Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

A Global Observer for Observable Autonomous Systems with Bounded Solution Trajectories

Hao Lei, Jianfeng Wei and Wei Lin

Department of Electrical Engineering and Computer Science
Case Western Reserve University, Cleveland, Ohio 44106

***Abstract*— The problem of global observer design for autonomous systems is investigated in this paper. A constructive approach is presented for the explicit design of global observers for completely observable systems whose solution trajectories are bounded from any initial condition. Since the bound of a solution trajectory depends on the initial condition and is therefore *not known a priori,* the idea of universal control is employed to tune the observer gains on-line, achieving global asymptotic convergence of the proposed high-gain observer.**

I. Introduction

In this paper, we consider the problem of designing a *global observer* that estimates the state $z(t) \in \mathbb{R}^n$ of the autonomous system

$$\begin{aligned} \dot{z} &= f(z), \qquad z(0) = z_0 \\ y &= h(z) \end{aligned} \tag{1.1}$$

from the observation of the system output $y(t)$. The vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathrm{R}$ are assumed to be smooth functions with $f(0) = 0$ and $h(0) = 0$.

The objective of this paper is to develop, under appropriate conditions, a constructive method for the explicit design of a global convergent observer for the nonlinear system (1.1). To achieve this goal, we assume that the autonomous system (1.1) is *globally observable* in the sense of [3]. That is, the mapping

$$\begin{aligned} \Phi \; : \; & \mathbb{R}^n \to \mathbb{R}^n \\ & z \to \Big(h(z), L_f h(z), \cdots, L_f^{n-1} h(z)\Big)^T \end{aligned} \tag{1.2}$$

is a global diffeomorphism.

Under the global observability condition (1.2), there exists a global change of coordinates

$$x = \Phi(z) = \Big(h(z), L_f h(z), \cdots, L_f^{n-1} h(z)\Big)^T \tag{1.3}$$

which transforms the autonomous system (1.1) into the observable system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= F(x), \qquad y = x_1, \end{aligned} \tag{1.4}$$

where $F(x)$ is a smooth function with $F(0) = 0$.

Obviously, if a global observer can be designed for the observable system (1.4), it is straightforward to find a global observer for the original nonlinear system (1.1) using the global inverse transformation $z = \Phi^{-1}(x)$. For this reason, we shall focus our attention, in the rest of this paper, on the question of how to explicitly construct a globally convergent observer for the observable system (1.4).

In the case when $F(x) = F(x_1)$, system (1.4) reduces to the so-called observer form [2], [9] for which the design of an observer is straightforward. In fact, for a long time in the literature a common observer design method has been finding a change of coordinates and output injection so that the nonlinear system (1.1) can be transformed into the observer form: $\dot{x} = Ax + \psi(y)$ and $y = Cx$ [2], [9], [15], [11], [19]. Notably, such an observer linearization technique requires not only the state equation be transformed into a linear system driven by a nonlinear output function, but also the output of system (1.1) be linearized in the new coordinates. The latter is restrictive and limits the applications of the observer design techniques of [9], [11].

This restriction has been removed recently in [6], where only a change of coordinates is sought transforming (1.1) into a linear system steered by an output injection, without linearizing the output map of (1.1). The paper [6] has resulted in a new observer design technique based on the solution of a first-order, singular, nonlinear PDE which can be solved approximately by a series expansion method. However, the method of [6] can only be applied to either locally asymptotically stable or totally unstable systems (1.1) (see Assumption 1 in [6]). By removing the restrictive Assumption 1 in [6], the result of [6] has further been extended to a wider class of observable systems (1.1) [12] or certain nonlinear systems whose linearization is not necessarily detectable [13].

This work was supported in part by the NSF under grants DMS-0203387 and ECS-0400413, and in part by the Air Force Research Laboratory under Grant FA8651-05-C-0110.

0-7803-9568-9/05/$20.00 ©2005 IEEE

In the work [3], a high-gain observer was presented for the observable system (1.4) under the additional requirement that $F(x)$ be globally Lipschitz in $\mathbb{R}^n$ [3]. In the paper [10], a recursive observer design method was proposed for the construction of a local observer whose gains are nonlinear functions of the estimated state. Recently, the existence of a global observer has been proved for the observable system (1.1) [1], under the hypotheses that system (1.4) is output-to-state stable (OSS) [18]. However, the proof given in [1] relies heavily on the information of the Lyapunov function $V(x)$ as well as the non-negative functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$, which are used to characterize the output-to-state stability of the nonlinear system (1.4). As a such, the result obtained in [1] is primarily an existence result. It is hard to be applied to the design of a global observer for the nonlinear system (1.4). In view of the discussions above, an important question arises naturally: *when can a global observer be explicitly designed for the observable system (1.1) or (1.4)?*

In this paper, we shall address this question and provide an answer under the following condition.

