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Parametric Approaches for Eigenstructure Assignment in High-order Descriptor Linear Systems

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***Abstract*— This paper considers eigenstructure assignment in high-order descriptor linear systems via proportional plus derivative feedback. It is shown that the problem is closely related with a type of so-called high-order Sylvester matrix equations. Through establishing two general parametric solutions to this type of matrix equations, two complete parametric methods for the proposed eigenstructure assignment problem are presented. Both methods give simple complete parametric expressions for the feedback gains and the closed-loop eigenvector matrices. The first one mainly depends one a series of singular value decompositions, and is thus numerically simple and reliable; the second one utilizes the right factorization of the system, and allows the closed-loop eigenvalues to be set undetermined and sought via certain optimization procedures. An example shows the effect of the proposed approaches.**

I. INTRODUCTION

This paper is concerned with the control of the following high-order dynamical linear system

$$A_m x^{(m)} + A_{m-1}x^{(m-1)} + \cdots + A_1\dot{x} + A_0 x = Bu, \quad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and the control vector, respectively; $A_i \in \mathbb{R}^{n\times n},\ i = 0, 1, 2, ..., m,$ and $B \in \mathbb{R}^{n\times r}$ are the system coefficient matrices, which satisfy the following assumption.

Assumption A1. $\text{rank}(A_m) = n_0,\ 0 \neq n_0 \leq n,$ $\text{rank}(B) = r.$

The above system (1) reduces to a second-order descriptor linear system and a first-order descriptor linear system when m takes the value of 2 and 1, respectively.

As a special case of (1), second-order linear systems have found applications in many fields, such as vibration and structural analysis, spacecraft control and robotics control, and hence have attracted much attention ([1]-[10]). However, concerning the control of second-order linear systems, most of the results are focused on stabilization (for e.g. [2] and [3]) and pole assignment ([4]-[7]). Regarding eigenstructure assignment in second-order linear systems, there have been only a few results ([8], [9], [10]). Reference [8] considers eigenstructrue assignemnt in a special class of second-order linears systems using inverse eigenvalue methods. Reference [9] proposes an algorithm for eigenstructure assignment in second-order linear systems, with the system coefficient matrices satisfying certain symmetric positivity condition. This algorithm is attractive because it utilizes only the original system data. Very recently, eigenstructure assignment in second-order linear systems using a proportional-plus-derivative feedback controller is considered in [10]. Simple, general, and complete parametric expressions in direct closed forms for both the closed-loop eigenvector matrix and the feedback gains are established. As in [9], the approach utilizes directly the original system data, and involves manipulations on only n-dimensional matrices. However, the approach has the disadvantage that it requires the controllability of the matrix pair (A_1, B), which is not satisfied in some applications.

Control design of the high-order descriptor linear system (1) can be realized by converting the system into the following corresponding extended first-order state-space descriptor system model

$$E_e\dot{z} = A_e z + B_e u, \quad (2)$$

where

$$z = \begin{bmatrix} x^T & \dot{x}^T & \cdots & \left(x^{(m-1)}\right)^T \end{bmatrix},$$

$$E_e = \text{Blockdiag}\left(I_n, \cdots, I_n, A_m\right), \quad (3)$$

$$A_e = \begin{bmatrix} 0 & I_n & & \\ & & \ddots & \\ & & & I_n \\ -A_0 & -A_1 & \ldots & -A_{m-1} \end{bmatrix},\ B_e = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B \end{bmatrix}. \quad (4)$$

As a consequence, the results will eventually involve manipulations on mn dimensional matrices E_e, A_e and B_e.

In this paper we consider eigenstructure assignment in the high-order descriptor linear system (1) via proportional plus derivative coordinate control. The intension is to provide simple direct methods which utilizes only the original system coefficients $A_i,\ i = 0, 1, 2, \cdots, m.$ Based on a series of singular value decompositions and the right factorization of the system, two complete parametric approaches are proposed. Very simple, complete parametric expressions for both the closed-loop eigenvector matrices and the feedback gains are established. These expressions contain a group of parameter vectors which represent the design degrees of freedom, which can be properly further chosen to produce a closed-loop system with some desired system specifications. The second method, which employs

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the right factorization of the system, happens to be a natural generalization of the parametric method in [13], [14] proposed for eigenstructure assignment in first-order state-space descriptor linear systems. With this method, besides the group of parameter vectors, the closed-loop eigenvalues may also be treated as part of the design degrees of freedom since they appear directly in the expressions of the eigenvector matrix and the feedback gains, and hence are not necessarily chosen *a priori*.

