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Extremal trajectories in P-time Event Graphs: application to control synthesis with specifications

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***Abstract*— This paper presents a modelling and an analysis of P-time Event Graphs in the field of topical algebra. A particular serie of matrices is introduced whose evolution determines the system behavior and the existence of a trajectory without token deaths. The extremal trajectories obeying to an interval of desired output are deduced. If every event is controllable, the Just-In-Time control of Timed Event Graph is solved when additional specifications are given by a P-time Event Graph.**

Keywords: P-time Petri Nets, Timed Event Graph, (max,+) algebra, token death, Kleene'star, control synthesis, fixed point.

I. Introduction

In an algebraic point of view, P-time Event Graphs can be modeled by a new class of systems called interval descriptor system [9] [10] for which the time evolution is not strictly deterministic but belongs to intervals. For interval descriptor system, lower and upper bounds of the intervals depends on the maximization, minimization and addition operations, simultaneously in the general case. The algebraic model of P-time Event Graphs corresponds to the semantic "And" of Time Stream Event Graph [8] and includes P-Timed Event Graphs.

An important characteristic of P-time Event Graphs is the possible deaths of tokens if a synchronization is not fulfilled. In this case, the initial algebraic model in the topical algebra, cannot be used. Some authors apply performance evaluation to determine the set of constraints guaranteeing the liveness of tokens in the strongly connected case [1]. Analysis of token liveness can be realized through the spectral vector [9] [10] in the general case.

Let us assume that a desired behavior of some transitions of the interval descriptor system is given by a sequence of intervals of execution times. We wish to slow down or accelerate the system without causing any event to occur later than the upper limits of this sequence and earlier than the lower limits. In other words, any trajectory which does not satisfy these specifications is forbidden. So, the problem is:

- to determine whether there exists trajectories which restrict the system to that behavior
- to obtain the extremal state trajectories, if they exist, which satisfy the desired output
- to calculate the corresponding input trajectories.

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In this paper, no hypothesis is taken on the structure of the Event Graph which can be a non-strongly connected graph. The initial marking must only satisfy the classical liveness condition and the usual hypothesis that places must be First In First Out (FIFO) is taken.

The paper is structured as follows. Notations and some previous results are first given. We then introduce the modelling of P-time Event Graphs in the (max,+) algebra in the "dater" form. We study its behavior with the help of a special serie of matrices and the extremal trajectories are deduced from this part. Lastly, a simple example illustrates the approach.

II. Preliminaries

A monoid is a couple $(S, \oplus)$ where the operation $\oplus$ is associative and presents a neutral element. A semi-ring S is a triplet $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoids, $\oplus$ is commutative, $\otimes$ is distributive relatively to $\oplus$ and the zero element ε of $\oplus$ is the absorbing element of $\otimes$ ($\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$) . A dioid D is an idempotent semi-ring (the operation $\oplus$ is idempotent, that is $a \oplus a = a$). Let us notice that contrary to the structures of group and ring, monoid and semi-ring do not have a property of symmetry on S. The unit $\mathbb{R} \cup \{-\infty\}$ provided with the maximum operation denoted $\oplus$ and the addition denoted $\otimes$ is an example of dioid. We have : $\mathbb{R}_{max}$ = $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$. The neutral elements of $\oplus$ and $\otimes$ are represented by $\varepsilon = -\infty$ and $e = 0$ respectively. The partial order denoted $\leq$ is defined as follows: $x \leq y \Longleftrightarrow x \oplus y = y \Longleftrightarrow x \wedge y = x \Longleftrightarrow x_i \leq y_i$, for i from 1 to n in $\mathbb{R}^n$. Notation $x < y$ means that $x \leq y$ and $x \neq y$. A dioid D is complete if it is closed for infinite sums and the distributivity of the multiplication with respect to addition applies to infinite sums : ($\forall\ c \in D$) ($\forall\ A \subseteq D$) $c \otimes (\bigoplus_{x \in A} x) = \bigoplus_{x \in A} c \otimes x$. For example, $\overline{\mathbb{R}}_{max}$ = $(\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \oplus, \otimes)$ is complete. The set of $n.n$ matrices with entries in a complete dioid D provided with the two operations $\oplus$ and $\otimes$ is also a complete dioid which is denoted $D^{n.n}$. The elements of the matrices in the (max,+) expressions (respectively (min,+) expressions) are either finite or ε (respectively T). We can deal with nonsquare matrices if we complete by rows or columns with entries equal to ε (respectively T). The mapping f is said residuated if for all $y \in D$, the least upper bound of the subset $\{x \in D \mid f(x) \leq y\}$ exists and lies in this subset. The mapping $x \in (\overline{\mathbb{R}}_{max})^n \mapsto A \otimes x$ defined over $\overline{\mathbb{R}}_{max}$is residuated (see [2]) and the left $\otimes$−residuation of B by A is denoted by: $A \backslash B = \max\{x \in (\overline{\mathbb{R}}_{max})^n$ such that

$A \otimes x \leq B\}$.

