Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On Output Tracking Using Dynamic Output Feedback Discrete-Time Sliding Mode Controllers

Nai One Lai, Christopher Edwards & Sarah K. Spurgeon

Abstract— **An output feedback based discrete time sliding mode control scheme is proposed. It incorporates a tracking requirement and is dynamic in nature. Previous work has shown that, with an appropriate choice of sliding surface, discrete time sliding mode control can be applied to non-minimum phase systems. The original scheme employed static output feedback and this imposed restrictions on the class of systems to which it was applicable. The scheme proposed in this paper includes a compensator and so some of the restrictions relating to stabilizability by outputs alone can be removed.**

I. INTRODUCTION

Many conventional (continuous time) sliding mode control design schemes assume that all the states of the plant are directly accessible. In real systems, it often happens that not all system states are fully available or measurable. One solution is to use an observer to reconstruct the system states [21], [25], [7]. Alternatively output feedback strategies can be employed in which the control law only requires knowledge of measured outputs [6], [9], [1]. There are, however, inherent system restrictions on using output feedback sliding mode control design procedures. Normally, the most serious limitation of output feedback sliding mode control is that the system must be relative degree one and minimum phase [8]. The minimum phase restriction in particular arises from the fact that the system zeros appear amongst the poles governing the sliding motion.

Compared with continuous time sliding mode strategies, the design problem in discrete time has been much less studied. With the exception of the early work in [20], most of the literature assumes all states are available [11], [12], [13], [14], [24]. Schemes which have restricted themselves to output measurements alone have invariably utilised observers [23]. Recent exceptions have been the work of [15], [18] and the discrete time versions of certain higher-order sliding mode control schemes in [3], [4]. In particular [15] considered an output tracking problem for an uncertain linear system using sliding mode ideas which requires output information only. It was shown in [15] that the requirements of relative degree and minimum phaseness could be overcome by the use of a novel sliding surface. In order that a stable (ideal) discrete time sliding motion exists, necessary and sufficient conditions were given in terms of the stabilizability by static output feedback of a fictitious system triple obtained from the real system. This fictitious system can easily be isolated once the real system is transformed into a special canonical form described in [15]. The stabilizability condition is the only significant restriction on the class of systems to which the results are applicable, but of course for general multivariable systems this condition can, at best, only be tested numerically. In [15] a static output feedback structure was considered and so the fact that there is a limitation on the class of systems to which it is applicable is not surprising. This paper builds on this earlier work and proposes a specific compensator structure to circumvent this restriction. The resulting controller is applied to a planar vertical takeoff and landing (PVTOL) aircraft model. The PVTOL has previously elicited interest from the control community [2], [17], [10] because the simplified two degree of freedom dynamics are nonlinear and nonminimum phase. It is an interesting theoretical problem associated with a nonlinear system clearly motivated by an application. Simulation results are presented to verify the robustness of this method.

N. O. Lai, C. Edwards and S. K. Spurgeon are all with the Control and Instrumentation Research Group, Department of Engineering, University of Leicester, University Road, Leicester, LE1 7RH, UK nol1,ce14,eon@le.ac.uk

II. PROBLEM FORMULATION

Consider the discrete time *square system* with matched uncertainties

$$x_p(k+1) = Gx_p(k) + H(u(k) + \xi(k)) \quad (1)$$
$$y(k) = Cx_p(k) \quad (2)$$

where $x_p \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ with $m = p < n$. Assume that the input and output distribution matrices H and C are full rank. In addition, assume the triple (G, H, C) is minimal. The matched uncertainties, $\xi(k)$, are unknown but are assumed to belong to a balanced set[1] $\mathcal{F}$.

The objective of this paper is to determine an appropriate sliding surface $\mathcal{S}$ formed from a linear combination of the states, and a control law which depends only on the measured outputs such that:

- for the nominal linear system when $\xi \equiv 0$ an *ideal sliding motion* is obtained in finite time i.e. $x_p(k) \in \mathcal{S}$ for all $k > k_s$;
- for uncertain systems the effect of the matched uncertainty ξ is minimized and an appropriate bounded motion about $\mathcal{S}$ is maintained.

The discrete time sliding mode situation is quite different from its continuous time counterpart: in continuous time an appropriate discontinuous control strategy can be employed to maintain ideal sliding in the presence of bounded matched uncertainty and to ensure its effect is

[1]The set $\mathcal{F}$ is balanced if $\xi(k) \in \mathcal{F} \Rightarrow -\xi(k) \in \mathcal{F}$ [5].

0-7803-9568-9/05/$20.00 ©2005 IEEE

completely rejected; in discrete time, complete rejection of the uncertainty is not possible. Furthermore, the inclusion of a switched term as part of the control law is detrimental to the performance of the controller [22], [13], [14].