The paper is composed of six sections. Section 2 gives the formulation of the eigenstructure assignment problem for high-order descriptor linear systems, and also relates it to a problem of solving a type of high-order Sylvester matrix equations. Section 3 proposes two complete parametric solutions to the type of m-order Sylvester matrix equations. Based on these solutions proposed in Section 3, two parametric methods are proposed in Section 4 for the formulated eigenstructure assignment problem. An illustrative example is presented in Section 5.

II. Problem Formulation

For the high-order dynamical system (1), by choosing the following control law

$$u = F_0 x + F_1 \dot{x} + \cdots + F_{m-1} x^{(m-1)},\ F_i \in \mathbb{R}^{r\times n}, \quad (5)$$

we obtain the closed-loop system as follows:

$$A_m x^{(m)} + A_{m-1}^c x^{(m-1)} + \cdots + A_1^c \dot{x} + A_0^c x = 0. \quad (6)$$

with and

$$A_i^c = A_i - BF_i,\ i = 0, 1, \cdots, m-1.$$

which can be written in the first-order state-space form

$$E_{ec}\dot{z} = A_{ec} z, \quad (7)$$

with

$$E_{ec} = \text{Blockdiag}\left(I_n, ..., I_n, A_m\right), \quad (8)$$

$$A_{ec} = \begin{bmatrix} 0 & I_n & & \\ & & \ddots & \\ & & & I_n \\ -A_0^c & -A_1^c & \ldots & -A_{m-1}^c \end{bmatrix}. \quad (9)$$

Definition 1: The high-order dynamical system (1) is called R-controllable (I-controllable) if and only if the corresponding extended first-order state-space representation (2)-(4) is R-controllable (I-controllable).

Recall the fact that a nondefective matrix possesses eigenvalues which are less sensitive to the parameter perturbations in the matrix, we here require the closed-loop matrix pair (E_{ec}, A_{ec}) to be nondefective, that is, the Jordan form of the matrix pair (E_{ec}, A_{ec}) possesses a diagonal form. Further, following the pole assignment theory for first-order descriptor linear systems, under the R- and I- controllability of system (1),

$$n_e = n(m-1) + n_0 \quad (10)$$

finite eigenvalues can be assigned to the closed-loop system (7)-(9). Therefore, the desired Jordan form of the matrix pair (E_{ec}, A_{ec}) takes the form

$$\Lambda = \text{diag}\left(s_1, s_2, \cdots, s_{n_e}\right), \quad (11)$$

where $s_i, i = 1, 2, \cdots, n_e$, are clearly the eigenvalues of the matrix pair (E_{ec}, A_{ec}).

Lemma 1: Let E_{ec} and A_{ec} be given by (8) and (9), and Λ be given by (11). Then

1) There exist matrices $V_i \in \mathbb{C}^{n\times n_e}, i = 1, 2, ..., m-1$, and $V = V_0 \in \mathbb{C}^{n\times n_e}$, satisfying

$$A_{ec}\begin{bmatrix} V \\ V_1 \\ \vdots \\ V_{m-1}\end{bmatrix} = E_{ec}\begin{bmatrix} V \\ V_1 \\ \vdots \\ V_{m-1}\end{bmatrix}\Lambda \quad (12)$$

if and only if

$$\begin{aligned} &A_m V\Lambda^m + \left(A_{m-1} - BF_{m-1}\right)V\Lambda^{m-1} + \cdots \\ &+ \left(A_1 - BF_1\right)V\Lambda + \left(A_0 - BF_0\right)V = 0 \end{aligned} \quad (13)$$

and

$$V_i = V_{i-1}\Lambda,\ i = 1, 2, ..., m. \quad (14)$$

2) There exist matrices $V_\infty \in R^{n\times(n-n_0)}$, $V_\infty' \in R^{n(m-1)\times(n-n_0)}$ satisfying

$$E_{ec}\begin{bmatrix} V_\infty' \\ V_\infty \end{bmatrix} = 0,\ \text{rank}\left(\begin{bmatrix} V_\infty' \\ V_\infty \end{bmatrix}\right) = n - n_0 \quad (15)$$

if and only if $V_\infty' = 0$ and

$$A_m V_\infty = 0,\ \text{rank}\left(V_\infty\right) = n - n_0. \quad (16)$$

Proof: The second conclusion is obvious, here we only show the first one.