Kleene's star is defined by: $A^* = \bigoplus_{i=0}^{+\infty} A^i$. Denoted as $G(A)$, an induced graph of a square matrix A is deduced from this matrix by associating: a node i with the column i and the line i ; an arc from the node j towards the node i with $A_{ij} \neq \varepsilon$. The weight of a path p, $\mid p \mid_w$ is the sum of the labels on the edges in the path. The length of a path p, $\mid p \mid_l$ is the number of edges in the path. A circuit is a path which starts and ends at the same node.

Theorem 2.2 (Theorem 4.75 part 1 in [2]). Given A and B in a complete dioid D, A^*B is the least solution of the equation $x = A \otimes x \oplus B$, and the inequality $x \geq A \otimes x \oplus B$.

Theorem 2.3 (Theorem 4.73 part 1 in [2]). Given A and B in a complete dioid D, $A^* \backslash B$ is the greatest solution of the equation $x = A \backslash x \wedge B$, and the inequality $x \leq A \backslash x \wedge B$.

Denoted as $G_h(A, B)$, a dynamic induced graph of the square matrices A and B on an horizon h is deduced from these matrices by associating: a node j_k with the column j and a node i_k with the row i for $k = 0$ to h; an arc from the node j_{k-1} towards the node i_k with $A_{ij} \neq \varepsilon$; an arc from the node j_{k+1} towards the node i_k with $B_{ij} \neq \varepsilon$.

III. P-TIME EVENT GRAPHS

A. Definition and modelling

The P-time Petri nets makes it possible to model the discrete event dynamic systems with time constraints of stay of the tokens inside the places. Consistent with the dioid $\overline{\mathbb{R}}_{max}$, we associate for each place a temporal interval defined in $\mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$.

Definition 3.1 (p-time Petri nets) A P-time Petri net is a pair $< R, IS >$ where R is a marked Petri nets

$IS : P \longrightarrow \mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$

$p_i \longrightarrow IS_i = [a_i, b_i]$ with $0 \leq a_i \leq b_i$

IS_i is the static interval of residence time or duration of a token in place p_i belonging to the set of places P. The token must stay in the place p_i during the minimum residence duration a_i . Before this duration, the token is in state of unavailability to firing the transition t_j. The value b_i is a maximum residence duration after which the token must thus leave the place p_i. If not, the system is found in a token-dead state. So, the token is available to firing the transition t_j in the interval time $[a_i, b_i]$.

For Event Graphs, we will express the interval of shooting of each transition from the system which will guarantee an functioning without token-dead state. The set ${}^\bullet p$ is the set of input transitions of P, $p^\bullet$ is the set of output transitions of P. The set ${}^\bullet t_i$ (respectively $t_i^\bullet$) is the set of the input (respectively output) places of the transition t_i. Let us consider the variable $x_i(k)$ as the date of the kth firing of transition t_i . For each place p_k, we associate an interval $[a_{ij}, b_{ij}]$ with a_{ij} the lower bound and b_{ij} the upper bound with $t_i \in^\bullet p$ and $t_j \in p^\bullet$. As $|{}^\bullet p| = |p^\bullet| = 1$, the set of upstream (respectively downstream) transitions of t_i is noted ${}^\leftarrow t_i ={}^\bullet ({}^\bullet t_i)$ (respectively $t_i^\rightarrow = (t_i^\bullet)^\bullet$).

We consider the "dater" type in (max,+) algebra : each variable $x_i(k)$ represents the date of the kth firing of transition x_i. The usual assumption of functioning FIFO of the places is taken: it guarantees the condition of nonovertaking of the tokens between them and the correct numbering of the events. So, the evolution is described by the following inequalities which expresses relations between the dates of firing of transitions:

$(\forall\ t_j \in^\leftarrow t_i\ )\ x_i(k) \geq x_j(k - m_{ij}) + a_{ij}$

with a_{ij} the lower bound of an upstream place of t_i and m_{ij} the corresponding number of tokens present initially.

Respectively, $(\forall\ t_j \in^\leftarrow t_i)\ x_i(k) \leq x_j(k - m_{ij}) + b_{ij}$

with b_{ij} the upper bound of an upstream place of x_i,which is equivalent to $(\forall\ j \in t_i^\rightarrow)\ x_j(k + m_{ji}) - b_{ji} \leq x_i(k)$

Consequently, the model can be described by the following expression in the (max,+) dioid.