As in [22] the class of sliding surfaces will be restricted to those which can be expressed in the form

$$\mathcal{S} = \{x_p \in \mathbb{R}^n \;\; : \;\; H^T P x_p = 0\} \tag{3}$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite (s.p.d.) matrix. Associate with P a candidate Lyapunov function $V(k) = x_p(k)^T P x_p(k)$ and define the corresponding Lyapunov difference function by

$$\Delta V(k) = V(k+1) - V(k) \tag{4}$$

Consider initially a nominal regulation problem where no uncertainty is present ($\xi(k) \equiv 0$). In the absence of uncertainty an ideal sliding motion can be attained on $\mathcal{S}$ and it follows from (1) that

$$H^T P x_p(k+1) = H^T P G x_p(k) + H^T P H u(k) = 0$$

The equivalent control action necessary to maintain an ideal sliding motion is given by

$$u_{eq}(k) = -(H^T P H)^{-1} H^T P G x_p(k) \tag{5}$$

If P is such that the closed-loop system, obtained from using the control law (5) in (1), satisfies $\Delta V(k) < 0$ for all k, then from standard Lyapunov theory the closed-loop system is asymptotically stable [19]. It is clear that $\Delta V(k) \equiv -x(k)^T Q x(k)$ where

$$Q := P - G_c^T P G_c \tag{6}$$

and the closed-loop system matrix

$$G_c := G - H(H^T P H)^{-1} H^T P G$$

Thus if $Q > 0$, the closed-loop system is stable.

Proposition 1: For the uncertain discrete time system in (1), the control law (5), with P chosen so that Q from (6) is s.p.d, has the property that it:

a) induces an ideal sliding motion on $\mathcal{S}$ in finite time when $\xi(k) \equiv 0$;
b) minimizes the effect of $\xi(k)$ on the closed loop dynamics in a min-max sense i.e. the control law in (5) minimises over all possible state feedback control laws the effect of the worst case uncertainty $\xi(k)$ on the Lyapunov difference $\Delta V(k)$.
c) minimizes in a min-max sense the deviation from the ideal sliding surface $\mathcal{S}$.

Proof

(a) follows from the fact that (5) is the discrete time equivalent control and so in the absence of uncertainty will ensure $H^T P x(k) \equiv 0$ for all $k > 1$.

(b) follows from the fact that (5) can also be shown to be a discrete min-max controller for $\Delta V(k)$: see for example [5], [22].

(c) is proved in [13].

■

If it is possible to solve

$$H^T P G = F C \tag{7}$$

for some matrix $F \in \mathbb{R}^{m\times m}$ then provided $\det G \neq 0$ the controller from (5) can be realised through outputs alone so that

$$u(k) = -(FCG^{-1}H)^{-1} F y(k) \tag{8}$$

Throughout the paper it will be assumed that:

A1) the plant state transition matrix G is nonsingular
A2) the matrix $CG^{-1}H$ has rank m.

Remark Assumption A1 is not a particularly strong one and most discrete systems satisfy this requirement. Assumption A2 is a system property and is independent of the choice of coordinates in the state space representation. It is in fact a necessary condition to find a s.p.d. matrix P and an $F \in \mathbb{R}^{m\times m}$ to solve (7). This can be seen as follows: Assuming A1 is satisfied, (7) is satisfied if and only if $H^T P = FCG^{-1}$ and consequently $H^T P H = FCG^{-1}H$. Since P is s.p.d. and H is full rank, rank$(H^T P H) = m$ and hence rank$(FCG^{-1}H) = m$. This implies both F and $CG^{-1}H$ must be rank m.

Based on assumptions A1 and A2, a change of coordinates will be introduced which facilitates insight into the class of systems for which this problem is solvable. Define a new matrix

$$S := CG^{-1} \tag{9}$$

This matrix will take the role of the output distribution matrix for a new, purely fictitious system (G, H, S), which will be necessary for the theoretical developments. In order to facilitate the analysis, a change of coordinates will be introduced for the fictitious system (G, H, S). By definition, and from assumption A2, rank$(SH) = m$. As argued in [6], since rank$(SH) = m$, there exists a change of coordinates such that

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \quad H = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \quad S = \begin{bmatrix} 0 & T \end{bmatrix} \tag{10}$$

where $G_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $H_2 \in \mathbb{R}^{m\times m}$ and is nonsingular and $T \in \mathbb{R}^{m\times m}$ is orthogonal. Partition the state vector x_p conformably as col(x_1, x_2) where $x_1 \in \mathbb{R}^{(n-m)}$. It follows from the canonical form (10) that the true output distribution matrix

$$C = SG = \begin{bmatrix} TG_{21} & TG_{22} \end{bmatrix} \tag{11}$$

The work in [16] shows that necessary and sufficient conditions for establishing a s.p.d matrix P and an F satisfying (6) and (7), are that assumptions A1 and A2 are satisfied and, for the square case considered here, that G_{11} is stable. Clearly this imposes a limitation on the class of systems for which the results are applicable. In order to circumvent this restriction a compensator based approach will be adopted here. Furthermore to incorporate a tracking element, integral action will also be included.

