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# Constructive feedback linearization of underactuated mechanical systems with 2-DOF

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*Abstract*— **In the last years, the control community has developed several and powerful methods to control nonlinear systems, especially for underactuated mechanical systems. Thus, methods based on passivity, like Interconnection and damping assignment passivity–based control (IDA-PBC) and Controlled Lagrangians have solved many interesting control problems for particular full classes of systems. Usually, the solutions of these methods relies on solving a set of partial differential equations (PDEs), which is not always possible.**

**This paper presents a constructive methodology to control underactuated mechanical systems with 2-DOF, by means of classical feedback linearization and Lyapunov design. The steps of the design are presented following a simple pseudo-code[1], that allows us to redesign a proposed fictitious output in a constructive way. The methodology has been tested with three very well-known underactuated mechanical systems: the inertia wheel pendulum, the pendulum on a cart and the rotary pendulum. The obtained solution for the inertia wheel pendulum takes into account the friction, since recent works have shown that it cannot be neglected. In the case of the planar pendulum on a cart, the solution is quite similar to the one obtained by Controlled Lagrangians but with better performance, and, furthermore, our planar pendulum solution paves the way to obtain a new solution for the rotary pendulum, or the so–called Furuta pendulum, that, to the best of our knowledge, has the largest attraction basin presented and experimentally tested so far. The attraction basin tends to the whole upper half plane by increasing a control gain. The only thing that hampers us to do this, is the actual saturation limit of the control input. In spite of the latter, successful experimental results for this rotary pendulum solution are given.**

## I. INTRODUCTION

The widely–used geometric control has solved successfully many classes of dynamical control problems. Moreover these differential geometric techniques have been also used to understand input-output approach and full state linearization, normal forms and zero dynamics of nonlinear systems (see, e.g. [1], [3]). In this sense, the flatness approach has been related to the feedback linearization, and, for instance, in the case of a system with a single control input the corresponding output is called differentially flat when the system is fully linearizable via static state feedback (see [2] and [5]). In the cases in which it is not possible to linearize the whole system by a static feedback, a partial feedback linearization is still possible. This partial feedback linearization transforms the system dynamics into two parts: the external dynamics, which becomes linear; and the internal nonlinear dynamics (locally characterized by the zero dynamics). This paper presents a methodology to control 2-DOF underactuated mechanical systems by means of output redefinition, Feedback Linearization and Lyapunov design. The methodology is based on the construction of a fictitious output through which the asymptotic stabilization at the origin of the system is achieved. We present three applications to show how to construct a suitable fictitious output jointly with the design of an external controller by means of Lyapunov's theory. The technique yields excellent results in the theoretical and experimental issues. In fact, from the theoretical point of view, the classical–state–feedback linearization jointly with the proposed methodology yields relations with other control techniques, which will be pointed out through the paper.

The methodology has been tested with three very well-known underactuated mechanical systems: the inertia wheel pendulum, the pendulum on a cart and the rotary pendulum. In the case of the inertial wheel pendulum, the model proposed takes into account the friction. The solution given in [8] did not take into account the friction and the implementation of the controller did not succeed as expected. On the other hand, in [19] the importance of the friction to stabilize this system was shown. Even though, with friction, the solution proposed shows a clear relation with the Differential Flatness approach [2], [5].

For the pendulum on a cart, we start, motivated by a cyclic variable[2], choosing as the fictitious output a function quite similar to the integral of the conjugate momentum. Then, this could have a relation between the chosen output and the phase shift [13] and, therefore, with the Controlled Lagrangians method [6]. Since with this output, we cannot fulfil the objective of stabilizing asymptotically the zero dynamics, therefore we redesign this output using the constructive approach proposed. Due to this redesign the solution obtained improves the performances obtained in [6] by Controlled Lagrangians and, in [10] and [15] by Forwarding. This improvement is due to a new control term which makes the zero dynamics converge faster. The region of attraction obtained is quite similar. The largest region of attraction for this pendulum was given in [10], [21] by IDA–PBC, and it was exactly the whole upper half plane, but, nevertheless, we only use this application to pave the way to obtain the solution of the rotary pendulum.

The last application we propose, is the natural extension of the one before, the rotary pendulum. The main difference

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[1] Friendly called "cook-book" by a reviewer.