Assumption 1.1: For every initial state $x(0) = x_0 \in \mathbb{R}^n$, the corresponding solution trajectory $x(t, x_0)$ of the observable system (1.1) or (1.4) is well-defined over the interval $[0, +\infty)$. Moreover, $x(t, x_0)$ is globally bounded. That is, there exists a constant $C \geq 0$ depending on the initial condition x_0, such that

$$\|x(x_0, t)\| \leq C \qquad \forall t \in [0, \infty). \tag{1.5}$$

Remark 1.2: Note that Assumption 1.1 encompasses an important class of dynamic systems such as the well-known Van der Pol equation and Duffing oscillator [4], [12], both of them have the origin as an *unstable* equilibrium and thus are *not stable in the sense of Lyapunov.* Yet, they have globally bounded solution trajectories from any initial condition.

Assumption 1.1 requires essentially that all the solution trajectories of the autonomous system (1.4) or (1.1) would not blow up for all $t \geq 0$. This appear to be a mild requirement for nonlinear systems without control inputs. However, the boundedness condition does limit the class of nonlinear systems under consideration. Consequently, the global observers proposed in this paper cannot be applied to nonlinear systems with unbounded solutions or having a finite escape time. This is a restriction of our work.

Under Assumption 1.1, we shall present in the next section a global observer that is based on the traditional high-gain observer with a subtle modification. Since the bound of the solution trajectory of the nonlinear system (1.4) or (1.1), namely $C = C(x_0)$, is usually *unknown,* a universal-type gain tuning law will be introduced, which is inspired by the recent work [17], where the idea of universal control is integrated with the non-separational principle based output feedback design method, yielding a solution to the problem of global output feedback stabilization of nonlinear systems with unknown parameters. In sharp contrast to the high-gain observer [8], the observer gain in this paper consists of two components, both of them are not constant and need to be tuned on-line in an adaptive manner. Another new ingredient of our global observer is to saturate the estimated states. However, substantially different from the paper [7] where the saturation level is a prescribed constant, the saturation level used in our global observer is not known a priori (due to the unknown bound $C(x_0)$), and hence must be updated delicately.

It should be pointed out that the explicit observer design method proposed in the next section will depend only on the knowledge of the system structure, i.e., the information of $F(x)$. There is no extra requirement on the nonlinear function $F(x)$, such as global Lipschitz or growth conditions. Moreover, the construction of the global observer involves no knowledge of any kind of Lyapunov function. All of this makes the proposed global observer easily implementable, as illustrated by the two examples in section 3.

II. Globally Convergent Observers

In this section, we show that under Assumption 1.1, a globally observable system (1.1), or equivalently, (1.4) permits a globally convergent observer. Moreover, it is possible to explicitly design a universal-like high-gain observer whose gains are adaptively updated. To make the presentation easy to follow, we give a constructive design procedure in section II-A, while in section II-B the stability analysis and the proof of convergence are included.

A. Explicit Design of Universal-like High-Gain Observers

To introduce the main result of this paper, we first recall the definition of a unit saturation function.

Definition 2.1: A unit saturation function $sat(s)$ is defined as

$$\mathrm{sat}(s) = \begin{cases} 1 & \text{if } \; s > 1 \\ s & \text{if } \; |s| \leq 1 \\ -1 & \text{if } \; s < -1 \end{cases} \tag{2.1}$$

According to the definition, it is not difficult to see that a unit saturation function has the following useful property to be used in the sequel.

Lemma 2.2: Given real numbers s_1, s_2 and $m > 0$, suppose that $|s_1| \leq m$. Then,

$$|s_1 - m\mathrm{sat}(\frac{s_2}{m})| \leq |s_1 - s_2|. \tag{2.2}$$

Theorem 2.3: Under Assumption 1.1, there exists a global observer for the observable system (1.4). In particular, a globally convergent observer can be constructed as

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + (MN)a_1(y - \hat{x}_1) \\
&\vdots \\
\dot{\hat{x}}_{n-1} &= \hat{x}_n + (MN)^{n-1}a_{n-1}(y - \hat{x}_1) \\
\dot{\hat{x}}_n &= F\big(\mathrm{sat}_N(\hat{x})\big) + (MN)^n a_n(y - \hat{x}_1) \\
\dot{N} &= \gamma\big(\frac{y - \hat{x}_1}{MN}\big)^2, \qquad N(0) = 1 \\
\dot{M} &= -M + \Delta(N), \qquad M(0) = 1
\end{aligned} \tag{2.3}$$

where $a_i > 0, i = 1, \cdots, n$ are the coefficients of the Hurwitz polynomial $s^n + \sum_{i=1}^n a_i s^{n-i}$, $\gamma \geq 1$ is a prescribed constant and $\text{sat}_N(\hat{x}) := \big(N\text{sat}(\frac{\hat{x}_1}{N}), \cdots, N\text{sat}(\frac{\hat{x}_n}{N})\big)$.

