22}^2} - m(m-1)x_{22}^{m-2} \right\} \frac{\partial}{\partial x_{1(q+2)}} = 0. \end{aligned} \tag{4.11}$$

Since $-\frac{\partial \alpha_1}{\partial x_{22}} = 0$, α_1 is a function of x_{21} only, i.e. $\alpha_1 = \alpha_1(x_{21})$.

We should solve the following set of partial differential equations:

$$\frac{\partial \alpha_{j+1}}{\partial x_{22}} = 2\frac{\partial \alpha_j}{\partial x_{21}} + 2x_{21}\frac{\partial \alpha_j}{\partial x_{22}} + x_{22}\frac{\partial^2 \alpha_j}{\partial x_{22}\partial x_{21}} + x_{21}x_{22}\frac{\partial^2 \alpha_j}{\partial x_{22}^2} \tag{4.12}$$

for $j = 1, \cdots, q$ where we set $\alpha_{q+1} \triangleq m x_{22}^{m-1}$. Let $P^M(x_{22})$ be the set of the polynomials in x_{22} with coefficients that are C^∞ functions of x_{21}, where M is the degree with respect to x_{22}. We claim that $\alpha_j \in P^{j-1}(x_{22})$ for $j = 1, \cdots, q$.

Proof of the claim) The claim is true for $j = 1$. If the claim is true for $j = \ell$, $1 \le \ell < q$, right-hand side of (4.12) belongs to $P^{\ell-1}(x_{22})$. Since $\frac{\partial \alpha_{\ell+1}}{\partial x_{22}} \in P^{\ell-1}(x_{22})$, we have $\alpha_{\ell+1} \in P^{\ell}(x_{22})$.
End of the proof of the claim.

Since $\alpha_{q+1} = m x_{22}^{m-1}$ should be in $P^q(x_{22})$, the equation (4.12) does not have a solution if $q < m-1$. Moreover, if $q = m-1$, it can be shown that there exists a solution of (4.12) using the following power series method. Set $q = m-1$. Since $\alpha_i \in P^{i-1}(x_{22})$ for $i = 1, \cdots, m-1$, let α_i be in the form

$$\alpha_j = \sum_{k=0}^{j-1} b_{j,k}(x_{21}) x_{22}^k$$

for some smooth functions $b_{j,k}$ for $j = 1, \cdots, m-1; k = 0, \cdots, j-1$. For simplicity, we set $b_{j,k} = 0$ if $k < 0$ or $k > j-1$ for $j = 1, \cdots, m-1$. The equation (4.12) becomes

$$\begin{aligned} \sum_{k=0}^{j} k b_{j+1,k} x_{22}^{k-1} &= \sum_{k=0}^{j-1} 2\frac{db_{j,k}}{dx_{21}} x_{22}^k + \sum_{k=0}^{j-1} 2k x_{21} b_{j,k} x_{22}^{k-1} \\ &+ \sum_{k=0}^{j-1} k \frac{db_{j,k}}{dx_{21}} x_{22}^k + \sum_{k=0}^{j-1} k(k-1) x_{21} b_{j,k} x_{22}^{k-1} \end{aligned} \tag{4.13}$$

for $j = 0, \cdots, m-1$. If we equate the coefficient of x_{22}^k in (4.13), we have

$$\begin{aligned} (k+1)b_{j+1,k+1} &= (k+2)\frac{db_{j,k}}{dx_{21}} \\ &+ 2(k+1)x_{21}b_{j,k+1} + (k+1)k x_{21} b_{j,k+1} \end{aligned} \tag{4.14}$$

for $j=0,\cdots,m-1; k=0,\cdots,j-1$ with $b_{m,0}=b_{m,1}=\cdots=b_{m,m-2}=0$ and $b_{m,m-1}=m$. Given $b_{m,0},b_{m,1},\cdots,b_{m,m-2}$ and $b_{m,m-1}$, we solve (4.14) to obtain $b_{m-1,0},b_{m-1,1},\cdots,b_{m-1,m-3}$ and $b_{m-1,m-2}$:

$$\begin{aligned}&(m-1)b_{m,m-1}=m(m-1)=m\frac{db_{m-1,m-2}}{dx_{21}},\\&j=m-1;k=m-2\\&0=(k+2)(k+1)x_{21}b_{m-1,k+1}(x_{21})+(k+2)\frac{db_{m-1,k}}{dx_{21}},\\&j=m-1;k=m-3,m-4,\cdots,0.\end{aligned}\tag{4.15}$$

Therefore, we can compute $b_{m-1,k}$'s recursively (we set all integration constants to 0):

$$\begin{aligned}&b_{m-1,m-2}=(m-1)x_{21}\\&\frac{db_{m-1,k}}{dx_{21}}=-(k+1)x_{21}b_{m-1,k+1}(x_{21}),\\&k=m-3,m-4\cdots,0.\end{aligned}\tag{4.16}$$