[2] It does not appear in the equations of motion [20].

between the pendulum on a cart and the rotary pendulum is that the latter does not have a feedforward form, then it is not possible to control it with a cascade technique as the one proposed in [16], due to that the main growth–restriction assumption is not fulfilled. We only are interested in stabilizing the upper position of the pendulum, not to swing it up from any position, whose solution was given in [18]. To prove stability we make use of an explicit Lyapunov function (unknown so far for this system). The solution proposed enlarges the largest region of attraction obtained so far for this system in [7], which depends on the physical parameters of the system. The solution can stabilize the upper position from any point over the upper half plane[3]. The excellent performance and a large region of attraction was checked in the actual laboratory pendulum. For others applications we refer the reader to [22].

The outline of the paper is: Section II describes the constructive–design pseudo-code. In section III, the well-known systems commented above are controlled by means of this methodology. The experiments are given at the end of this section. And, finally, a conclusion section.

## II. CONSTRUCTIVE OUTPUT

The class of underactuated mechanical systems considered is the one with a single control input and two degrees of freedom (2-DOF). A simple iterative pseudo-code can be stated in order to find an output, in a constructive way, which allows to linearize the system by means of a static state feedback. For, we start from the system obtained by applying the collocated partial feedback linearization or so–called Spong's normal form [23]. The generalized coordinates for the corresponding 2-DOF are: $\theta$ for the non–actuated joint, and $x$ for the actuated one.

For this kind of systems, we propose the following general structure for the fictitious outputs:

$$\begin{aligned} \text{a)} \quad \eta &= \theta + f(\theta, x) \\ \text{b)} \quad \eta &= x + f(\theta). \end{aligned}$$

Next, a pseudo-code that explains how to design a controller through a fictitious output, that in turn has to be designed, is stated:

1. Check if the system is full state linearizable [1].
   1.1 If so, choose an output of the form a).
   1.2 If not, choose the option b) as the output.
2. Derive the output $r$ times until the control signal appears explicitly.
   2.1 If $r = n$, with $n$ the dimension of the system, the system is full state linearizable by a static feedback of the output, which will represent the flat output of the system. Go to 5.
   2.2 If $r < n$, then the system can be only partially linearized and the resulting system is characterized by its zero dynamics. Go to 3.
   2.3 If $r$ is not defined, then another function has to be chosen as the output candidate. Go to 2.
3. Analyze the stability of the zero dynamics (*ZD*)
   3.1 If the *ZD* is unstable, then another function has to be chosen as the output candidate. Go to 2.
   3.2 If the *ZD* is asymptotically stable, go to 6.
   3.3 If the *ZD* is stable but not asymptotically, go to 4.
4. At this point there are two possibilities,
   4.1 Add to the output candidate a term derived from the passive output of the system, in such a way that a Lyapunov function can be obtained to prove asymptotic stability of the zero dynamics. If this is not possible to achieve or we are also interested in local exponential stability (LES), then go to 4.2. In other case go to 4.3.
   4.2 To guarantee Local exponential stability, the Jacobian of the zero dynamics will be computed and modified in order to be Hurwitz. This will imply to add to the output candidate an appropriate term derived from the new Jacobian. Go to 4.3.
   4.3 At this point, integrate the output derivatives until get the new output. If necessary define new states but taking into account the relative degree of the system. Redefine if necessary a new output from a derivative of the output candidate. Go to 6 to achieve local asymptotic stability for the full system, or go to 7 to try to achieve global asymptotic stability.
5. A linear feedback of the external state, given by the output and its derivatives, stabilizes global and asymptotically the origin of the full system [3].
6. A linear feedback of the external state, given by the output and its derivatives, stabilizes local and asymptotically the origin of the full system [3]. In order to try to achieve global asymptotic stability go to 7.
7. The following Lyapunov function candidate is proposed $V = E_0 + \eta_v^T \eta_v$, where $E_0$ is the energy of the zero dynamics sub–system and $\eta_v$ is the new external state which is related to the output. Compute $\dot{V}$ and choose if possible an appropriate $u$ that makes $\dot{V}$ be negative definite.