$x_i(k) \geq \bigoplus_{j \in^\leftarrow t_i} x_j(k - m_{ij}) \otimes a_{ij}$

$x_i(k) \geq \bigoplus_{j \in t_i^\rightarrow} x_j(k + m_{ji}) \otimes (-b_{ji})$ or

$$x_i(k) \geq \bigoplus_{j \in^\leftarrow t_i} a_{ij}^- \otimes x_j(k-m_{ij}) \oplus \bigoplus_{j \in t_i^\rightarrow} a_{ij}^+ \otimes x_j(k+m_{ji}) \quad (1)$$

with $a_{ij}^- = a_{ij}$ and $a_{ij}^+ = -b_{ji}$, $a_{ij}^- \in \mathbb{R}^+$, $a_{ij}^+ \in \mathbb{R}^-$

Let us notice the above set is also, equivalent to the following "interval descriptor system" : $\bigoplus_{j \in^\leftarrow t_i} a_{ij} \otimes x_j(k - m_j) \leq x_i(k) \leq \bigwedge_{j \in^\leftarrow t_i} b_{ij} \odot x_j(k - m_j)$ with m_j the number of the present tokens in each place p_j at the instant $t = 0$ (initial marking). The lower bound (respectively upper bound) is a (max, +) function (respectively (min, +) function) and this model is an example of $((max, +), (min, +))$ type of interval descriptor system. This form can be used but need the use of two dioids which complicates its treatment.

Some transitions can be considered as inputs. They are usually associated to transitions i such that ${}^\leftarrow t_i = \emptyset$ and describe for instance the input of a part. Similarly, some transitions can be considered as outputs. They are usually associated to transitions i such that $t_i^\rightarrow = \emptyset$ and describe for instance the departure of a finished product. In relation to these transitions, the following additions to the initial Event Graph do not modify its behavior and make it possible to alleviate the notations and expressions without reduction of generality.

To each input transition, an input place and its input transition denoted u are added such that the place is without token and has an interval $[0, 0]$.

For $j \in t_i^\rightarrow$, $x_j(k) \geq u_i(k - m_{ij}) + a_{ij}$ with $a_{ij} = 0$

and $x_j(k) \leq u_i(k - m_{ij})+b_{ij}$ with $b_{ij} = 0$

or $u_i(k) \geq x_j(k + m_{ij}) - b_{ji}$

Similarly, to each output transition, an output place and its corresponding output transition denoted y are added such that the place is without token and has an interval $[0, 0]$.

For $j \in^\leftarrow t_i$, $y_i(k) \geq x_j(k - m_{ji}) + a_{ji}$ with $a_{ji} = 0$

and $y_i(k) \leq x_j(k - m_{ji}) + b_{ji}$ with $b_{ji} = 0$

or $x_j(k) \geq y_i(k + m_{ji}) - b_{ji}$

Naturally, for each input transition i, $|^\leftarrow t_i \ | = 0$ and $|\ t_i^\rightarrow \ | = 1$ and for each output transition i, $|^\leftarrow t_i \ | = 1$ and $|\ t_i^\rightarrow \ | = 0$. In the (max,+) algebra, an equivalent inequality set is:

$(j \in t_i^{\rightarrow})$ $x_i(k) \geq b_{ij}^- \otimes u_j(k)$, $u_i(k) \geq b_{ij}^+ \otimes x_j(k)$
$(j \in^{\leftarrow} t_i)$ $y_i(k) \geq c_{ij}^- \otimes x_j(k)$, $x_i(k) \geq c_{ij}^+ \otimes y_j(k)$
with $b_{ij}^- = a_{ij} = 0$, $b_{ij}^+ = -b_{ji}$=0, $c_{ij}^- = a_{ij} = 0$, $c_{ij}^+ = -b_{ji} = 0$.

B. Models in (max,+) algebra

One can represent the date sequence $x(k) \in \overline{\mathbb{R}}_{max}$ with $k \in \mathbb{Z}$ by the following formal power series in one variable γ and coefficients in $\overline{\mathbb{R}}_{max}$: $x(\gamma) = \bigoplus_{k \in \mathbf{Z}} x(k)\gamma^k$. Variable γ may be regarded as the backward shift operator in event domain (formally, $\gamma x(k) = x(k-1)$) and γ-transform of functions is analogous to the Z-transform used in discrete-time classical control theory. Denoted $\overline{\mathbb{R}}_{max}[[\gamma]]$, the set of formal series in γ constitutes a dioid which brings a synthetic representation of trajectory $x(k) \in \overline{\mathbb{R}}_{max}$ with $k \in \mathbb{Z}$.