Moreover, all the states of the closed-loop system (1.4)-(2.3) are well-defined and bounded on $[0, \infty)$. In addition,

$$\lim_{t\to\infty} \|x(x_0, t) - \hat{x}(\hat{x}_0, t)\| = 0, \quad \forall (x_0, \hat{x}_0) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Remark 2.4: Notably, (2.3) is an adapted high-gain observer that is inspired by the idea of universal control [20], [5]. The observer gain $L = MN$ is composed of two parts. One is $N(t)$ — the level of saturation — which is updated in a way similar to the one suggested in [17]. The other one is the gain $M(t)$ to be tuned through a linear ODE driven by a nonlinear function of $N(t)$. The novelty of the global observer (2.3) lies in the introduction of the moving saturation level $N(t)$ enabling one to overcome the difficulty caused by the lack of the knowledge of $C(x_0)$ — the bound of solution trajectories of the observable system (1.1) or (1.4).

It is important to point out that $\Delta(N)$ in the observer (2.3) can be calculated directly based on the observable system (1.4), in particular, by the nonlinear function $F(x)$. To see how, we introduce the following technical lemma whose proof can be carried out in a fashion similar to that of Lemma 2.2 in [16], and hence is omitted here.

Lemma 2.5: Let $g : \mathbb{R}^n \to \mathbb{R}$ be a C^1 real-valued function. Then, there exist two smooth functions $\alpha, \beta : [0, +\infty) \to [1, +\infty)$, such that $\forall x, z \in \mathbb{R}^n$,

$$|g(x) - g(z)| \leq \alpha(||x||)\beta(||z||)\big(\sum_{i=1}^{n} |x_i - z_i|\big). \qquad (2.4)$$

With the help of inequality (2.5), $\big|F(x) - F(\text{sat}_N(\hat{x}))\big|$ can be estimated as follows. By Assumption 1.1, $||x(t, x_0)|| \leq C$, $\forall t \geq 0$. Since $||\text{sat}_N(\hat{x})|| \leq N$, by Lemma 2.5 there exist two smooth positive functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that

$$\begin{aligned} &\big|F(x) - F(\text{sat}_N(\hat{x}))\big| \\ &\leq \alpha(C)\beta(N)\big(\sum_{i=1}^{n} \big|x_i - N\text{sat}(\frac{\hat{x}_i}{N})\big|\big) \end{aligned} \qquad (2.5)$$

Using this estimation, one can simply choose

$$\Delta(N) = \beta^2(N) \geq 1. \qquad (2.6)$$

In the next subsection, it will be shown that such a choice of $\Delta(N)$ suffices to ensure the dynamic system (2.3) being a globally convergent observer of system (1.4). The reader is also referred to the two examples in section 4 for further details on how $\Delta(N)$ in the observer (2.3) can be explicitly determined from the function $F(x)$.

To sum up, a global observer for the observable system (1.4) with bounded solutions trajectories can be constructed in three steps:

Step 1. Pick a suitable $\gamma > 0$ and choose constants $a_i > 0$, $i = 1, \cdots, n$, such that $p(s) = s^n + \sum_{i=1}^n a_i s^{n-i}$ is Hurwitz;

Step 2. Use inequality (2.5) to estimate $\big|F(x) - F(\text{sat}_N(\hat{x}))\big|$ and find $\beta(N) \geq 1$. Then, compute $\Delta(N) = \beta^2(N)$;

Step 3. With the obtained parameters γ, a_i's and $\Delta(N)$, design the observer (2.3).

B. Analysis of Boundedness and Convergence

We now show that Theorem 2.3 holds. That is, the observer (2.3) designed in the previous subsection works and it is indeed a globally convergent observer for the observable system (1.4).

We begin with the proof by examining the property of the error dynamics. Let $e_i = x_i - \hat{x}_i$, $i = 1, 2, \cdots, n$ be the estimate errors and denote $L = MN$. Then, the error dynamics is given by

$$\begin{aligned} \dot{e}_i &= e_{i+1} - L^i a_i e_1, \qquad i = 1, 2, \cdots, n-1 \\ \dot{e}_n &= F(x) - F(\text{sat}_N(\hat{x})) - L^n a_n e_1 \end{aligned} \qquad (2.7)$$

Similar to the routine in the analysis of high-gain observers [8], we introduce the change of coordinates

$$\varepsilon_i = \frac{e_i}{L^i}, \qquad i = 1, 2, \cdots, n. \qquad (2.8)$$

By construction, $L(t) = M(t)N(t) \geq 1$ $\forall t \geq 0$. This is because $N(t)$ is a non-decreasing function with $N(0) = 1$. Moreover, $M(t) \geq 1$ due to the choice of $\Delta(N) \geq 1$ and $M(0) = 1$.