## III. APPLICATIONS

### A. Inertia Wheel Pendulum With Friction

As a motivating application we propose the Inertia Wheel Pendulum. The equations of motion can be expressed as follows:

$$(1-\varepsilon)\ddot{\varphi} = (\sin\varphi - k\dot{\varphi} - u) \tag{1}$$

$$\varepsilon(1-\varepsilon)\ddot{\gamma} = (-\varepsilon\sin\varphi + \varepsilon k\dot{\varphi} + u) \tag{2}$$

where $\varphi$ ($\theta$ in the pseudo-code) is the non-actuated variable, $\gamma$ is the actuated variable ($x$ in the pseudo-code), $\varepsilon < 1$ is the inertia relation and $k > 0$ the friction coefficient.

In this case, the system is full state linearizable considering the state to be stabilized as the vector $(\dot{\varphi}, \varphi, \dot{\gamma})$. Thus, the fictitious output will be like the one in the option a).

$$\begin{aligned} \eta &= \varphi + f(\cdot) \\ \dot{\eta} &= \dot{\varphi} + \dot{f}(\cdot) \\ (1-\varepsilon)\ddot{\eta} &= (\sin\varphi - k\dot{\varphi} - u) + (1-\varepsilon)\ddot{f}(\cdot) \end{aligned}$$

[3] In [17] the existence of a semiglobal stabilization was pointed out.

Since the system is full-state linearizable, the input control can only appear at the $n^{th}$ derivative. Designing $\ddot{f}(\cdot)$ to cancel the friction and the input control signal yields

$$(1-\varepsilon)\ddot{f}(\cdot) = k\dot{\varphi} + u \tag{3}$$

In this way, the second derivative of the chosen output does not depend on the $u$ and the third derivative can be compute without introducing a new state variable.

$$\begin{aligned}
&(1-\varepsilon)\ddot{\eta} = \sin\varphi \\
&(1-\varepsilon)\eta^{(3)} = \dot{\varphi}\cos\varphi \\
&(1-\varepsilon)\eta^{(4)} = \left(\frac{\sin\varphi - k\dot{\varphi} - u}{1-\varepsilon}\right)\cos\varphi - \dot{\varphi}^2\sin\varphi = \nu
\end{aligned}$$

It can be noticed that choosing $\ddot{f}(\cdot)$ the system has relative degree 4. In this case, a linear controller is intended to be applied in order to control a cascade of 4 integrators, where the external law is given by: $\nu = -k_1\eta - k_2\dot{\eta} - k_3\ddot{\eta} - k_4\eta^{(3)}$, where $k_i;\ i = 1, ..., 4$, are positive constants. In order to apply this control law $\eta$ and $\dot{\eta}$ has to be computed, which depend on $f(\cdot)$ and $\dot{f}(\cdot)$ respectively. To obtain $f(\cdot)$, we use the expression (3) and the equations of motion (1–2) of the system. In this way we get,

$$(1-\varepsilon)\ddot{f}(\cdot) = \varepsilon\ddot{\varphi} + \varepsilon\ddot{\gamma} + k\dot{\varphi}$$

Integrating twice with respect to the time, the above expression yields,

$$(1-\varepsilon)f(\cdot) = \varepsilon\varphi + \varepsilon\gamma + k\int_0^t \varphi dt$$

Substituting from the definition of $\eta$ the equations for $\eta$ and $\dot{\eta}$ read

$$\begin{aligned}
(1-\varepsilon)\eta &= \varphi + \varepsilon\gamma + k\int_0^t \varphi dt \\
(1-\varepsilon)\dot{\eta} &= \dot{\varphi} + \varepsilon\dot{\gamma} + k\varphi
\end{aligned}$$

It can be seen that in the output $\eta$ and in $\dot{\eta}$ the variables $\gamma$, $\dot{\gamma}$ and $\int_0^t \varphi dt$ appear, where the last one is a new state variable. If we are interesting in stabilizing asymptotically $\dot{\gamma}$ the following change of coordinates is proposed.

$$F = \varepsilon\gamma + k\int_0^t \varphi dt$$

In this way the equations given by $\eta$ and its derivatives yield:

$$\begin{aligned}
&(1-\varepsilon)\eta = \varphi + F \\
&(1-\varepsilon)\dot{\eta} = \dot{\varphi} + \dot{F} \\
&(1-\varepsilon)\ddot{\eta} = \sin\varphi \\
&(1-\varepsilon)\eta^{(3)} = \dot{\varphi}\cos\varphi \\
&(1-\varepsilon)\eta^{(4)} = \left(\frac{\sin\varphi - k\dot{\varphi} - u}{1-\varepsilon}\right)\cos\varphi - \dot{\varphi}^2\sin\varphi = \nu
\end{aligned}$$

To recapitulate, there are 4 derivatives of the output and 4 state variables to control, $(\varphi, \dot{\varphi}, \gamma, F)$, therefore, the considered dynamics is full state linearizable by a static state feedback.