The state inequalities are deduced from 1 with the following notations. As the trajectory x is non-decreasing, condition $x \geq \gamma^1 x$ is introduced into A_1^-

$M = \bigoplus_{i \in P} m_i$; for $k = m(^\bullet(t_i))$, if $t_j \in^{\leftarrow} t_i$,$(A_k^-)_{ij} = e \oplus a_{ij}^-$ if $k = 1$, $i = j$ and $(A_k^-)_{ij} = a_{ij}^-$ otherwise; for $k = m((t_i)^\bullet)$, $(A_k^+)_{ij} = a_{ij}^+$ if $t_j \in t_i^{\rightarrow}$.

$$x \geq \bigoplus_{0 \leq i \leq M} A_i^- \otimes \gamma^i x \oplus \bigoplus_{0 \leq i \leq M} A_i^+ \otimes \gamma^{-i} x$$
$$= A_0 x \oplus \bigoplus_{1 \leq i \leq M} (A_i^- \otimes \gamma^i x \oplus A_i^+ \otimes \gamma^{-i} x)$$
with $A_0 = A_0^- \oplus A_0^+$.

This inequation has a solution in $\mathbb{R}_{max}$ if A_0^* converges in $\mathbb{R}_{max}$ and we can note:

$$x \geq A_0^* \otimes \bigoplus_{1 \leq i \leq M} (A_i^- \otimes \gamma^i x \oplus A_i^+ \otimes \gamma^{-i} x)$$

The right hand term represents the least solution of this ARMA equation.

If the Event Graph is live, there is no circuit without token and consequently the matrices $(A_0^-)^*$ and $(A_0^+)^*$ converges. The convergence of these previous matrices is a necessary condition but not a sufficient condition of convergence of A_0^*. The temporal liveness of the p-time event graph needs the convergence of A_0^*.

This expression can classically be simplified by increasing the vector state. In this state inequality, the new state vector is denoted $\mathcal{X}$.

$\mathcal{X} \geq (\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)\mathcal{X}$

The input and output inequalities are respectively:

$\mathcal{X} \geq B^- \otimes u$ and $u \geq B^+ \otimes \mathcal{X}$;

$y \geq C^- \otimes \mathcal{X}$ and $\mathcal{X} \geq C^+ \otimes y$

The matrix B^- (respectively, B^+) is composed of card(u) non null rows (respectively, columns)which contains once non null component $b_{ij}^- = a_{ij} = 0$ (respectively, $b_{ij}^+ = -b_{ji} = 0$). The matrix C^+ (respectively, C^-) is composed of card(u) non null rows (respectively, columns) which contains once non null component $c_{ij}^+ = -b_{ji} = 0$ (respectively, $c_{ij}^- = a_{ij} = 0$).

Remark 3.2 This form generalizes the classical state equation of the Timed Event Graphs: if $\mathcal{A}^+ = \varepsilon$, $B^+ = \varepsilon$ and $C^+ = \varepsilon$, the system becomes $\begin{cases} \mathcal{X} \geq \gamma^1.\mathcal{A}^- \mathcal{X} \oplus B^- \otimes u \\ y \geq C^- \otimes \mathcal{X} \end{cases}$.

These expressions describe the "lower" constraints on $\mathcal{X}$ produced by the model which can maximize it. Symmetrically, as $(\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)$ is residuated, the following form expresses every "upper" constraint on $\mathcal{X}$ which can minimize it.

$\mathcal{X} \leq (\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+) \backslash \mathcal{X}$
$u \leq B^- \backslash \mathcal{X}$ and $\mathcal{X} \leq B^+ \backslash u$
$\mathcal{X} \leq C^- \backslash y$ and $y \leq C^+ \backslash \mathcal{X}$

The two models show a dualism if we remark that $(\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)\mathcal{X} = \gamma^1.\mathcal{A}^- \mathcal{X} \oplus \gamma^{-1}.\mathcal{A}^+ \mathcal{X}$ and $(\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+) \backslash \mathcal{X} = (\gamma^{-1}.\mathcal{A}^+) \backslash \mathcal{X} \wedge (\gamma^1.\mathcal{A}^-) \backslash \mathcal{X}$ (property f3 in [2] part 4.4.4)

Symbols $\geq$, $\oplus$ and $\otimes$,correspond respectively to $\leq$, $\wedge$ and $\backslash$. Symbol γ^1 is replaced by γ^{-1} and reciprocally. Each lower (upper) matrix correspond respectively to upper (lower) matrix with the same notation.

C. Existence of a state trajectory without deaths of token

An acceptable functioning of a system can be defined by any functioning which guarantees the liveness of tokens and which does not lead to any deadlock situation, consequently. As this behavior can be represented by a state trajectory which verifies the algebraic model, the aim of this part is to study the existence of a state trajectory.