III. Main Results

To incorporate integral action the difference equations

$$x_r(k+1) = x_r(k) + \tau(r(k) - Cx(k)) \qquad (12)$$

will be added where τ represents the sample interval. The quantity $r(k)$ represents the signal to be tracked by the output. Furthermore assume $r(k) = r_s = \text{const}$ for $k > k_s$. Also introduce additional states $x_c \in \mathbb{R}^{(n-m)}$, which under certain circumstances (which will be explained later) represent an estimate of the states x_1.

The intention is to induce an ideal sliding motion on the surface

$$\mathcal{S} = \{(x_1, x_c, x_r, x_2) : K_1 x_c + K_r x_r + x_2 + S_r r_s) = 0\} \qquad (13)$$

where $K_1 \in \mathbb{R}^{m\times(n-m)}$ and $K_r \in \mathbb{R}^{m\times m}$ together with $S_r \in \mathbb{R}^{m\times m}$ represent design freedom.

Let the compensator take the form

$$x_c(k+1) = G_{11}x_c + G_{12}x_2 + L(y - \hat{y}) \qquad (14)$$

where

$$\hat{y}(k) = TG_{21}x_c(k) + TG_{22}x_2(k) \qquad (15)$$

and $L \in \mathbb{R}^{(n-m)\times m}$ is a design variable.

During an ideal sliding motion, from (13)

$$x_2(k) = -K_1 x_c(k) - K_r x_r(k) - S_r r(k)$$

and so after some algebraic manipulation

$$x_c(k+1) = \Phi x_c(k) + \Gamma_1 y(k) + \Gamma_2 x_r(k) + \Gamma_3 r(k) \qquad (16)$$

where

$$\begin{aligned} \Phi &= G_{11} - LTG_{21} - G_{21}K_1 + LTG_{22}K_1 && (17)\\ \Gamma_1 &= L && (18)\\ \Gamma_2 &= -G_{12}K_r + LTG_{22}K_r && (19)\\ \Gamma_3 &= -G_{12}S_r + LTG_{22}S_r && (20) \end{aligned}$$

It is assumed as part of the design process that L is chosen to guarantee that $\det \Phi \neq 0$.

Augment the system in (10) with the integral and compensator states from (12) and (16) to obtain:

$$x_a(k+1) = G_a x_a(k) + H_a(u(k) + \xi(k)) + H_r r(k) \qquad (21)$$

where $x_a = \text{col}(x_1, x_c, x_r, x_2)$. (At first sight this represents a non-intuitive arrangement of the states but it leads to a simplification in the presentation.)

The measurable outputs associated with this system are $y_a = \text{col}(x_c, x_r, y)$. It is easily verified that

$$G_a = \begin{bmatrix} G_{11} & 0 & 0 & G_{12} \\ \Gamma_1 TG_{21} & \Phi & \Gamma_2 & \Gamma_1 TG_{22} \\ -\tau TG_{21} & 0 & I_m & -\tau TG_{22} \\ G_{21} & 0 & 0 & G_{22} \end{bmatrix} H_r = \begin{bmatrix} 0 \\ \Gamma_3 \\ \tau I_m \\ 0 \end{bmatrix}$$

$$H_a^T = \begin{bmatrix} 0 & 0 & 0 & H_2^T \end{bmatrix} \qquad (22)$$

and the output distribution matrix

$$C_a = \begin{bmatrix} 0 & I_{n-m} & 0 & 0 \\ 0 & 0 & I_m & 0 \\ TG_{21} & 0 & 0 & TG_{22} \end{bmatrix} \qquad (23)$$

where $y_a := C_a x_a$.

Modify the control law (8) to include the reference signal so that

$$u(k) = -(FC_aG_a^{-1}H_a)^{-1}FC_ax_a(k) + F_r r(k) \qquad (24)$$

where now both F and $F_r \in \mathbb{R}^{m\times m}$ are to be determined (in terms of L, K_1, K_r and S_r).