Isolating u, it is easy to notice that the linearizing feedback law is not valid when $\cos\varphi = 0$, therefore it is only useful to stabilize asymptotically the origin with a domain of attraction in the set $(\varphi, \dot{\varphi}, \dot{\gamma}, F) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^3$.

If an appropriate trajectory in $\varphi$ is imposed, such that its integral value goes to zero, then $\gamma$ will go to a neighbourhood of zero. This could be achieved taking advantage of the flatness property of the output. Since the system is locally linearizable by a static state feedback and in this system there is a single input control signal, the chosen output is locally flat (see [5]). In this way we can express the state variables as functions of the output and its derivatives. In this case expressing $\varphi = \arcsin((1-\varepsilon)\ddot{\eta})$ and imposing an appropriate trajectory in $\ddot{\eta}$, $\gamma$ could be stabilized in a neighbourhood of zero.

### *B. Pendulum on a cart*

The dynamic equations of the pendulum on a cart can be rewritten (see [11]) using the collocated partial state feedback from [23] as

$$\ddot{\theta} = a\sin\theta - b\cos\theta u \tag{4}$$
$$\ddot{x} = u, \tag{5}$$

where $a = \frac{g}{l}$, $b = \frac{1}{l}$, $l$ is the length of the pendulum, $g$ is the gravity acceleration and $u$ is the new control input. It can be proved that the system is not full-state linearizable. Thus, following the pseudo-code in section II and choosing the output b) $\eta_1 = x + f(\theta)$, the second derivative reads after several manipulations [4]

$$\ddot{\eta}_1 = u(1 - f'(\theta)b\cos\theta) + f'(\theta)a\sin\theta + f''(\theta)\dot{\theta}^2 = \nu \tag{6}$$

where $\nu$ is the external control. The linearizing controller is obtained isolating the input control $u$, thus,

$$u = \frac{-\nu + f'(\theta)a\sin\theta + f''(\theta)\dot{\theta}^2}{f'(\theta)b\cos\theta - 1}.$$

The corresponding zero dynamics is obtained zeroing the output $\eta$, its derivatives and $\nu$, yielding

$$\begin{aligned}
\ddot{\theta} &= a\sin\theta - b\cos\theta\left(\frac{f'(\theta)a\sin\theta + f''(\theta)\dot{\theta}^2}{f'(\theta)b\cos\theta - 1}\right) \\
u_0 &= \frac{f'(\theta)a\sin\theta + f''(\theta)\dot{\theta}^2}{f'(\theta)b\cos\theta - 1},
\end{aligned} \tag{7}$$

where $u_0$ is the control input $u$ for the zero dynamics. Now we must choose the free function $f(\theta)$ in order to stabilize the zero dynamics. Furthermore, we know by the output definition if $\eta_1 = 0$ and $f(0) = 0$ then $x$ must be zero. It is very interesting to enlighten that choosing $f(\theta) \propto \sin\theta$, the linearizing controller $u$ with $\nu = 0$ is essentially the conservative part of the controller obtained in [6], [10] and [15][5]. At this point we choose $f(\theta) = k_1\sin\theta$. Now, it is

[4]For functions with one scalar argument we will use $(\cdot)'$ to denote differentiation with respect to its argument.