Since

$$\begin{aligned} &A_{ec}\begin{bmatrix} V \\ V_1 \\ \vdots \\ V_{m-1}\end{bmatrix} \\ &= \begin{bmatrix} 0 & I_n & & \\ & & \ddots & \\ & & & I_n \\ -A_0^c & -A_1^c & \ldots & -A_{m-1}^c \end{bmatrix}\begin{bmatrix} V \\ V_1 \\ \vdots \\ V_{m-1}\end{bmatrix} \\ &= \begin{bmatrix} V_1 \\ \vdots \\ V_{m-1} \\ -\sum_{i=0}^{m-1}\left(A_i - BF_i\right)V_i \end{bmatrix}, \end{aligned}$$

and

$$E_{ec}\begin{bmatrix} V \\ V_1 \\ \vdots \\ V_{m-1} \end{bmatrix}\Lambda = E_{ec}\begin{bmatrix} V\Lambda \\ V_1\Lambda \\ \vdots \\ V_{m-1}\Lambda \end{bmatrix} = \begin{bmatrix} V\Lambda \\ V_1\Lambda \\ \vdots \\ A_m V_{m-1}\Lambda \end{bmatrix},$$

the equation (12) is clearly equivalent to the relations in (14) and

$$-\sum_{i=0}^{m-1}(A_i - BF_i)V_i = A_m V_{m-1}\Lambda,$$

which can be equivalently converted into

$$\sum_{i=0}^{m-1}(A_i - BF_i)V_i + A_m V_{m-1}\Lambda = 0. \tag{17}$$

Using the relations in (14), we can obtain the relations

$$V_i = V\Lambda^i,\ i = 1, 2, ..., m-1.$$

Substituting these relations into (17) yields the equation (13). ■

The first conclusion of the above lemma states that the Jordan matrix of the matrix pair (E_{ec}, A_{ec}) is Λ if and only if there exists a matrix $V \in \mathbb{C}^{n\times n_e}$ satisfying (17), and in this case the corresponding eigenvector matrix of the matrix pair (E_{ec}, A_{ec}) is given by

$$V_{ec}^f = \begin{bmatrix} V \\ V\Lambda \\ \vdots \\ V\Lambda^{m-1} \end{bmatrix}. \tag{18}$$

The second conclusion of the above lemma states that the infinite eigenvector matrix of the matrix pair (E_{ec}, A_{ec}) is given by

$$V_{ec}^{\infty} = \begin{bmatrix} 0 \\ V_\infty \end{bmatrix}, \tag{19}$$

where V_∞ satisfies (16). Therefore, the entire eigenvector matrix of the matrix pair (E_{ec}, A_{ec}) is

$$V_{ec} = \begin{bmatrix} V & 0 \\ V\Lambda & \vdots \\ \vdots & 0 \\ V\Lambda^{m-1} & V_\infty \end{bmatrix} \tag{20}$$

With the above understanding, the problem of eigenstructure assignment in the high-order descriptor dynamical system (1) via the proportional plus derivative feedback law (5) can be stated as follows.

Problem ESA (Eigenstructure assignment) Given the system (1) satisfying Assumption A1, and the matrix Λ =diag$(s_1, s_2, \cdots, s_{n_e})$, with $s_i, i = 1, 2, \cdots, n_e$, being a group of self-conjugate complex numbers (not necessarily distinct), find a general parametric form for the matrices $F_i \in \mathbb{R}^{r\times n}$, $i = 0, 1, 2, ..., m-1$, and $V \in \mathbb{C}^{n\times n_e}$ such that the matrix equation (13) and the condition

$$\det\begin{bmatrix} V & 0 \\ V\Lambda & \vdots \\ \vdots & 0 \\ V\Lambda^{m-1} & V_\infty \end{bmatrix} \neq 0 \tag{21}$$

are satisfied.