Proposition 3.3

Given $w_k = \mathcal{A}^- \otimes (w_{k-1})^* \otimes \mathcal{A}^+$ with $w_1 = \mathcal{A}^- \otimes \mathcal{A}^+$, a necessary condition of existence in $\mathbb{R}$ of a state trajectory on an infinite horizon is that the matrices w_k have only negative or null circuits.

The following property gives a graphical interpretation of the serie of matrices w_k .

Property 3.4

The matrix $(w_k)^*$ represents the maximum of all the paths from vertices i_k to vertices i_k in the dynamic induced graph $G_k(\mathcal{A}^-, \mathcal{A}^+)$ developed on the horizon k. Each element of the diagonal $(w_k)_{ii}^*$ represents the maximum between the greatest circuit and zero.

Property 3.5 The serie $w_0 = \varepsilon$ and $w_k = \mathcal{A}^- \otimes (w_{k-1})^* \otimes \mathcal{A}^+$ for $k \geq 1$ is nondecreasing.

Example

$$\mathcal{A}^- = \begin{pmatrix} \varepsilon & 1 & \varepsilon \\ 2 & \varepsilon & \varepsilon \\ 3 & \varepsilon & 4 \end{pmatrix} \text{ and } \mathcal{A}^+ = \begin{pmatrix} \varepsilon & -8 & -7 \\ -6 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & -5 \end{pmatrix}$$
$$w_1 = \begin{pmatrix} -5 & \varepsilon & \varepsilon \\ \varepsilon & -6 & -5 \\ \varepsilon & -5 & -1 \end{pmatrix} ; w_2 = \begin{pmatrix} -5 & \varepsilon & -9 \\ \varepsilon & -6 & -5 \\ -7 & -5 & -1 \end{pmatrix}$$

; $w_2 = \begin{pmatrix} -5 & -19 & -9 \\ -18 & -6 & -5 \\ -7 & -5 & -1 \end{pmatrix}$; $w_4 = w_3$;

As the serie w_k is nondecreasing, positive circuits can be found in $G_h(\mathcal{A}^-, \mathcal{A}^+)$ which entails the nonexistence of trajectory without token deaths on an infinite horizon. The opposite conclusion can be made if it exists k_1 such that $w_k = w_{k-1}$ for $k \geq k_1$ with $(w_k)_{ij} \in \mathbb{R}_{max}$ because the circuits of $G_h(\mathcal{A}^-, \mathcal{A}^+)$ have only negative weight for any horizon h.

In the following part, as a behavior without token deaths is chosen, the process presents no loss of resources.

D. Extremal trajectories

Let us assume that the desired behavior of the output transitions y of the interval descriptor system is given by a sequence of intervals of execution times $[z^-, z^+]$. The aim of this part is the determination of the greatest and lowest trajectories $(\mathcal{X}, u, y)$ satisfying this desired output: $y \in [z^-, z^+]$. Based on the special forms of the matrices B^-, B^+, C^+ and C^-, the following property will permit to facilitate the determination of the trajectories.

Property 3.6

$B^- \otimes B^+ \leq I$ and $C^+ \otimes C^- \leq I$

The problem can be reformulated as follows. The greatest (respectively, lowest) trajectories of $(\mathcal{X}, u, y)$ are denoted $(\mathcal{X}^+, u^+, y^+)$ (respectively, $(\mathcal{X}^-, u^-, y^-)$) .

Property 3.7

The greatest trajectories $(\mathcal{X}^+, u^+, y^+)$ are given by the determination of the greatest solution $(\mathcal{X}, u, y)$ of the following inequality set

$$\begin{cases} \mathcal{X} \leq (\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)\backslash\mathcal{X} \wedge C^-\backslash z^+ \\ u \leq B^-\backslash\mathcal{X} \\ y \leq C^+\backslash\mathcal{X} \end{cases} \quad (2)$$

under the condition $(\mathcal{X}, u, y) \geq (\mathcal{X}^-, u^-, y^-)$.

Symmetrically, the lowest trajectories $(\mathcal{X}^-, u^-, y^-)$ are given by the determination of the lowest solution $(\mathcal{X}, u, y)$ of the following inequality set

$$\begin{cases} \mathcal{X} \geq (\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)\mathcal{X} \oplus C^+ \otimes z^- \\ u \geq B^+ \otimes \mathcal{X} \\ y \geq C^- \otimes \mathcal{X} \end{cases}$$

under the condition $(\mathcal{X}, u, y) \leq (\mathcal{X}^+, u^+, y^+)$

Consequently, the resolution of the greatest solution $(\mathcal{X}, u, y)$ is sequential: the state trajectory is first calculated and the control and the output are simply deduced. The resolution of the state trajectories $\mathcal{X}$ is independent of the determination of the control u and the output y by reason of the special form of the matrices of the matrices B^-, B^+, C^+ and C^-. The following part will consequently consider only the resolution of the following inequalities $\mathcal{X} \leq (\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)\backslash\mathcal{X} \wedge C^-\backslash z^+$ and $\mathcal{X} \geq (\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)\mathcal{X} \oplus C^+ \otimes z^-$.