The objective is to select F and a parameter $F_2 \in \mathbb{R}^{m\times m}$ so that the surface

$$\mathcal{S}_a = \{x_a \ : \ FC_aG_a^{-1}x_a + F_2S_rr_s = 0\} \qquad (25)$$

is identical to the surface $\mathcal{S}$ in (13), and then to select K_1, K_r and L to ensure a stable ideal sliding motion when confined to $\mathcal{S}$.

Providing the design matrix F is chosen to ensure the eigenvalues of

$$G_c = G_a - H_a(FC_aG_a^{-1}H_a)FC_a \qquad (26)$$

are inside the unit disk, $(I - G_c)$ is invertible. Define $x_s = (I - G_c)^{-1}(H_r + H_aF_r)r_s$ then using (24) and defining $e(k) = x_a(k) - x_s$ it follows from simple algebraic manipulation that

$$e(k+1) = G_ce(k) + H_a\xi(k) \qquad (27)$$

In the absence of uncertainty $e(k) \to 0$ as $k \to \infty$, and since steady state is achieved, it follows from (12) that $y_p(k) = r_s$ and so tracking is achieved. Furthermore it can be shown that

$$\begin{aligned} FC_aG_a^{-1}x_s &= FC_aG_a^{-1}(I - G_c)^{-1}(H_r + H_aF_r)r_s \\ &\equiv FC_aG_a^{-1}(H_r + H_aF_r)r_s \end{aligned}$$

and consequently if F_r is chosen as

$$F_r = -(FC_aG_a^{-1}H_a)^{-1}\left(FC_aG_a^{-1}H_r + F_2S_r\right) \qquad (28)$$

then $FC_aG_a^{-1}x_s + F_2S_rr_s = 0$ and so $x_s \in \mathcal{S}$. The control law can then be written

$$\begin{aligned} u(k) = -(FC_aG_a^{-1}H_a)^{-1}\big(&FC_ax_a(k) \\ &+ (FC_aG_a^{-1}H_r + F_2S_r)r(k)\big) \end{aligned} \qquad (29)$$

From (27) the problem is therefore to find an F and a s.p.d. matrix $P_a \in \mathbb{R}^{2n\times 2n}$ such that

$$FC_a = H_a^TP_aG_a \qquad (30)$$

and

$$G_c^TP_aG_c - P_a < 0 \qquad (31)$$

As in §II define for the augmented system $S_a := C_a G_a^{-1}$. After some algebra

$$S_a = \left[\begin{array}{c|ccc} 0 & \Phi^{-1} & -\Phi^{-1}\Gamma_2 & -\Phi^{-1}\Gamma_1 T - \tau\Phi^{-1}\Gamma_2 T \\ 0 & 0 & I_m & \tau T \\ 0 & 0 & 0 & T \end{array}\right] =: \left[\begin{array}{cc} 0 & T_a \end{array}\right] \tag{32}$$

where $T_a \in \mathbb{R}^{(n+m)\times(n+m)}$ and $\det T_a \neq 0$. Define a matrix

$$F = \left[\begin{array}{ccc} F_2K_1\Phi & F_2K_1\Gamma_2 + F_2K_r & F_2K_1\Gamma_1 - F_2K_r\tau + F_2T^T \end{array}\right] \tag{33}$$

where $F_2 \in \mathbb{R}^{m\times m}$ and is nonsingular. This variable has no effect on the dynamics of the reduced order sliding motion but is required to solve the constraint (30).
After a little algebra it can be shown that

$$FS_a = FC_aG_a^{-1} = F_2\left[\begin{array}{cccc} 0 & K_1 & K_r & I_m \end{array}\right] \tag{34}$$

and so the sliding surface $\mathcal{S}_a$ in (25) is identical to the one in (13) because by definition F_2 is nonsingular.
To facilitate choosing the parameters L, K_1 and K_r change coordinates according to the transformation $x_a \mapsto \tilde{T}x_a =: \tilde{x}$ where

$$\tilde{T} := \left[\begin{array}{cccc} I_{n-m} & -I_{n-m} & 0 & 0 \\ 0 & I_{n-m} & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & K_1 & K_r & I_m \end{array}\right] \tag{35}$$