[5]The controllers given in [6], [10] and [15] were based on energy shaping methods, and were compound of two parts, the conservative and dissipative ones.

easy to see that this zero dynamics has the following energy function

$$E_0 = \frac{1}{2}(k_1 b \cos^2\theta - 1)\dot{\theta}^2 + a(1-\cos\theta). \tag{8}$$

Notice that this energy function is a Lyapunov function candidate if and only if the set $\{\theta : k_1 b\cos^2\theta - 1\}$ is positive. Using this Lyapunov function candidate, the asymptotically stability at the origin $(\theta, \dot{\theta}) = (0,0)$ cannot be proved. According to the proposed pseudo-code the output $\eta$ will be modified to achieve asymptotic stability of the origin. Since the zero dynamics (7) is the dynamical equation of a simple pendulum with energy (8), we only need to add a term in the output $\eta$ such that the derivative of the energy function yields the passive output [12]. Thus, we split the control action into two terms, i.e. $u_0 + u_1$, being $u_1$ the corresponding term added to the output $\eta_1$. Thus, it yields

$$\dot{E}_0 = -(k_1 b\cos^2\theta - 1)b\cos\theta\dot{\theta}u_1, \tag{9}$$

and then choosing $u_1$ such that $\dot{E}_0 \leq 0$, it has a zero dynamic asymptotically stable. Thus, we choose

$$u_1 \triangleq k_2 \frac{\dot{\theta}\cos\theta}{(k_1 b\cos^2\theta - 1)}. \tag{10}$$

Now, the new output $\eta_2$ is built from the new zero dynamics as follows,

$$\begin{aligned} \ddot{\eta}_2 &= \ddot{\eta}_1 + k_2\dot{\theta}\cos\theta \\ \dot{\eta}_2 &= \dot{\eta}_1 + k_2\sin\theta \end{aligned}$$

At this point, we have a zero dynamics with the origin asymptotically stable. Furthermore, it can be proved that choosing suitable control gains, the origin of the zero dynamics is also LES[6]. Notice that the system considered is of order 3, because we do not care about the $x$–coordinate and then we will only need to use one derivative of the new output redefined as $\dot{\eta}_2$. In this way, the output is derived once and the system is of third order. Therefore the zero dynamics is of order 2.

*Proposition 1:* The pendulum on a cart system (4–5) with the state feedback given by

$$u = \frac{k_1\sin\theta\left(a\cos\theta - \dot{\theta}^2\right)}{k_1 b\cos^2\theta - 1} + \frac{k_2\dot{\theta}\cos\theta}{(k_1 b\cos^2\theta - 1)}, \tag{11}$$

and the control gains $k_1 > 1/b > 0$ and $k_2 > 0$, ensures local exponential stability of the origin of its zero dynamics $(\theta, \dot{\theta}) = (0,0)$.

*Proof:* By definition of the zero dynamics $\dot{\eta}_2 = \ddot{\eta}_2 = 0$. Linearizing this zero dynamics around the origin $(\theta, \dot{\theta}) = (0,0)$, it is straightforward to see that all eigenvalues $\lambda_i$, $i = 1,2$, of the characteristic polynomial satisfy $Re[\lambda_i] < 0$, with the control gains given in the proposition. ∎

[6]Locally Exponentially Stable

*Local Asymptotic stability*

Now, we need to design the external controller $\nu$. Taking into account that $x$ is a cyclic variable of the system, the new state to be stabilized will be chosen as $[\theta, \dot{\theta}, \dot{\eta}_2]$.

A linear external controller

$$\nu = -k_3\dot{\eta}_2 \tag{12}$$

where $\dot{\eta}_2 = \dot{x} + k_1\dot{\theta}\cos\theta + k_2\sin\theta$, would be sufficient to achieve local asymptotic stability (LAS) for the whole system (4–5), since the origin of the zero dynamics is asymptotically stable [3], furthermore in this case is also LES. Since the main objective is to enlarge the region of attraction, the external controller will be stabilized using a Lyapunov function.

*External controller based on Lyapunov*

We proposed the following Lyapunov function candidate composed of two terms: the energy function $E_0$ above mentioned, given by equation (8), and from (12) a quadratic function of $\dot{\eta}_2$. Therefore the Lyapunov function candidate reads

$$V = E_0 + \frac{1}{2}k_3\dot{\eta}_2^2. \tag{13}$$

*Proposition 2:* The origin $(\theta, \dot{\theta}, \dot{x}) = (0,0,0)$ of the pendulum on a cart system (4–5), with $k_1$,$k_2$ as defined in Proposition 1 and positive $k_3$ and $k_4$, is locally asymptotically stable with a domain of attraction in the set $(\theta, \dot{\theta}, \dot{x}) \in (-\arccos(1/\sqrt{k_1 b}), \arccos(1/\sqrt{k_1 b})) \times \mathbb{R}^2$. Furthermore the function (13) is a Lyapunov function in this set with the state feedback given by

$$\begin{aligned} u &= \frac{-\nu + k_1\sin\theta\left(a\cos\theta - \dot{\theta}^2\right) + k_2\dot{\theta}\cos\theta}{k_1 b\cos^2\theta - 1} \\ \nu &= -k_4\left(b\dot{\theta}\cos\theta + k_3\dot{\eta}_2\right) \end{aligned} \tag{14}$$