In the new coordinates, the error dynamics (2.7) can be expressed in the following compact form

$$\dot{\varepsilon} = LA\varepsilon + \frac{1}{L^n} b_0 \big[F(x) - F\big(\text{sat}_N(\hat{x})\big)\big] - \frac{\dot{L}}{L} D\varepsilon \qquad (2.9)$$

where

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}, \qquad A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}$$

$$b_0 = [0, \cdots, 0, 1]^T, \quad D = \text{diag}(1, 2, \cdots, n)$$

Since a_i's are the coefficients of a stable polynomial, A is a Hurwitz matrix. Therefore, by [14] (see inequality (6)) there exist a positive definite matrix $P = P^T$ and real constants $c_2 > c_1 > 0$, such that

$$\begin{aligned} A^T P + PA &\leq -I \\ c_1 I \leq DP + PD &\leq c_2 I \end{aligned} \qquad (2.10)$$

Now, consider the Lyapunov function $V(\varepsilon) = \varepsilon^T P \varepsilon$ for the error dynamics (2.9). A direct calculation gives

$$\begin{aligned} \dot{V} \leq & -L\|\varepsilon\|^2 - \frac{\dot{L}}{L}\varepsilon^T(DP + PD)\varepsilon \\ & + 2\varepsilon^T P b_0 \frac{1}{L^n}\big[F(x) - F(\text{sat}_N(\hat{x}))\big] \end{aligned} \qquad (2.11)$$

Observe that

$$\begin{aligned}\frac{\dot{L}}{L} &= \frac{\dot{M}}{M}+\frac{\dot{N}}{N}, \qquad\qquad \dot{N}=\gamma\varepsilon_1^2\geq 0\\ \dot{M} &= -M+\beta^2(N), \quad N\geq 1 \ \text{ and } M\geq 1.\end{aligned}$$

With these in mind and using (2.10) and (2.5), we deduce from (2.11) that

$$\begin{aligned}\dot{V} &\leq -L\|\varepsilon\|^2-\frac{\dot{M}}{M}\varepsilon^T(DP+PD)\varepsilon\\ &\quad +\frac{2}{L^n}\varepsilon^T Pb_0\big[F(x)-F(\text{sat}_N(\hat{x}))\big]\\ &\leq -M(N-c_2)\|\varepsilon\|^2-c_1\frac{\beta^2(N)}{M}\|\varepsilon\|^2 \qquad (2.12)\\ &\quad +\frac{2\|Pb_0\|\alpha(C)\beta(N)}{L^n}\sum_{i=1}^{n}\big|x_i-N\text{sat}(\frac{\hat{x}_i}{N})\big|\cdot\|\varepsilon\|.\end{aligned}$$

Using inequality (2.12), we can prove that starting from any initial condition $(x_0,\hat{x}_0)\in\mathbb{R}^n\times\mathbb{R}^n$ and $M(0)=N(0)=1$, system (1.4)-(2.3) has the following properties:

- **(i)** All the states of the dynamic system (1.4)-(2.3) are well defined and globally bounded on $[0,+\infty)$;
- **(ii)** $\lim_{t\to\infty}e(t)=0$, $\lim_{t\to+\infty}M(t)=M_\infty$, and $\lim_{t\to+\infty}N(t)=N_\infty$.

Since $x(t,x_0)$ of the observable system (1.4) is bounded by C, the property (i) follows immediately if one can show that the error signal $e(t)=x(t)-\hat{x}(t)$ and the observer gain $(M(t),N(t))$ are well-defined and globally bounded on $[0,+\infty)$. In what follows, we use a contradiction argument to prove that this is indeed the case.

Consider the error dynamics (2.9) or, equivalently, (2.7) and assume that it has a solution $X(t):=(N(t),M(t),e(t))$ which is neither well defined nor globally bounded on $[0,+\infty)$. Then, there is a maximal time interval $[0,t_f)$ on which $X(t)$ is well defined. In addition,

$$\lim_{t\to t_f}||(N(t),M(t),e(t))||=+\infty. \qquad (2.13)$$

That is, $t_f>0$ is a finite escape time of the dynamic system (2.3)-(2.9).

Claim 1: $N(t)$ *cannot escape at* $t=t_f$.

If $N(t)$ has a finite escape time t_f, $\lim_{t\to t_f}N(t)=+\infty$. By construction, $\dot{N}\geq 0$ and $N(t)$ is a monotone nondecreasing function. Thus, there exists a time $t_1^*\in[0,t_f)$ such that

$$N(t)\geq C\geq|x_i(t)|, \qquad t\in[t_1^*,t_f).$$

This, together with Lemma 2.2, yields

$$\big|x_i-N\text{sat}(\frac{\hat{x}_i}{N})\big|\leq|e_i|, \quad t\in[t_1^*,t_f). \qquad (2.14)$$

Using (2.14) and $\varepsilon_i=e_i/L^i$ with $L\geq 1$, we have

$$\begin{aligned}&\frac{1}{L^n}\|2Pb_0\|\alpha(C)\beta(N)\sum_{i=1}^{n}\big|x_i-N\text{sat}(\frac{\hat{x}_i}{N})\big|\cdot\|\varepsilon\|\\ \leq\ &\|2Pb_0\|\alpha(C)\beta(N)\frac{1}{L^n}\sum_{i=1}^{n}|e_i|\cdot\|\varepsilon\|\\ \leq\ &\big(\frac{M}{c_1}(c_0\alpha(C))^2+\frac{c_1\beta^2(N)}{M}\big)\|\varepsilon\|^2, \qquad t\in[t_1^*,t_f)\end{aligned}$$

where $c_0>0$ is a suitable real constant.