Similarly, if $m>2$, given $b_{j,0},\cdots,b_{j,j-1}$, we can compute $b_{j-1,0},\cdots,b_{j-1,j-2}$ for $j=m-1,m-2\cdots,2$ by solving

$$\begin{aligned}&b_{j-1,j-2}=\frac{j-1}{j}\int b_{j,j-1}(x_{21})dx_{21}\\&\frac{db_{j-1,k}}{dx_{21}}=-(k+1)x_{21}b_{j-1,k+1}(x_{21})+\frac{k+1}{k+2}b_{j,k+1}(x_{21}),\\&k=j-3,j-4,\cdots,0.\end{aligned}\tag{4.17}$$

For example, if $m=2$, the equation (4.16) has a solution of the form $b_{1,0}=x_{21}$. Hence α_1 is given by

$$\alpha_1(x_{21},x_{22})=x_{21}.$$

For $m=3$, if we solve (4.16) and (4.17), we have

$$\alpha_1(x_{21},x_{22})=\frac{1}{2}x_{21}^2,\ \alpha_2(x_{21},x_{22})=-\frac{2}{3}x_{21}^3+2x_{21}x_{22}.$$

We can obtain the state transformation by solving the following partial differential equation [9]:

$$\frac{\partial T}{\partial x}=[ad_{-F}^3\bar{g}_1,\ ad_{-F}^2\bar{g}_1,\ ad_{-F}\bar{g}_1,\ \bar{g}_1,\ ad_{-F}\bar{g}_2,\ \bar{g}_2]^{-1}.$$

Hence, the following diffeomorphism

$$z=T(x)=\begin{bmatrix}x_{11}\\x_{12}-\frac{1}{6}x_{21}^3\\x_{13}-\frac{1}{2}x_{21}^2x_{22}+\frac{7}{24}x_{21}^4\\x_{14}-x_{21}x_{22}^2-\frac{2}{15}x_{21}^5+\frac{2}{3}x_{21}^3x_{22}\\x_{21}\\x_{22}-\frac{1}{2}x_{21}^2\end{bmatrix}$$

transforms (4.10) ($q=2$ and $m=3$) into

$$\dot{z}=\left(\begin{array}{cccc|cc}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\\hline 0&0&0&0&0&1\\0&0&0&0&0&0\end{array}\right)z+\begin{pmatrix}\frac{1}{6}y_{e2}^3\\-\frac{7}{24}y_{e2}^4\\\frac{2}{15}y_{e2}^5\\0\\\frac{1}{2}y_{e2}^2\\0\end{pmatrix},$$

$$[y_{e1},y_{e2}]^T=\left(\begin{array}{cccc|cc}1&0&0&0&0&0\\\hline 0&0&0&0&1&0\end{array}\right)z.$$

We conclude that the system (4.9) is not dynamically observer error linearizable via less than $m-1$ integrators but is dynamically observer error linearizable via $m-1$ integrators added on the first output channel. The example shows that the number of integrators needed for the dynamic observer error linearization via integrators can not be bounded by a function of the system dimension and the number of outputs. In contrast to our result, there exist bounds of the number of the integrators that depend on the number of states and inputs in the dynamic feedback linearization results of [4] and [5].

V. Conclusion

One may think that there may be an upper bound of the number of integrators needed for the dynamic observer error linearization using integrators since there exists an upper bound for dynamic feedback linearization using integrators. To show this conjecture is not true, we have provided an example which shows that the number of integrators needed for the dynamic observer error linearization using integrators can not be bounded by a function of the dimension of the system and the number of outputs. In addition, some necessary condition was presented for a system to be dynamic observer error linearizable under the assumption of equal observability indices. Constructive algorithms for the dynamic observer error linearization do not exist until now and this issue may be studied in the future.

References

[1] B. Charlet, J. Lévine, and R. Marino, "Sufficient conditions for dynamic state feedback linearization," SIAM Journal of Control and Optimization, vol.29, pp.38–57, 1991.

[2] B. Charlet, J. Lévine, and R. Marino, "On dynamic feedback linearization," Systems & Control Letters, vol.13, pp.143–151, 1989.

[3] D. Noh, N. H. Jo, and J. H. Seo, "Nonlinear observer design by dynamic observer error linearization," IEEE Transactions on Automatic Control, vol.49, no.10, pp.1746–1750, 2004.

[4] H. G. Lee, Y. M. Kim, and H. T. Jeon, "On the linearization via a restricted class of dynamic feedback," IEEE Transactions on Automatic Control, vol.45, pp.1385–1391, 2000.

[5] W. M. Sluis and D. M. Tilbury, "A bound on the number of integrators needed to linearize a control system," Systems & Control Letters, vol.29, pp.43–50, 1996.

[6] John M. Lee, Introduction to Smooth Manifolds, Springer-Verlag, 2000.

[7] A. Isidori, Nonlinear Control Systems, Springer-Verlag, third edition, 1995.

[8] A. J. Krener and W. Respondek, "Nonlinear observers with linearizable error dynamics," SIAM J. Control Optim., vol.23 no.2, pp197–216, 1985.

[9] X. H. Xia and W. B. Gao, "Nonlinear observer design by observer error linearization," SIAM J. Control Optim., vol.27, no.1, pp.199–216, 1989.