*Proof:* The Lyapunov function candidate (13) is positive definite in the set $\{\theta : \cos^2\theta > 1/k_1 b\}$ and its derivative along the trajectories of the system (4–5) yields

$$\dot{V} = \nu\left(b\dot{\theta}\cos\theta + k_3\dot{\eta}_2\right) - k_2 b\dot{\theta}^2\cos^2\theta$$

and choosing the $\nu$ proposed, yields

$$\dot{V} = -k_4\left(b\dot{\theta}\cos\theta + k_3\dot{\eta}_2\right)^2 - k_2 b\dot{\theta}^2\cos^2\theta \leq 0$$

It only remains to prove, from La Salle, that the largest invariant set in which $\dot{V} = 0$ is the origin. It is easy to see that in order to make $\dot{V} = 0$, the two terms have to be zero, and from the second one $\dot{\theta}$ has to be zero, since the $\cos\theta \neq 0$ in the set given in the Proposition. If $\dot{\theta} = 0 \Rightarrow \dot{V} = -k_4(k_3\dot{\eta}_2)^2 = 0$ and then $\dot{\eta}_2 = 0$. The fact that $\dot{\eta}_2 = 0$ implies $\dot{x} = -k_2\sin\theta$. The derivative with respect to the time of the latter identity yields $\ddot{x} = 0$, so $\dot{x} = cst$, and therefore $u = 0$ from the equations of motion. Now, $u = 0$ implies $u_0 = 0$ and therefore either $\sin\theta = 0$ or $\cos\theta = 0$. Since $\cos\theta \neq 0$, the only possibility is $\sin\theta = 0 \Rightarrow \theta = 0$ and therefore $\dot{x} = 0$. ∎

### C. Rotary pendulum

Consider the pendulum shown in Fig. 3. The arm shaft (corresponding to the angle $\varphi$) is subject to a torque, while no torque is applied directly to the pendulum shaft (angle $\theta$). Therefore, this is an underactuated system. The system coordinates are: $\theta$ = angle of pendulum from the upward vertical, $\varphi$ = angle of arm from a fixed vertical plane. As pointed out in [10], we can rewrite the system using the collocated partial state feedback [23]. Thus the equations of motions are

$$\ddot{\theta} = a\sin\theta - b\cos\theta u + \frac{1}{2}\dot{\varphi}^2\sin 2\theta \qquad (15)$$

$$\ddot{\varphi} = u, \qquad (16)$$

with $a = mgl/J$, $b = mrl/J$ and $m$ being the mass of the pendulum, $2l$ the length of the pendulum, $r$ the radius of the arm, $J$ moment of inertia of the pendulum with respect to the pivot. Following the same procedure as described in the pendulum on a cart, we choose an output equivalent to $\dot{\eta}_2$ of the pendulum on a cart, i.e. $\dot{\eta} = \dot{\varphi} + k_1\dot{\theta}\cos\theta + k_2\sin\theta$.

*Proposition 3:* The rotary pendulum system (15–16) with the state feedback given by

$$u = (\text{rhs of (11)}) + \frac{k_1\sin\theta\cos^2\theta}{(k_1 b\cos^2\theta - 1)}\dot{\varphi}^2, \qquad (17)$$

and the control gains $k_1 > 1/b > 0$ and $k_2 > 0$, ensures local exponential stability of the origin of its zero dynamics $(\theta, \dot{\theta}) = (0, 0)$.