This effectively forces the last m states of the new coordinates to represent (what in continuous time sliding mode control) would be called the 'switching function' $\sigma = K_r x_r + K_1 x_c + x_2$ associated with $\mathcal{S}$ in (13). It follows $\tilde{G} = \tilde{T}G_a\tilde{T}^{-1}$, $\tilde{H} = \tilde{T}H_a$, $\tilde{H}_r = \tilde{T}H_r$, $\tilde{C} = C_a\tilde{T}^{-1}$ and $\tilde{S} = S_a\tilde{T}^{-1}$. After some straightforward algebra

$$\tilde{G} = \left[\begin{array}{cccc} G_{11} - LTG_{21} & 0 & 0 & G_{12} - LTG_{22} \\ LTG_{21} & G_{11} - G_{12}K_1 & -G_{12}K_r & LTG_{22} \\ -\tau TG_{21} & -\tau TG_{21} + \tau TG_{22}K_1 & I_m + \tau TG_{22}K_r & -\tau TG_{22} \\ * & * & * & * \end{array}\right]$$

where the matrices $*$ play no explicit part in the analysis which follows. Also

$$\tilde{H} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ H_2 \end{array}\right], \qquad \tilde{H}_r = \left[\begin{array}{c} -\Gamma_3 \\ \Gamma_3 \\ \tau I_m \\ K_1\Gamma_3 + \tau K_r \end{array}\right] \tag{36}$$

From equation (34)

$$F\tilde{S} = \left[\begin{array}{cccc} 0 & 0 & 0 & F_2 \end{array}\right] \tag{37}$$

Some algebra reveals the closed-loop system matrix

$$\tilde{G}_c = \tilde{G} - \tilde{H}(F\tilde{S}\tilde{H})^{-1}F\tilde{C} =: \left[\begin{array}{cc} \tilde{G}_{11} & \tilde{G}_{12} \\ 0 & 0 \end{array}\right] \tag{38}$$

where

$$\tilde{G}_{11} = \left[\begin{array}{c|c} G_{11} - LTG_{21} & 0 \\ \hline LTG_{21} & \\ -\tau TG_{21} & \tilde{G}_m \end{array}\right] \tag{39}$$

and

$$\tilde{G}_m := \left[\begin{array}{cc} G_{11} - G_{12}K_1 & -G_{12}K_r \\ -\tau TG_{21} + \tau TG_{22}K_1 & I_m + \tau TG_{22}K_r \end{array}\right]$$

This is most easily seen from the definition of $\tilde{S} = \tilde{C}\tilde{G}^{-1}$ and the fact that $\tilde{G}_c = (I - \tilde{H}(F\tilde{S}\tilde{H})^{-1}F\tilde{S})\tilde{G}$. From (37) and (36) it can be easily shown that

$$(I - \tilde{H}(F\tilde{S}\tilde{H})^{-1}F\tilde{S}) = \text{diag}(I_{n-m}, I_{n-m}, I_m, 0_{m\times m})$$

and hence the structure in (38) follows immediately.
It is clear from (38) that

$$\sigma(\tilde{G}_c) = \{0\}^m \cup \sigma(G_{11} - LTG_{21}) \cup \sigma(\tilde{G}_m)$$

where

$$\tilde{G}_m = \underbrace{\left[\begin{array}{cc} G_{11} & 0 \\ -\tau TG_{21} & I_m \end{array}\right]}_{G_{11}^a} - \underbrace{\left[\begin{array}{c} G_{12} \\ -\tau TG_{22} \end{array}\right]}_{G_{12}^a}\left[\begin{array}{cc} K_1 & K_r \end{array}\right] \tag{40}$$

Since the matrix pair (G_{11}, G_{21}) is observable (see for example [16]) and T is nonsingular, the pair (G_{11}, TG_{21}) is observable. Consequently an L can always be found which makes $(G_{11} - LTG_{21})$ stable. Likewise it can be shown that provided (G, H, C) does not have any invariant zeroes at unity, the pair (G_{11}^a, G_{12}^a) is controllable and hence the choice of the parameters K_1 and K_r constitutes a state-feedback problem. Consequently K_1, K_r and L can be chosen to make $\tilde{G}_{11}$ from (39) stable.
In the new set of coordinates $\tilde{x}$, let the Lyapunov matrix be represented by $\tilde{P}$. Using the definition of $\tilde{S}$, equation (30) becomes

$$\tilde{H}^T\tilde{P} = F\tilde{C}\tilde{G}^{-1} = F\tilde{S} \tag{41}$$

In order to show that $\tilde{P}$ is a Lyapunov matrix for $\tilde{G}_c$ it must be established that

$$\tilde{Q} := \tilde{P} - \tilde{G}_c^T\tilde{P}\tilde{G}_c > 0 \tag{42}$$

It can be seen from the structures of $\tilde{H}$ and $F\tilde{S}$ in (36) and (37) and from the fact that $\det H_2 \neq 0$ that in order to satisfy (41) $\tilde{P}$ must have a block diagonal structure:

$$\tilde{P} = \left[\begin{array}{cc} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{array}\right] \tag{43}$$

where $\tilde{P}_1 \in \mathbb{R}^{(2n-m)\times(2n-m)}$, $\tilde{P}_2 \in \mathbb{R}^{m\times m}$ and

$$F_2 = H_2^T\tilde{P}_2 \tag{44}$$

In terms of the partition in (38), (42) can be written as

$$\tilde{Q} = \left[\begin{array}{cc} \tilde{P}_1 - \tilde{G}_{11}^T\tilde{P}_1\tilde{G}_{11} & -\tilde{G}_{11}^T\tilde{P}_1\tilde{G}_{12} \\ -\tilde{G}_{12}^T\tilde{P}_1\tilde{G}_{11} & \tilde{P}_2 - \tilde{G}_{12}^T\tilde{P}_1\tilde{G}_{12} \end{array}\right] \tag{45}$$