Letting

$$W = F_{m-1}V\Lambda^{m-1} + \cdots + F_1 V\Lambda + F_0 V = \begin{bmatrix} F_0 & F_1 & \cdots & F_{m-1} \end{bmatrix}\begin{bmatrix} V \\ V\Lambda \\ \vdots \\ V\Lambda^{m-1} \end{bmatrix}, \tag{22}$$

then (13) becomes

$$A_m V\Lambda^m + \cdots + A_1 V\Lambda + A_0 V = BW. \tag{23}$$

Clearly, equation (23) becomes the type of generalized Sylvester matrix equation investigated in [13], [14], [15] when $A_i = 0,\ i = 2, 3, ..., m$. Due to this fact, we call the equation (23) the m-th order generalized Sylvester matrix equation.

It follows from the above deduction that, to solve Problem ESA, the key step is to solve the following problem.

Problem HSE (High-order Sylvester equation) Given the matrices $A_i \in \mathbb{R}^{n\times n},\ i = 0, 1, ..., m,\ B \in \mathbb{R}^{n\times r}$ satisfying Assumption A1, and a diagonal matrix

$$\Lambda = \text{diag}(s_1, s_2, \cdots, s_q) \in \mathbb{C}^{q\times q}, \tag{24}$$

find a parameterization for all the matrices $V \in \mathbb{C}^{n\times q}$ and $W \in \mathbb{C}^{r\times q}$ satisfying the m-th order Sylvester matrix equation (23).

It should be noted that the number of columns of the matrices V,W and Λ in the above Problem HSE are changed into q because this makes the Problem HSE more general.

III. Solution to Problem HSE

Denote

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_q \end{bmatrix}, \tag{25}$$

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_q \end{bmatrix}, \tag{26}$$

then, in view of (24), we can convert the high-order Sylvester matrix equation (23) into the following column form

$$(s_i^m A_m + \cdots + s_i A_1 + A_0)v_i = Bw_i, \tag{27}$$

$$i = 1, 2, \cdots, q.$$

A. Case of prescribed $s_i,\ i=1,2,\cdots,q$

The equations in (27) can be further written in the following form

$$\Pi_i \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0,\ i=1,2,\cdots,q, \tag{28}$$

where

$$\Pi_i = \begin{bmatrix} s_i^m A_m + \cdots + s_i A_1 + A_0 & -B \end{bmatrix},. \tag{29}$$
$$i=1,2,\cdots,q.$$

This states that

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} \in \ker \Pi_i,\ \ i=1,2,\cdots,q. \tag{30}$$

The following algorithm produces two sets of constant matrices N_i and D_i, $i=1,2,\cdots,q$, to be used in the representation of the solution to the matrix equation (23).

Algorithm P1 (Solving N_i and D_i, $i=1,2,\cdots,q$)

1) Through applying SVD to the matrix Π_i, $i=1,2,\cdots,q$, obtain two sets of unitary matrices $P_i \in \mathbb{C}^{n\times n}$ and $Q_i \in \mathbb{C}^{(n+r)\times(n+r)}$, $i=1,2,\cdots,q$, satisfying

$$P_i \Pi_i Q_i = \begin{bmatrix} \operatorname{diag}(\sigma_1,\sigma_2,\cdots,\sigma_{n_i}) & 0 \\ 0 & 0 \end{bmatrix}, \tag{31}$$
$$i=1,2,\cdots,q,$$

where $\sigma_k>0, k=1,2,\cdots,n_i$, are the singular values of Π_i, and

$$n_i = \operatorname{rank}\begin{bmatrix} s_i^m A_m + \cdots + s_i A_1 + A_0 & B \end{bmatrix}, \tag{32}$$
$$i=1,2,\cdots,q.$$

2) Obtain the matrices $N_i \in \mathbb{R}^{n\times(n+r-n_i)}$ and $D_i \in \mathbb{R}^{r\times(n+r-n_i)}$, $i=1,2,\cdots,q$, by partitioning the matrix Q_i as follows:

$$Q_i = \begin{bmatrix} * & N_i \\ * & D_i \end{bmatrix},\ i=1,2,\cdots,q. \tag{33}$$

As a result of (31) and (33), the matrices $N_i \in \mathbb{R}^{n\times(n+r-n_i)}$ and $D_i \in \mathbb{R}^{r\times(n+r-n_i)}$, $i=1,2,\cdots,q$, obtained through the above Algorithm P1 satisfy

$$\Pi_i \begin{bmatrix} N_i \\ D_i \end{bmatrix} = 0,\ \ \operatorname{rank}\begin{bmatrix} N_i \\ D_i \end{bmatrix} = n+r-n_i, \tag{34}$$
$$i=1,2,\cdots,q.$$

This indicates that the columns of $\begin{bmatrix} N_i \\ D_i \end{bmatrix}$ form a set of basis for $\ker \Pi_i$.