Remark 3.8 The above formulation generalizes the classical backward equation of the Timed Event Graphs: if $\mathcal{A}^+ = \varepsilon$, $B^+ = \varepsilon$ and $C^+ = \varepsilon$, the system 2 becomes $\begin{cases} \mathcal{X} \leq (\gamma^1.\mathcal{A}^-)\backslash\mathcal{X} \wedge C^-\backslash z^+ \\ u \leq B^-\backslash\mathcal{X} \end{cases}$ and the greatest solution satisfies the corresponding equalities.

Greatest state trajectory

The control of the system is on the horizon h and the process starts at $k = 0$ and can stop after h. So, only the constraints of the process on the horizon h are considered.

First, the process starts at $k = 0$ and the constraints before zero cannot be considered. So, the only constraint on $\mathcal{X}(k)$ for $k = 0$ is $\mathcal{X}(k) \leq \mathcal{A}^-\backslash\mathcal{X}(k+1) \wedge \mathcal{C}^-\backslash z(k)$. Let us notice that the assumption $\mathcal{X}(-1) = \mathcal{X}(0)$ entails that $\mathcal{X}(0) \leq \mathcal{A}^+\backslash\mathcal{X}(-1)$ is satisfied because the components of $\mathcal{A}^+$ belong to $\mathbb{R}^-$.

Symmetrically, as the process can stop after h , the only constraint on $\mathcal{X}(k)$ for $k = h$ is $\mathcal{X}(k) \leq \mathcal{A}^+\backslash\mathcal{X}(k-1) \wedge \mathcal{C}^-\backslash z(k)$. Let us notice that the hypothesis $x(h+1) = T = +\infty$ is usually taken for the classical "backward" equations of Timed Event Graphs (part 5.6.2 in [2]) and consequently, $\mathcal{X}(h) \leq (\mathcal{A}^-)\backslash\mathcal{X}(h+1)$ is satisfied.

Theorem 3.9 If the process operates on the horizon h, the greatest state trajectory is given by the following forward/backward algorithm of the determination of the greatest trajectory.

Coefficients by forward iteration

a) Initialization: $w_0 = \varepsilon$ and $\beta_0^+ = \mathcal{C}^-\backslash z(0)$

for $k = 1$ to h, $w_k = \mathcal{A}^- \otimes (w_{k-1})^* \otimes A^+$ and $\beta_k^+ = ((w_{k-1})^* \otimes \mathcal{A}^+)\backslash\beta_{k-1}^+ \wedge \mathcal{C}^-\backslash z(k)$,

b) Trajectory $\mathcal{X}^+$ by backward iteration

$\mathcal{X}^+(h) = (w_h)^*\backslash\beta_h^+$

for $k = h-1$ to 0, $\mathcal{X}^+(k) = (w_k)^*\backslash[\mathcal{A}^-\backslash\mathcal{X}(k+1) \wedge \beta_k^+]$

Lowest state trajectory

Theorem 3.10 If the process operates on the horizon h, the lowest state trajectory is given by the following forward/backward algorithm of the determination of the lowest trajectory.

a) Coefficients by forward iteration

Initialization: $w_0 = \varepsilon$ and $\beta_0^+ = \mathcal{C}^+ \otimes z^-(0)$

for $k = 1$ to h, $w_k = \mathcal{A}^- \otimes (w_{k-1})^* \otimes A^+$ and $\beta_k^- = \mathcal{A}^- \otimes (w_{k-1})^* \otimes \beta_{k-1}^- \oplus \mathcal{C}^+ \otimes z^-(k)$,

b) Trajectory $\mathcal{X}^+$ by backward iteration

$\mathcal{X}^-(h) = (w_h)^* \otimes \beta_h^-$

for $k = h-1$ to 0, $\mathcal{X}^-(k) = (w_k)^* \otimes [\mathcal{A}^+ \otimes \mathcal{X}(k+1) \oplus \beta_k^-]$

Remark 3.11 The two previous algorithms use the same serie of matrices w_k and show a dualism.