A family of solutions $(\tilde{P}_1, \tilde{P}_2)$ will now be shown to exist to make $\tilde{Q} > 0$. Specifically, let $\tilde{P}_1 > 0$ be a solution to

$$\tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11} > 0 \qquad (46)$$

Such a solution $\tilde{P}_1$ is guaranteed to exist since $\tilde{G}_{11}$ is stable. Then from the Schur complement, inequality (45) is satisfied if and only if

$$\begin{aligned} \tilde{P}_2 &> \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{11} (\tilde{P}_1 - \tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{11})^{-1} (\tilde{G}_{11}^T \tilde{P}_1 \tilde{G}_{12}) \\ &\quad + \tilde{G}_{12}^T \tilde{P}_1 \tilde{G}_{12} \qquad (47) \end{aligned}$$

Any pair $(\tilde{P}_1, \tilde{P}_2)$ satisfying (46) and (47) ensures $\tilde{P}$ from (43) satisfies (41) and (42).

Remark If $\xi(k)$ represents an exogenous disturbance then it can easily be shown that, for the closed-loop system, the states $x(k)$ evolve in such a way that

$$F C_a G_a^{-1} x_a(k+1) - F_2 S_r r_s = H_a^T P_a H_a \xi(k) \qquad (48)$$

for $k = 1, 2, \ldots$ and thus $\|H_a^T P_a H_a \xi(k)\|$ represents the deviation from the ideal sliding surface

$$\mathcal{S}_a = \{x_a \mid F C_a G_a^{-1} x_a - F_2 S_r r_s = 0\}$$

If $\xi(k)$ is bounded then $\max_{\xi(k)\in\mathcal{F}} \|H_a^T P_a H_a \xi(k)\|$ represents the boundary layer about $\mathcal{S}_a$ into which the states $x_a(k)$ ultimately enter. As argued in [13], the choice of the control law in (24) minimizes the worst case deviation from $\mathcal{S}_a$ over all possible controllers.

IV. PVTOL AIRCRAFT SIMULATIONS

The planar vertical takeoff and landing (PVTOL) aircraft is an example of a nonlinear, nonminimum phase system [2], [17], [10]. In this paper, the coupling between the rolling moment and the lateral acceleration of the aircraft has been taken into account, where ϵ is the coefficient representing the coupling. The inputs of the system are the thrust acceleration (u_1) and the roll acceleration (u_2). The two outputs are the horizontal position, $x(k)$, and the vertical position, $y(k)$ (altitude). (The inputs, outputs and tracking configuration used here are the standard framework for the PVTOL system.) The equations are given by

$$\begin{aligned} \ddot{x} &= -\sin\theta u_2 + \epsilon \cos\theta u_1 \qquad (49) \\ \ddot{y} &= \cos\theta u_2 + \epsilon \sin\theta u_1 - g \qquad (50) \\ \ddot{\theta} &= u_1 \qquad (51) \end{aligned}$$

A linearization of the aircraft system about the equilbirium point $x = y = \theta = \dot{x} = \dot{y} = \dot{\theta} = 0$ and $u_1 = 0$, $u_2 = g$ yields system matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \epsilon & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $g = 9.81$ is the acceleration of gravity and $\epsilon \neq 0$. This linearization has zeros at ± 3.3015 and so is (significantly) non-minimum phase. The system was discretized at a sample rate of $t = 0.2$sec and the controller was designed with a nominal value of $\epsilon = 0.9$. The discrete-time zeros are at $\{-1, -1, 1.9359, 0.5166\}$ so the discretization is also significantly non-minimum phase. The control task, as in [2], is to track the reference trajectories, which are step commands: $x(k) \rightarrow 20m$ and $y(k) \rightarrow 30m$ whilst for the roll angle, $\theta \rightarrow 0$. The control law, denoted by $u(k) = -G_y y(k) - G_r r(k)$ is given by