In view of this estimation, it follows from (2.12) that

$$\dot{V}\leq -M\big(N-c_2-\frac{(c_0\alpha(C))^2}{c_1}\big)\|\varepsilon\|^2, \quad t\in[t_1^*,t_f).$$

Since $\lim_{t\to t_f}N(t)=+\infty$, there is a $t_2^*\in[t_1^*,t_f)$ such that

$$N(t)\geq 1+c_2+\frac{c_0\alpha(C))^2}{c_1}, \quad t\in[t_2^*,t_f).$$

Using the last two inequalities and noting that $M(t)\geq 1$, we arrive at

$$\dot{V}\leq-\|\varepsilon\|^2\leq-\varepsilon_1^2=-\frac{1}{\gamma}\dot{N}, \quad \forall t\in[t_2^*,t_f). \qquad (2.15)$$

Consequently,

$$+\infty=N(t_f)\leq\gamma V(\varepsilon(t_2^*))+N(t_2^*)=constant, \quad (2.16)$$

which is a contradiction.

Therefore, Claim 1 is true and $N(t)$ is well-defined and bounded on $[0,t_f]$.

Claim 2 : $M(t)$ *is well-defined and bounded on* $[0,t_f]$.

By the boundedness of $N(t)$, there exists a real constant $d>1$ such that $\Delta(N(t))\leq d,\ t\in[0,t_f]$. Hence,

$$\dot{M}=-M+\Delta(N)\leq -M+d, \quad \forall t\in[0,t_f],$$

which implies that $M(t)$ is bounded on $[0,t_f]$.

Claim 3 : $e(t)$ *is well-defined and bounded on* $[0,t_f]$.

To show this claim, we rescale system (2.7) by introducing the transformation

$$\xi_i=\frac{e_i}{L^{*i}}, \quad L^*=MN^*, \quad i=1,\cdots,n, \qquad (2.17)$$

where $N^*>0$ is a constant to be determined later.

In the ξ-coordinates, the error dynamic system (2.7) can be expressed as

$$\begin{aligned}\dot{\xi} &= L^*A\xi+L^*a\xi_1-L\Gamma a\xi_1-\frac{\dot{M}}{M}D\xi\\ &\quad +b_0\frac{1}{L^{*n}}\big[F(x)-F(\text{sat}_N(\hat{x}))\big] \qquad (2.18)\end{aligned}$$

where $\xi=(\xi_1,\cdots,\xi_n)^T$, $a=(a_1,\cdots,a_n)^T$ and $\Gamma=\text{diag}(1,\frac{N}{N^*},\cdots,(\frac{N}{N^*})^{n-1})$.

For the rescaled system (2.18), consider the Lyapunov function $W(\xi)=\xi^TP\xi$ with P satisfying (2.10). Then, a straightforward calculation gives

$$\begin{aligned}\dot{W} &\leq -(L^*-c_2)\|\xi\|^2-c_1\frac{\Delta(N)}{M}\|\xi\|^2+2L^*\xi^TPa\xi_1\\ &\quad -2L\xi^TP\Gamma a\xi_1+\frac{2\xi^TPb_0}{L^{*n}}\Big(\big[F(x)-F(\text{sat}_C(\hat{x}))\big]\\ &\quad +\big[F(\text{sat}_C(\hat{x}))-F(\text{sat}_N(\hat{x}))\big]\Big) \qquad (2.19)\end{aligned}$$

Using the boundedness of $N(t)$ and $M(t)$ and Lemma 2.5, we can obtain the following estimations:

$$\begin{aligned}
&|2L^*\xi^T Pa\xi_1| \le \|\xi\|^2 + (L^*\|Pa\|)^2\xi_1^2 \\
&|2L\xi^T P\Gamma a\xi_1| \le \|\xi\|^2 + L^2\|P\Gamma a\|^2\xi_1^2 \\
&\left|2\xi^T Pb_0 \frac{1}{L^{*n}}\left[F(x) - F(\text{sat}_C(\hat{x}))\right]\right| \le \theta_1\|\xi\|^2 \\
&\left|2\xi^T Pb_0 \frac{1}{L^{*n}}\Big(F(\text{sat}_C(\hat{x})) - F(\text{sat}_N(\hat{x}))\Big)\right| \\
&\quad \le \theta_2\|\xi\| \le \theta_2^2\|\xi\|^2 + 1
\end{aligned}$$

where θ_1 is a positive constant depending on C, while $\theta_2 > 0$ is a constant depending on C and the bound of $N(t)$.