The above deduction clearly yields the following result.

Theorem 2: Let $n_i, i=1,2,...,q$, be defined by (32), and $N_i \in \mathbb{R}^{n\times(n+r-n_i)}$ and $D_i \in \mathbb{R}^{r\times(n+r-n_i)}$, $i=1,2,\ldots,q$, be obtained via Algorithm P1. Then all the matrices V and W satisfying the high-order Sylvester matrix equation (23) can be parameterized by columns as follows:

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} N_i \\ D_i \end{bmatrix} f_i,\ i=1,2,\ldots,q, \tag{35}$$

where $f_i \in \mathbb{C}^{n+r-n_i}, i=1,2,\ldots,q$, are a set of arbitrary parameter vectors.

Regarding the controllability of system (1), we have the following basic result, which is an extension of the well-known HPB criterion.

Lemma 3: The high-order dynamical system (1) is controllable if and only if

$$\operatorname{rank}\begin{bmatrix} s_i^m A_m + \cdots + s_i A_1 + A_0 & B \end{bmatrix} = n, \forall s \in \mathbb{C}. \tag{36}$$

Proof: By the well-known HPB criterion, we need only to show that condition (36) is equivalent to

$$\operatorname{rank}\begin{bmatrix} A_e - sE_e & B_e \end{bmatrix} = n_e, \forall s \in \mathbb{C},$$

where E_e, A_e and B_e are given by (3) and (4).

Since

$$\begin{aligned} & \operatorname{rank}\begin{bmatrix} A_e - sE_e & B_e \end{bmatrix} \\ = \ & \operatorname{rank}\begin{bmatrix} -sI_n & I_n & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & -sI_n & I_n & 0 \\ -A_0 & -A_1 & \ldots & -A_{m-1} - sI_n & B \end{bmatrix} \\ = \ & \operatorname{rank}\begin{bmatrix} 0 & \Xi_s & 0 \\ -\sum_{i=0}^m A_i s^i & A_{1\text{-}m}(s) & B \end{bmatrix} \\ = \ & n(m-1) + \operatorname{rank}\begin{bmatrix} \sum_{i=0}^m A_i s^i & B \end{bmatrix}, \end{aligned}$$

where

$$A_{1\text{-}m}(s) = \begin{bmatrix} -A_1 & \ldots & -A_{m-1} & -sA_m \end{bmatrix},$$

$$\Xi_s = \begin{bmatrix} -sI_n & I_n & & \\ & \ddots & \ddots & \\ & & -sI_n & I_n \end{bmatrix}.$$

The conclusion clearly follows. ■

Based on the above lemma, the following corollary of Theorem 2 can be immediately derived.

Corollary 4: Let system (1) be controllable, and Λ be given by (24), then the degrees of freedom existing in the general solution to the high-order Sylvester matrix equation (23) is qr.

B. Case of undetermined $s_i,\ i=1,2,\cdots,q$

By performing the right factorization of

$$G(s) = (s^m A_m + \cdots + sA_1 + A_0)^{-1} B,$$

we can obtain a pair of polynomial matrices $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ satisfying

$$(s^m A_m + \cdots + sA_1 + A_0)^{-1} B = N(s) D^{-1}(s). \tag{37}$$

Theorem 5: Let the system (1) be R-controllable, and $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ satisfy the right factorization (37). Then

1) The matrices V and W given by (25), (26) and

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} N(s_i) \\ D(s_i) \end{bmatrix} f_i,\ i = 1, 2, ..., q \qquad (38)$$

satisfy the high-order Sylvester matrix equation (23) for all $f_i \in \mathbb{C}^r, i = 1, 2, ..., q$.

2) When

$$\text{rank} \begin{bmatrix} N(s_i) \\ D(s_i) \end{bmatrix} = r,\ i = 1, 2, ..., q \qquad (39)$$

hold, (38) gives all the solutions to Problem HSE.