$$\begin{aligned} G_y &= \left[\begin{matrix} -0.0353 & 0.0019 & -3.4299 & 1.9681 \\ -0.4279 & 0.1825 & -13.2014 & 20.6632 \end{matrix} \right. \\ &\qquad \left. \begin{matrix} -0.0006 & -0.6894 & -6.5105 & 0.0042 \\ -1.0565 & -8.9925 & -78.9691 & 10.7684 \end{matrix} \right] \\ G_r &= \begin{bmatrix} 0.0112 & -0.0005 \\ 0.3957 & -0.4868 \end{bmatrix} \end{aligned}$$

produces stable results with good tracking. The closed-loop poles are associated with the matrix $G_{11} - LTG_{21}$ are $\{0.8, 0.82, 0.86, 0.88\}$ and the ones associated with the matrix $\tilde{G}_m$ are $\{0.8, 0.85, 0.875, 0.825, 0.9, 0.95\}$. The simulation results are shown in Figures 1 - 4. From the graphs, as ϵ varies from its nominal value, the horizontal position is affected very minimally. In particular, in Figure 2, the parameter ϵ is allowed to vary sinusoidally about 0.9 with amplitude 0.1 and frequency 1 rad/s. It is observed that tracking performance is good and θ is well regulated.

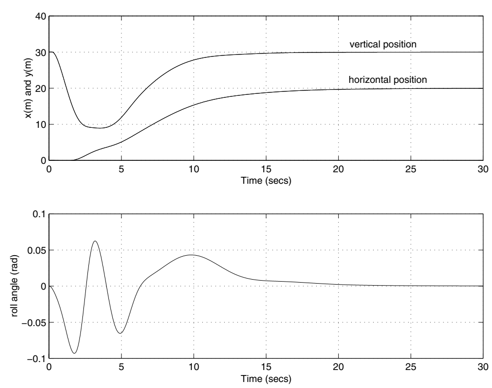

Fig. 1. *Tracking control of the PVTOL for $\epsilon = 0.9$*

V. CONCLUSIONS

This paper has proposed a new output feedback based discrete time sliding mode control scheme. It incorporates a tracking requirement and is dynamic in nature. Previous work has shown that, with an appropriate choice of surface, discrete time sliding mode control can be applied to non-minimum phase systems. This is significant because the majority of the sliding mode literature which develops schemes using only output information requires the system to be minimum phase. The original scheme [15] was effectively a static output feedback one and so inherently

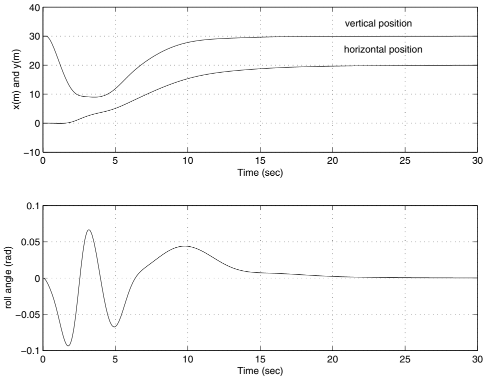

Fig. 2. *Tracking control of the PVTOL for time-varying* ϵ

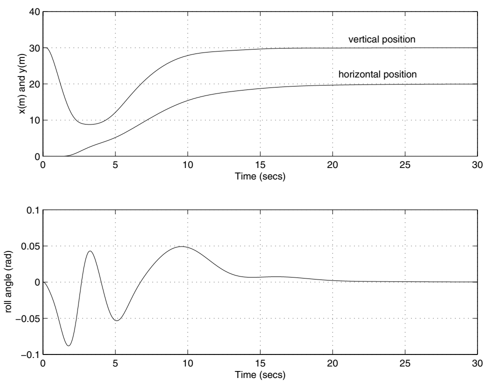

Fig. 3. *Tracking control of the PVTOL for* $\epsilon = 0.5$

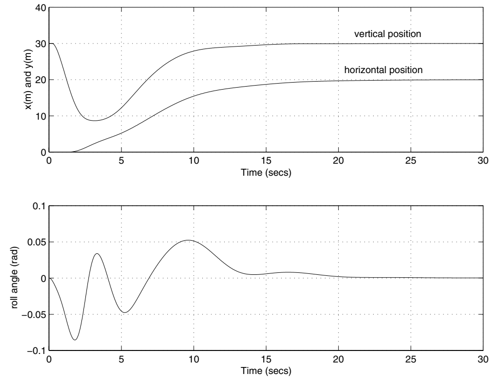

Fig. 4. *Tracking control of the PVTOL for* $\epsilon = 0.25$

this imposed restrictions on the class of systems to which it was applicable. The scheme which has been proposed here includes a compensator and so some of the output feedback restrictions have been removed. The key aspect of the new scheme proposed here is that it is applicable to non-minimum phase systems and the restrictions on the systems for which the results are applicable are very mild. As a demonstration it has been applied to a discretized version of a linearization of the PVTOL aircraft. The resulting discrete time controller law has been tested on the nonlinear model and very good results have been obtained.