Substituting the four estimations into (2.19) and choosing $N^* = \max\{3 + c_2 + \theta_2^2 + \theta_1, N(t_f)\}$, one has

$$\begin{aligned}
\dot{W} &\le -\|\xi\|^2 + \big((L^*\|Pa\|)^2 + L^2\|P\Gamma a\|^2\big)\xi_1^2 + 1 \\
&\le -\mu W(\xi) + \mu_1\varepsilon_1^2 + 1, \qquad t \in [0, t_f) \qquad (2.20)
\end{aligned}$$

where μ and μ_1 are suitable positive constants.

From (2.20) it follows that $\frac{d(e^{\mu t}W)}{dt} \le e^{\mu t}(\mu_1\varepsilon_1^2 + 1)$. Hence,

$$W(\xi(t)) \le W(\xi(0)) + \mu_1\int_0^{t_f}\varepsilon_1^2 ds + \frac{1}{\mu}, \ \forall t \in [0, t_f).$$

Note that $\dot{N} = \gamma\varepsilon_1^2$. Thus, the boundedness of $N(t)$ on $[0, t_f]$ guarantees the boundedness of $\int_0^{t_f}\varepsilon_1^2(t)dt$. With this in mind, it is concluded that $W(\xi)$ is bounded on $[0, t_f]$, so is $\xi(t)$. In view of (2.17) and (2.8), both $e(t)$ and $\varepsilon(t)$ are bounded on $[0, t_f]$ as well.

Putting the three claims together, we arrive at the conclusion that all signals $N(t), M(t), e(t)$ are well-defined and bounded on $[0, t_f]$, which contradicts to the assumption (2.13). Therefore, the dynamics system (2.3)-(2.9) has no finite escape time over $[0, +\infty)$ and the property (i) holds.

By construction, $N(t)$ is a monotone nondecreasing function. This, together with the boundedness property, implies that $\lim_{t\to\infty} N(t) = N_\infty$. Consequently, it follows immediately from the last equation of (2.3) that $\lim_{t\to\infty} M(t) = M_\infty$.

Finally, we show that $\lim_{t\to\infty} e(t) = 0$. To this end, observe that both ε_1 and $\dot{\varepsilon}_1$ are bounded on $[0, +\infty)$. Moreover, $\varepsilon_1 \in L_2$ as $N(t)$ is globally bounded. By the Barbalat's Lemma, we have $\lim_{t\to\infty}\varepsilon_1 = \lim_{t\to\infty} e_1 = 0$.

On the other hand, boundedness of all the states guarantees that the ω-limit set Ω of the dynamic systems composed of (1.4), (2.3) and (2.7) is nonempty, closed and invariant. In view of $\lim_{t\to\infty} e_1 = 0$, $\Omega \subseteq \{(e_1, e_2, \cdots, e_n, x, \hat{x}, M, N) | e_1 = 0\}$.

Note that $e_{i+1} = \dot{e}_i + L^i a_i e_1, \ i = 1, \cdots, n-1$. By the invariance of Ω, we conclude that $e_2 = e_3 = \cdots = e_n = 0$ on Ω. Therefore, $\lim_{t\to\infty}\|e(t)\| = 0$. This completes the proof of Theorem 2.3.

Using Theorem 2.3, it is easy to obtain an important corollary which is devoted to the design of a global observer for observable systems in the triangular form

$$\begin{aligned}
\dot{z}_i &= z_{i+1} + f_i(z_1, z_2, \cdots, z_i), \quad i = 1, 2, \cdots, n-1 \\
\dot{z}_n &= f_n(z), \qquad y = z_1 \qquad (2.21)
\end{aligned}$$

where $z = (z_1, \cdots, z_n)^T \in \mathbb{R}^n$, and $y \in \mathbb{R}$ are the system state and output, respectively, $f_i(z_1, \cdots, z_i), i = 1, \cdots, n$ are smooth functions with $f_i(0, \cdots, 0) = 0$.

Due to the lower-triangular structure, one can explicitly construct a global change of coordinates $x = \Psi(z)$ which renders system (2.21) globally diffeomorphic to system (1.4). As a consequence, we have the following conclusion.

Corollary 2.6: Assume that all the solution trajectories of the lower-triangular system (2.21) from any initial condition are well-defined and bounded on $[0, +\infty)$. Then, a global convergent observer exists and can be explicitly constructed.

III. Application and Simulation

In this section, we give two examples to illustrate the validity of the proposed global observer (2.3).