*Proof:* The linearization of the zero dynamics around the origin is exactly the same as in the pendulum on a a cart case. Thus, we can invoke the proposition 1. ∎

*External controller*

We proposed the same Lyapunov function candidate as in the pendulum on a cart system composed of the terms (8) and the quadratic function of $\dot{\eta}$. Thus, it yields

$$V = E_0 + \frac{1}{2}k_3\dot{\eta}^2. \qquad (18)$$

*Proposition 4:* Consider the rotary pendulum (15–16). Fix the positive constant $k_1$ as indicated in Proposition 3, and arbitrary positive constants $k_3,\ k_4$. Then, there exists a positive constant $k_2$ large enough such that the controller

$$u = \frac{-\nu + k_1\sin\theta\left(a\cos\theta - \dot{\theta}^2 + \cos^2\theta\dot{\varphi}^2\right) + k_2\dot{\theta}\cos\theta}{k_1 b\cos^2\theta - 1} \qquad (19)$$

$$\nu = -k_4\left(\dot{\theta}b\cos\theta + k_3\dot{\eta}\right)$$

ensures:

(i) The function (18) is a Lyapunov function in the set $\Omega \equiv \{(\theta, \dot{\theta}, \dot{\varphi}) \in (-\arccos(1/\sqrt{k_1 b}), \arccos(1/\sqrt{k_1 b})) \times \mathbb{R}^2\}$.

(ii) The origin $(\theta, \dot{\theta}, \dot{\varphi}) = (0, 0, 0)$ is (locally) asymptotically stable with the Lyapunov function $V$ given by equation (18).

(iii) The domain of attraction is in the set $\Omega$.

*Proof:* The Lyapunov function candidate (18) is positive definite in the set $\{\theta : \cos^2\theta > 1/k_1 b\}$ and its derivative along the trajectories of the system (15–16) yields

$$\dot{V} = \nu\left(\dot{\theta}b\cos\theta + k_3\dot{\eta}\right) - k_2\dot{\theta}^2\cos^2\theta - \sin\theta\cos\theta\dot{\theta}\dot{\varphi}^2.$$

In comparison with the pendulum on a cart control problem, here there is the extra-term $-\sin\theta\cos\theta\dot{\theta}\dot{\varphi}^2$. We need to prove that $\dot{V} \leq 0$. For, choosing the $\nu$ proposed in the proposition and taking into account that $\frac{1}{2}\frac{d}{dt}\left(\cos^2\theta\right) = -\sin\theta\cos\theta\dot{\theta}$, the derivative of the Lyapunov function yields

$$\dot{V} = -k_4\left(\dot{\theta}b\cos\theta + k_3\dot{\eta}\right)^2 + \psi(\theta, \dot{\theta}, \dot{\varphi}),$$

where the function $\psi(\cdot)$ is defined as

$$\psi(\theta, \dot{\theta}, \dot{\varphi}) \triangleq -k_2\dot{\theta}^2\cos^2\theta + \frac{1}{2}\frac{d}{dt}\left(\cos^2\theta\right)\dot{\varphi}^2.$$

To do $\dot{V} \leq 0$ we only need to prove that $\psi(\cdot) \leq 0$. To this end, first assume that $\dot{\varphi} \neq 0$, then we can write

$$\frac{\frac{d}{dt}\left(\cos^2\theta\right)}{\cos^2\theta} \leq 2k_2 b\frac{\dot{\theta}^2}{\dot{\varphi}^2}.$$

The solution for this differential inequality reads[7]

$$\cos^2\theta(t) \leq \cos^2\theta(0)e^{2k_2 b\int_0^t\left(\frac{\dot{\theta}(\tau)}{\dot{\varphi}(\tau)}\right)^2 d\tau}.$$

Notice that the integral is well defined for any initial condition $\forall\ t > 0$. The only possible point in which this integral is not well defined is $\dot{\varphi}(0) = 0$, but in this case the extra-term in $\dot{V}$ disappears and the system evolves with $\dot{V} \leq 0$. Therefore, for any initial condition there exists a ${k_2}^\star > 0$ such that for $k_2 > {k_2}^\star$ the function $\psi(\cdot) \leq 0,\ \forall\ t \geq 0$. The worst case, which occurs when $\dot{\theta}$ remains at zero, is studied jointly with the largest invariant set below.