REFERENCES

[1] S.H. Źak and S. Hui. On variable structure output feedback controllers for uncertain dynamical systems. *IEEE Transaction on Automatic Control*, 38(10):1509–1512, 1993.

[2] S.A. Al-Hiddabi and N.H. McClamroch. Output tracking for nonlinear non-minimum phase vtol aircraft. *Proc 37th IEEE Conference on Decision and Control, Florida*, pages 4573–4578, 1998.

[3] G. Bartolini, A. Pisano, and E. Usai. Digital sliding mode control with O(t^3) accuracy. In X. Yu and J. X. Xu, editors, *Advances in Variable Structure Systems - Analysis, Integration and Application*, pages 103–112. Proceedings of the 6th IEEE International Workshop on VSS, 2000.

[4] G. Bartolini, A. Pisano, and E. Usai. Digital second-order sliding mode control for uncertain nonlinear systems. *Automatica*, 37:1371–1377, 2001.

[5] M. Corless and J. Manela. Control of uncertain discrete-time systems. *Proc American Control Conference, Seattle*, pages 515–520, 1986.

[6] C. Edwards and S. K. Spurgeon. Sliding mode stabilization of uncertain systems using only output information. *International Journal of Control*, 62:1129–1144, 1995.

[7] C. Edwards and S.K. Spurgeon. Robust output tracking using a sliding mode controller/observer scheme. *International Journal of Control*, 64:967–983, 1996.

[8] C. Edwards and S.K. Spurgeon. *Sliding Mode Control: Theory and Applications*. Taylor & Francis, 1998.

[9] R. El-Khazali and R. DeCarlo. Output feedback variable structure control design. *Automatica*, 31(6):805–816, 1995.

[10] I. Fantoni, A. Zavala, and R. Lozano. Global stabilization of a pvtol aircraft with bounded thrust. *Proc 41st IEEE Conference on Decision and Control, Las Vagas*, pages 4462–4467, 2002.

[11] K. Furuta and Y. Pan. Variable structure control with sliding sector. *Automatica*, 36:211–228, 2000.

[12] G. Golo and C. Milosavljević. Robust discrete-time chattering free sliding mode control. *Systems and Control Letters*, 41:19–28, 2000.

[13] S. Hui and S.H. Źak. On discrete-time variable structure sliding mode control. *Systems and Control Letters*, 38:283–288, 1999.

[14] A. Jafari Koshkouei and A. S. I. Zinober. Sliding mode control of discrete-time systems. *Journal of Dynamic Systems, Measurement, and Control*, 122:793–802, 2000.

[15] N.O. Lai, C. Edwards, and S.K. Spurgeon. Discrete output feedback sliding mode based tracking control. *Proc 43rd IEEE Conference on Decision and Control, Bahamas*, pages 237–242, 2004.

[16] N.O. Lai, C. Edwards, and S.K. Spurgeon. On discrete time output feedback min-max controller. *International Journal of Control*, 77:554–561, 2004.

[17] X-Y Lu, S.K. Spurgeon, and I. Postlethwaite. Robust variable structire control of a pvtol aircraft. *International Journal of Systems Science*, 28(6):547–558, 1997.

[18] G. Monsees. *Discrete-Time Sliding Mode Control*. PhD thesis, Delft University of Technology, 2002.

[19] K. Ogata. *Discrete-Time Control Systems*. Prentice-Hall International, Inc., 1995.

[20] H. Sira-Ramirez. Non-linear discrete variable structure systems in quasi-sliding mode. *International Journal of Control*, 54:1171–1187, 1991.

[21] J. J. E. Slotine, J. K. Hedrick, and E. A. Misawa. On sliding observers for nonlinear systems. *Transactions of the ASME: Journal of Dynamic Systems, Measurement and Control*, 109:245–252, 1987.

[22] S.K. Spurgeon. Hyperplane design techniques for discrete-time variable structure control systems. *International Journal of Control*, 55:445–456, 1992.

[23] C.Y. Tang and E. Misawa. Discrete variable structure control for linear multivariable systems. *Journal of Dynamic Systems, Measurement and Control*, 122:783–792, 2000.

[24] C.Y. Tang and E. Misawa. Sliding surface design for discrete VSS using LQR technique with a preset eigenvalue. *Systems and Control Letters*, 45:1–7, 2002.

[25] B. L. Walcott and S. H. Źak. Combined observer-controller synthesis for uncertain dynamical systems with applications. *IEEE Transactions on Systems, Man and Cybernetics*, 18:88–104, 1988.