Example 3.1: Consider the Van der Pol oscillator

$$\begin{aligned}
&\dot{x}_1 = x_2 \\
&\dot{x}_2 = -x_1 + x_2(1 - x_1^2) \qquad (3.1) \\
&y = x_1
\end{aligned}$$

which is of the form (1.4).

It has been known that this oscillator is not stable at the origin but its solution trajectories are well-defined and globally bounded on $[0. + \infty)$. In other word, Assumption 1.1 holds. By Theorem 2.3, one can explicitly design a globally convergent observer for the planar system (3.1).

Following the algorithm given in section 2, we choose $a_1 = a_2 = 1$ and $\gamma = 5$. To determine $\Delta(N)$ from $F(x) = -x_1 + x_2(1 - x_1^2)$, we observe that

$$\begin{aligned}
&\left|F(x) - F(\text{sat}_N(\hat{x}))\right| = \Big|\sum_{i=1}^{2}\frac{\partial F}{\partial \xi_i}(\xi)(x_i - N\text{sat}(\frac{\hat{x}_i}{N}))\Big| \\
&\le (|-1 - 2\xi_1\xi_2| + |1 - \xi_1^2|)\Big(\sum_{i=1}^{2}\big|x_i - N\text{sat}(\frac{\hat{x}_i}{N})\big|\Big) \\
&\le \big(2 + 3(C+N)^2\big)\Big(\sum_{i=1}^{2}\big|x_i - N\text{sat}(\frac{\hat{x}_i}{N})\big|\Big) \qquad (3.2)
\end{aligned}$$

where C is the bound of $x(x_0, t)$ for any x_0, and $\xi \in \mathbb{R}^2$ is a point on the line between x and $\text{sat}_N(\hat{x})$, and hence $|\xi_i| \le C + N, i = 1, 2$. Noting that $\big(2 + 3(C+N)^2\big) \le 6(\frac{4}{3} + C^2)(N^2 + 1)$, one can choose $\beta(N) = N^2 + 1$, and hence $\Delta(N) = (N^2 + 1)^2$.

Substituting the parameters and function thus obtained into (2.3) , one can get a global observer for system (3.1).

Fig. 1 shows the transient responses of the obtained observer and Van der Pol oscillator, with the initial condition $(x_1(0), x_2(0), \hat{x}_1(0), \hat{x}_2(0)) = (1, 2, 3, -2)$.

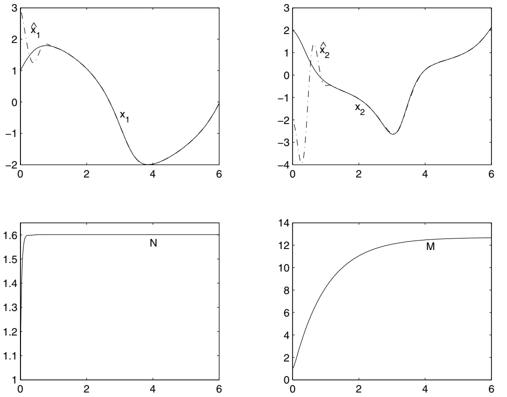

Fig. 1. Observation of Van der Pol oscillator

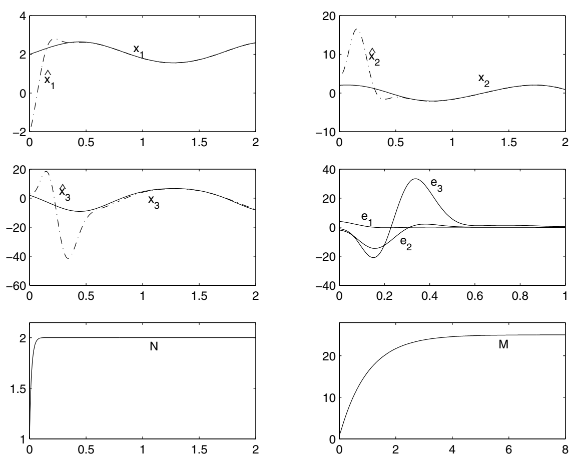

Fig. 2. Estimation of the observable system (3.3)

Example 3.2: Consider the three-dimension system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_2 - 3x_1^2 x_2, \qquad y \;=\; x_1 \end{aligned} \tag{3.3}$$

whose solution trajectories from any initial condition are closed orbits. In fact, it is easy to verify that for Lyapunov function $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \frac{1}{2}(x_1^3 + x_3)^2$, its derivative along the trajectories of system (3.3) is identical to zero. Thus, system (3.3) satisfies Assumption 1.1.

Using the observer design method in section 2, one can simply pick $a_1 = 1, a_2 = 3, a_3 = 1$ and $\gamma = 5$. Similar to what was done in the previous example, we can choose, from the structure of $F(x) = -x_2 - 3x_1^2 x_2$, $\beta(N) = N^2 + 1$ and hence $\Delta(N) = (1 + N^2)^2$. Consequently, a globally convergent observer of the form (2.3) can be obtained for the autonomous system (3.3).