In fact, from La Salle, now we prove that the largest invariant set in the set $\{\dot{V} = 0\} \cap \Omega$ is the origin. First, fix $k_2 > {k_2}^\star$. Then, as in the pendulum on a cart case, if $\dot{V} = 0$ then $\dot{\theta} = 0$ and $\dot{\eta} = 0$, which implies that $\dot{\varphi} = -k_2\sin\theta$. The derivative with respect to the time of the latter identity yields $\ddot{\varphi} = 0$, so $\dot{x} = cst$, and then $u = 0$. Now taking into account that $\cos\theta \neq 0$ in $\Omega$, $u = 0$ implies either $\sin\theta = 0$ or $a + {k_2}^2\cos\theta\sin^2\theta = 0$. For the first case, $\sin\theta = 0$ then $\theta = 0$ in $\Omega$, and therefore $\dot{\varphi} = 0$ and the pendulum is at the origin. The other case can be rewritten as $\cos\theta\sin^2\theta = -a/{k_2}^2$. The only solution for $\cos\theta$ is negative and therefore it is outside of the set $\Omega$. ∎

*Experimental system. Benchmark*

This subsection shows experimental results on the actual rotary pendulum depicted in Fig. 3. In order to approach to a Hamiltonian system, a LuGre model [14] to partially compensate the friction of the arm of the pendulum has been used. Particular values of the parameters used in the experimental framework can be found in [18].

[7]Alternatively we can invoke the Comparison Lemma starting from the solution of the differential equation $\psi(\cdot) = 0$.

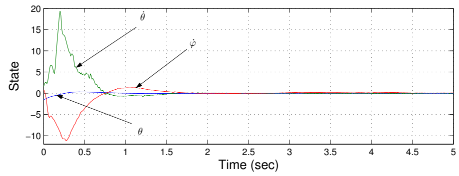


Fig. 1. First experiment.

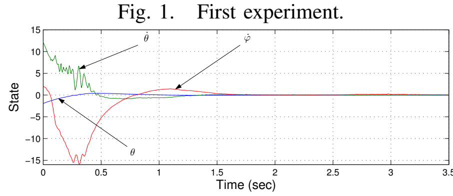


Fig. 2. Second experiment.

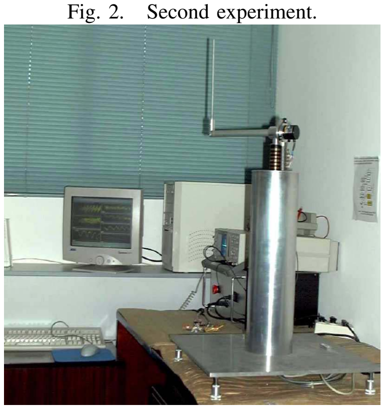

Fig. 3. Experimental rotary pendulum system.

We show two experiments. The constants were $k_1 = 100$, $k_2 = 500$, $k_3 = 30$ and $k_4 = 10$. In both experiments the initial condition for the angular position of the pendulum was $\theta(0) = 1.45$ rad. To the best of our knowledge this is the largest region of attraction achieved in experimental results to stabilize this kind of pendula. The maximum theoretical value of $\theta$ is given by Proposition 4 and for that value of $k_1$ is 1.5 rad. In fact, in [7] the maximum theoretical $\theta$ is given by the equation $\sin^2\theta = R^2/(R^2+L^2)$, where $R$ and $L$ are the radius of the arm and the length of the pendulum, respectively. This formula is not tunable, since depends only on the physical parameters, and gives rise in our pendulum to a maximum $\theta = 0.7$ rad, which is half the value presented above. Figure 1 shows an experiment with initial conditions for the velocities near to zero. In Fig. 2 the initial conditions for velocities are not near to zero. From the theoretical point of view, the region of attraction tends to the horizontal position of the pendulum, by increasing $k_1$. Unfortunately, the system saturates and it was not possible enlarge more this practical region of attraction.

## IV. Conclusions

This paper presents an easy constructive methodology to control underactuated mechanical systems with 2-DOF, by means of classical feedback linearization and Lyapunov design. The design follows a simple pseudo-code, that allows to redesign a proposed fictitious output in a constructive way. The obtained solution for the inertia wheel pendulum takes into account the friction. The solution for the planar pendulum on a cart paves the way to obtain a new solution for the rotary pendulum that, to the best of our knowledge, has the largest attraction basin presented and experimentally tested so far. Successful experimental results for this rotary pendulum solution are given.

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