Proof: It follows from (37) that

$$(s_i^m A_m + \cdots + s_i A_1 + A_0) N(s_i) - BD(s_i) = 0, \quad (40)$$
$$i = 1, 2, \ldots, q.$$

Using (38) and (40), yields

$$\begin{aligned} & (s_i^m A_m + \cdots + s_i A_1 + A_0) v_i - Bw_i \\ = \ & [(s_i^m A_m + \cdots + s_i A_1 + A_0) N(s_i) - BD(s_i)] f_i \\ = \ & 0, \ \ i = 1, 2, ..., q. \end{aligned}$$

This states that the equations in (27) holds. Therefore, the first conclusion of the theorem is true.

It follows from Corollary 4 that, under the controllability of system (1), the degrees of freedom existing in the general solution to the matrix equation (23), with Λ given by (24), is qr, while in the solution (38), the number of free parameters just equal to qr. Further, it is clear that all these parameters in the solution (38) have contributions when condition (39) holds. With this we complete the proof. ∎

The right factorization (37) performs a fundamental role in the solution (38). When s_i, $i = 1, 2, ..., q$, are chosen to be different from the zeros of $\det\left(s^m A_m + s^{m-1} A_{m-1} + \cdots + sA_1 + A_0\right)$, we can take

$$\begin{cases} N(s) = \text{Adj}\,(s^m A_m + \cdots + sA_1 + A_0)\, B \\ D(s) = \det\,(s^m A_m + \cdots + sA_1 + A_0)\, I_r \end{cases}.$$

For general numerical algorithms solving such right factorizations, one can refer to [16], [17]. The following simple procedure can also be used.

Algorithm P2 (Right coprime factorization)

1) Under the R-controllability of system (1), find a pair of unimodular matrices $P(s)$ and $Q(s)$, of appropriate dimensions, satisfying

$$\begin{aligned} & P(s) \begin{bmatrix} s^m A_m + \cdots + sA_1 + A_0 & -B \end{bmatrix} Q(s) \\ = \ & \begin{bmatrix} I_n & 0 \end{bmatrix}. \end{aligned}$$

2) Obtain the pair of polynomial matrices $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ by partitioning the unimodular matrix $Q(s)$ as follows:

$$Q(s) = \begin{bmatrix} * & N(s) \\ * & D(s) \end{bmatrix}.$$

It is worth pointing out that the pair of polynomial matrices $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ satisfying the right factorization (37) obtained from the above Algorithm P2 are right coprime since

$$\text{rank} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = r,\ \forall s \in \mathbb{C}.$$

This condition certainly implies the condition (39), which ensures the completeness of the solution (38).

To finish this section, let us finally give a remark on the extension of the results.

Remark 1: The main results in this section can be easily extended into the case that the matrix Λ is a general Jordan form.

IV. Solution to Problem ESA

Introducing the following auxiliary equation

$$W_\infty = F_{m-1} V_\infty, \qquad (41)$$

then we have

$$W_\infty = \begin{bmatrix} F_0 & F_1 & \cdots & F_{m-1} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ V_\infty \end{bmatrix}. \qquad (42)$$

Combining (42) with (22), gives

$$\begin{aligned} & \begin{bmatrix} W & W_\infty \end{bmatrix} \\ = \ & \begin{bmatrix} F_0 & F_1 & \cdots & F_{m-1} \end{bmatrix} \begin{bmatrix} V & 0 \\ V\Lambda & \vdots \\ \vdots & 0 \\ V\Lambda^{m-1} & V_\infty \end{bmatrix}. \end{aligned}$$

Therefore, when condition (21) holds, the feedback gain matrices can be obtained as

$$\begin{aligned} & \begin{bmatrix} F_0 & F_1 & \cdots & F_{m-1} \end{bmatrix} \\ = \ & \begin{bmatrix} W & W_\infty \end{bmatrix} \begin{bmatrix} V & 0 \\ V\Lambda & \vdots \\ \vdots & 0 \\ V\Lambda^{m-1} & V_\infty \end{bmatrix}^{-1}. \end{aligned} \qquad (43)$$

Following from the above deduction and the results in Section III, we can obtain the following two theorems regarding the solution to the Problem ESA.