Fig. 2 gives the simulation results of the observer and the autonomous system (3.3) starting from the initial condition $(x_1, x_2, x_3, \hat{x}_1, \hat{x}_2, \hat{x}_3) = (2, 2, 2, -2, 4, 3)$.

IV. Conclusions

Under the global boundedness and observability conditions, we have shown that a globally convergent observer can be explicitly designed for autonomous systems. The constructed observer is of high-gain type but different from the traditional one [8] in the sense that the observer gains here are composed of two time-varying components $M(t)$ and $N(t)$, both of them are adaptively updated in order to deal with the issue of the unknown bound of the solution trajectories. The gain update law is reminiscent from the recent work [17] on universal output control of nonlinear systems with unknown parameters. What is new here is the application of the saturated state estimates whose saturation level $N(t)$, compared with the one used in [7], is unknown and required to be tuned on-line.

Although Assumption 1.1 does not require Lyapunov stability of nonlinear systems, it prevents the observer (2.3) being applicable to the autonomous system (1.1) or (2.21) with unbounded solution trajectories. Relaxing Assumption 1.1 and extending Theorem 2.3 to nonlinear systems that may not satisfy Assumption 1.1 will be an interesting topic for future research.

References

[1] Astolfi, A. and L. Praly, Global complete observability and out-to-state stability imply the existence of a gloabally convergent observer, *Proc. of the 42th IEEE CDC, Maui, Hawaii,* 1562-1567 (2003).

[2] D. Bestle and M. Zeitz, Canonocal form observer design for nonlinear time variable systems, *Int. J. Control,* **38**, 419-431 (1983).

[3] J. P. Gauthier, H. Hammouri and S. Othman, A simple observer for nonlinear systems, applications to bioreactocrs, *IEEE Trans. Automat. Contr.,* **37**, 875-880 (1992).

[4] J. Guckenheimer and P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, Springer, NY (1983).

[5] A. Ilchmann, Non-identifier based high gain adaptive control, *Lecture Notes in Contr. & Inform. Sci.,* vol. 189, Springer-Verlag, NY (1984).

[6] Kazantzis N. and C. Kravaris, Nonlinear observer design using Lyapunov's Auxiliary theorem, *Syst. Contr. Lett*, **34**, 241-247 (1998).

[7] H. K. Khalil and F. Esfandiari, Semiglobal stabilization of a class of nonlinear systems using output feedback, *IEEE Trans. Automat. Contr.,* **38**, 1412 - 1415 (1993)

[8] H. K. Khalil and A. Saberi, Adaptive stabilization of nonlinear systems using high-gain feedback, *IEEE-TAC.*, **32,** 1031-1035 (1987).

[9] Krener, A. J. and A. Isidori, Linearization by output injection and nonlinear observer, *Syst. & Contr. Lett.,* **3**, 47-52 (1983).

[10] Krener, A. and W. Kang, Locally convergent nonlinear observers,*SIAM J. Contr. Optimiz.*, **42**, 155–177 (2003).

[11] Krener, A. J. and W. Respondek, Nonlinear observers with linearizable error dynamics, *SIAM J. Contr. Optimiz.,* **23**, 197–216 (1985).

[12] Krener, A. J. and M. Xiao, Nonlinear observers design in the Siegel domain, *SIAM J. Contr. Optimiz.,* **41**, 932–953 (2002).

[13] Krener, A. J. and M. Xiao, Observers for linearly unobservable nonlinear systems, *Syst. Contr. Lett.,* **46**, 281-288 (2002).

[14] Krishnamurthy, P. and E. Khorrami, Generalized adaptive output feedback form with unknown parameters, *Proc. of the 41st IEEE CDC*, Las Vages, NV, 1503–1508 (2002).

[15] W. Lin and C. I. Byrnes, Remarks on linearization of autonomous systems and nonlinear observer design, *SCL,* vol. 25, 31-40 (1995).

[16] W. Lin and R. Pongvuthithum, Global stabilization of cascade systems by C^0 partial state feedback, *IEEE Trans. Automat. Contr.,* vol. 47, pp. 1356-1362 (2002).

[17] H. Lei and W. Lin, Universal output feedback control of nonlinear systems with unknown growth rate, *Proc. of the 16th IFAC World Congress, Prague,* July 4-8 (2005). Also, to appear in Automatica.

[18] E. Sontag and Y. Wang, Output-to-state stability and detectability of nonlinear systems, *Syst. & Contr. Lett.*, **29**, 279-290 (1997).

[19] Xia, X. and W. B. Gao (1989). Nonlinear observer design by observer error linearization, *SIAM J. Contr. Optimiz.*, **27**, 199–216.

[20] J. C. Willems and C. I. Byrnes, Global adaptive stabilization in the absence of information on the sign of the high frequency gain, *Lecture Notes in Contr. & Info. Sci.,* **62,** Springer-Verlag, (1984).