IV. Control synthesis in Timed Event Graphs with specifications

Let us assume now, that a P-time Event Graph describes the specifications of a Timed Event Graph. So, the objective is to obtain the corresponding greatest optimal control. This problem is the generalization of the classical Just-In-Time control of Timed Event graphs where the "backward" equations express the optimal control [2] [3]. Let us recall that dater type equations give the least solution (the earliest times) of the process evolution (see system 3 below) and the greatest solution (the latest times) of the control problem is explicitly given by the "backward" recursive equations where the co-vector plays the role of the state vector.

$$\begin{cases} \mathcal{X}(k) = A\backslash\mathcal{X}(k+1)\wedge C\backslash z^{+}(k) \\ u(k) = B\backslash\mathcal{X}(k) \end{cases}$$

Let us assume that any events are stated as controllable, meaning that the corresponding transitions may be delayed from firing until some arbitrary time provided by a supervisor.

So, if $\mathcal{X}$ represents the earliest time of firing of transitions, the model of Timed Event Graph is given by the following equations. Equality arises from the assumption that there is no extra delay for firing transitions whenever tokens are all available.

$$\begin{cases} \mathcal{X} = A\gamma^{1}.\mathcal{X} \oplus B.u \\ y = C.\mathcal{X} \end{cases} \tag{3}$$

with $B = I$, and I is the identity matrix. The modelling of a transportation network with timetable leads to this type of model.

From the P-time Event Graph, dynamic specifications are deduced:

$\mathcal{X} \geq (\gamma^{1}.\mathcal{A}^{-} \oplus \gamma^{-1}.\mathcal{A}^{+})\mathcal{X}$

with the hypothesis of convergence in $\mathbb{R}_{max}$ of matrices w_k.

Theorem 4.1 If the serie $w_0 = \varepsilon$ and $w_k = (A \oplus \mathcal{A}^{-}) \otimes (w_{k-1})^{*} \otimes A^{+}$ converges in $\mathbb{R}_{max}$, the resolution of the following system by the algorithm of determination of greatest trajectories

$$\begin{cases} \mathcal{X} = (\gamma^{1}.(A \oplus \mathcal{A}^{-}) \oplus \gamma^{-1}.\mathcal{A}^{+})\backslash\mathcal{X}\wedge C\backslash z^{+} \\ u = \mathcal{X} \end{cases}$$

gives the optimal control.

The resolution can be deduced easily from the previous parts if $A \oplus \mathcal{A}^{-}$ and C replace respectively $\mathcal{A}^{-}$ and C^{-} . Naturally, the classical backward equations are included in the inequality set for $B = I$.

V. Example

The following example allows us to illustrate the theoretical results on extremal trajectories. Computation tests are made using maxplus toolboxes under Scilab. The mono-input/mono-output p-time events graph contains one input transition x_1, two internal transitions x_2 and x_3 and one output transition x_4. The Petri Net of the figure Fig.1 has already been completed with additionnal transitions u and y, places p_1 and p_9 and the relevant arcs.

The state is $\mathcal{X}(k) = \begin{pmatrix} x_1(k) & x_2(k) & x_3(k) & x_4(k) \end{pmatrix}^{t}$ (t transposed) and the corresponding matrices are given by:

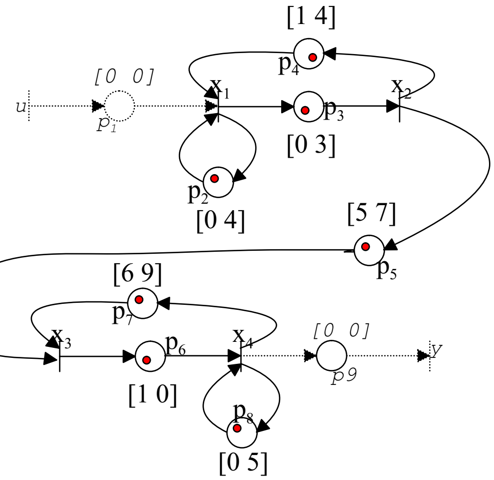

Fig. 1. P-time event graphs

$$A^{-} = \begin{pmatrix} e & 1 & \varepsilon & \varepsilon \\ e & e & \varepsilon & \varepsilon \\ \varepsilon & 5 & e & 6 \\ \varepsilon & \varepsilon & 1 & e \end{pmatrix}, \quad A^{+} = \begin{pmatrix} -4 & -3 & \varepsilon & \varepsilon \\ -4 & \varepsilon & -7 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -3 \\ \varepsilon & \varepsilon & -9 & -5 \end{pmatrix},$$

$C^{-} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix}$ $C^{+} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix}^{t}$

$B^{-} = \begin{pmatrix} e & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}^{t}$ $B^{+} = \begin{pmatrix} e & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$

with $e = 0$ and $\varepsilon = -\infty$ in the usual algebra.