Theorem 6: Let n_i, $i = 1, 2, ..., n_e$, be given by (32), and $N_i \in \mathbb{R}^{r\times(n+r-n_i)}$ and $D_i \in \mathbb{R}^{n\times(n+r-n_i)}$ $i = 1, 2, ..., n_e$, be given by Algorithm P1. Then

1) Problem ESA has a solution if and only if there exist a group of parameters $f_i \in \mathbb{C}^{n+r-n_i}, i = 1, 2, ..., n_e$, satisfying the following constraints:

Constraint C1: $f_i = \bar{f}_j$ if $s_i = \bar{s}_j$.
Constraint C2$_a$: $\det V_{ca} \neq 0$ with

$$V_{ca} = \begin{bmatrix} N_1 f_1 & \cdots & N_{n_e} f_{n_e} & 0 \\ s_1 N_1 f_1 & \cdots & s_{n_e} N_{n_e} f_{n_e} & \vdots \\ \vdots & \vdots & \vdots & 0 \\ s_1^{m-1} N_1 f_1 & \cdots & s_{n_e}^{m-1} N_{n_e} f_{n_e} & V_\infty \end{bmatrix}. \tag{44}$$

2) When the above condition is met, all the solutions to the Problem ESA are given by

$$V = \begin{bmatrix} N_1 f_1 & N_2 f_2 & \cdots & N_{n_e} f_{n_e} \end{bmatrix}, \tag{45}$$

and

$$\begin{aligned} & \begin{bmatrix} F_0 & F_1 & \cdots & F_{m-1} \end{bmatrix} \\ = & \begin{bmatrix} D_1 f_1 & \cdots & D_{n_e} f_{n_e} & W_\infty \end{bmatrix} V_{ca}^{-1}, \end{aligned} \tag{46}$$

where $f_i \in \mathbb{C}^{n+r-n_i}, i = 1, 2, ..., n_e$, are arbitrary parameter vectors satisfying Constraints C1 and C2$_a$, $W_\infty \in \mathbb{R}^{r \times (n-n_0)}$ is an arbitrary parameter matrix.

Theorem 7: Let system (1) be controllable, and $N(s) \in \mathbb{R}^{n \times r}[s]$ and $D(s) \in \mathbb{R}^{r \times r}[s]$ be a pair of polynomial matrices satisfying the right factorization (37) and condition (39). Then

1) Problem ESA has a solution if and only if there exist a group of parameters $f_i \in \mathbb{C}^r, i = 1, 2, ..., n_e$, satisfying Constraint C1 and
Constraint C2$_b$: $\det V_{cb} \neq 0$ with

$$V_{cb} = \begin{bmatrix} N(s_1) f_1 & \cdots & N(s_{n_e}) f_{n_e} & 0 \\ s_1 N(s_1) f_1 & \cdots & s_{n_e} N(s_{n_e}) f_{n_e} & \vdots \\ \vdots & \vdots & \vdots & 0 \\ s_1^{m-1} N(s_1) f_1 & \cdots & s_{n_e}^{m-1} N(s_{n_e}) f_{n_e} & V_\infty \end{bmatrix}. \tag{47}$$

2) When the above condition is met, all the solutions to the Problem ESA are given by

$$V = \begin{bmatrix} N(s_1) f_1 & N(s_2) f_2 & \cdots & N(s_{n_e}) f_{n_e} \end{bmatrix}, \tag{48}$$

and

$$\begin{aligned} & \begin{bmatrix} F_0 & F_1 & \cdots & F_{m-1} \end{bmatrix} \\ = & \begin{bmatrix} D(s_1) f_1 & \cdots & D(s_{n_e}) f_{n_e} & W_\infty \end{bmatrix} V_{cb}^{-1} \end{aligned} \tag{49}$$

where $f_i \in \mathbb{C}^r, i = 1, 2, ..., n_e$, are arbitrary parameter vectors satisfying Constraints C1 and C2$_b$, $W_\infty \in \mathbb{R}^{r \times (n-n_0)}$ is an arbitrary parameter matrix.

The proof of the above two theorems can be easily carried out based on the discussion in Section II and the results in Section III. The only thing which needs to be mentioned is that Constraint C1 is required because it is a necessary and sufficient condition for the matrices $F_i,\ i = 0, 1, 2, ..., m-1$, given by (46) or (49) to be real.

V. Concluding Remarks

This paper gives two complete parametric solutions to the Problem of eigenstructure assignment in high-order linear systems. Both solutions provide all the degrees of freedom, which can be sought to meet certain desired system performances.

The first solution utilizes only a series of singular value decompositions, and hence is numerically very simple and reliable. While the second one has the advantage that the closed-loop eigenvalues can be set undetermined and used as a part of extra design degrees of freedom.

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