a) Existence of a trajectory

The calculation of the matrices w_k shows a transitory mode from $k = 1$ to 4 and a constant matrix $w_k = w_4$ for all $k \geq 6$. If we modify the value of temporization associated with the place p_8 ($A^{+}(4,4) = -3$), the series w_k diverges towards the infinite for all $k \geq 2$ ($w_2 \longrightarrow +\infty$) . Thus, it is impossible to calculate an acceptable trajectory on the horizon: $2 \leq k \leq 6$. However, if we change the temporization related to the places p_4 or p_7 ($A^{+}(2,1) = -3$ or $A^{-}(3,4) = 7$) with $A^{+}(4,4) = -5$, the new serie diverges for $k = 7$ ($w_7 \longrightarrow +\infty$). As the control horizon is $1 \leq k \leq 6$ below, therefore the calculation of a trajectory is still possible in this horizon. In the following part, we consider the initial values of the matrices A^{-}, A^{+}, B^{-}, B^{+}, $C^{-}$and C^{+} .

b) Extremal trajectories

Given the following desired output,

k	1	2	3	4	5	6
z^{-}	0	2	5	8	10	13
z^{+}	5	13	19	22	26	30

we obtain the following table which summarizes the computation results of the acceptable trajectories of the control u and the output y:

k	1	2	3	4	5	6
u^{-}	6	10	13	17	17	21
u^{+}	13	13	17	20	24	27
y^{-}	2	7	12	15	19	22
y^{+}	5	10	15	20	24	29

As the numerical results found here check the relation $z^{-}(k) \leq y^{-}(k) \leq y^{+}(k) \leq z^{+}(k)$ for all $1 \leq k \leq 6$, the obtained extremal trajectories satisfy the problem.

Now, if we consider the following desired output z^- defined by the table

k	1	2	3	4	5	6
z^-	4	7	12	16	22	28

,

the lower and upper output are

y^-	7	12	17	20	24	28
y^+	5	10	15	20	24	29

.

Consequently, the objective $y(k) \in [z^-(k), z^+(k)]$ cannot be obtained because $y^-(k) \nleq y^+(k)$.

c) Control synthesis with specifications

Now, the P-time Event graph of Fig 1. (without the added subgraphs in dotted lines) describes the specifications of the Timed Event Graph of Fig 2. Any fires of transition are controllable. Transition $x_4 = y$ is the output.

$$A = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 \\ 5 & 3 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \end{pmatrix} B = I,\ C = C^- \text{ and}$$

$$A \oplus A^- = \begin{pmatrix} e & 1 & \varepsilon & \varepsilon \\ e & e & \varepsilon & 4 \\ 5 & 5 & e & 6 \\ \varepsilon & \varepsilon & 1 & e \end{pmatrix}$$

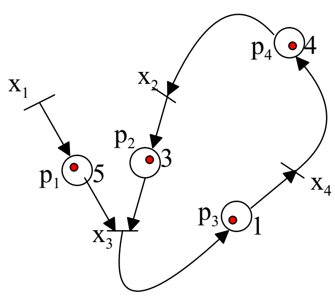

Fig. 2. Timed event graphs

The new serie w_k converges in $\mathbb{R}_{max}$ for $k = 4$. The desired output z^+ is taken as above. The following table gives the greatest trajectories of the state.

k	1	2	3	4	5	6
x_1^+	9	13	16	20	23	27
x_2^+	9	12	16	19	23	26
x_3^+	9	14	18	21	26	30
x_4^+	5	10	15	19	22	27

The values of x_1^+ and $x_4^+ = y^+$ are lower than or equal to the values of u^+ and $y^+$of problem b).

VI. Conclusion

P-time Event Graphs presents a nondeterministic behavior defined by lower and upper limits. In this paper, we have shown that it can be modeled under the special form of two models which use "noncausal" matrices (exponents can be negative. See definition 5.35 in [2]). This fact entails that lower and upper trajectories cannot easily be deduced by a simple forward iteration like in the state equation in Timed Event Graphs. The introduction of a nondecreasing serie of matrices makes it possible to determine these trajectories. Its convergence determines the existence of a trajectory without deaths of tokens. As the size of the matrices corresponds to the size of the forward/backward model which depends on the number of transitions and the initial marking, this serie gives an efficient way to calculate the circuit weights of the dynamic induced graph and to solve the token liveness problem. A perspective is naturally, the connection with the spectral vector in the (max,+) case [11] [6].

The determination of extremal trajectories satisfying a desired output trajectories introduces natural conditions of existence. Their calculations use a forward/backward iteration based on the serie of matrices. In the last part, the approach is applied to optimal control synthesis of Timed Event Graphs satisfying dynamic specifications modeled by a P-Time Event Graph. A perspective is the generalization to partially controllable events.

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