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Relaxation of an optimal design problem with an integral-type constraint

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***Abstract*— We study a new relaxation for a two-dimensional optimal design problem in conductivity consisting of determining how to mix two given conducting materials in order to minimize the amount of one of them, subject to a constraint on the efficiency of the conducting properties of the mixture. Our approach here is different from that obtained in [10], and based on a local reformulation of the optimal design problem by means of the introduction of new potentials. The concept of constrained quasiconvexification is used in an important way.**

I. INTRODUCTION

We would like to analyze the following optimal design problem. We have at our disposal two given conducting materials, one of them is a bad and cheap conductor and the other is better but also more expensive, and we want to fill out the domain Ω (a regular, open and simply-connected set of $\mathbb{R}^2$) mixing those materials in order to minimize the amount of the most expensive conductor. The respective conductivities are α and β with $0 < \alpha < \beta$. If $\chi(x)$ is the characteristic function of the set where we put the material with conductivity α, the conductivity function in Ω is

$$a(x) = \chi(x)\alpha + (1 - \chi(x))\beta.$$

The electric potential of the body is given by the solution of the diffusion equation

$$\left\{ \begin{aligned} -\operatorname{div}(a(x)\nabla u(x)) = 0, &\quad \text{in } \Omega, \\ a(x)\nabla u \cdot n = f, &\quad \text{on } \partial\Omega, \end{aligned} \right. \tag{1}$$

where $f \in H^{-\frac{1}{2}}(\partial\Omega)$ stands for the current flux on $\partial\Omega$ and n is the outer normal vector to $\partial\Omega$. We assume the compatibility condition

$$\int_{\partial\Omega} f\, ds = 0,$$

in order to guarantee the existence of solution of the (1). Recall that under that condition, (1) has a unique solution up to an additive constant. The optimal design problem we want to address consists of finding a layout of material, i.e., a characteristic function χ, minimizing the functional

$$I(\chi) = \int_\Omega (1 - \chi(x))\, dx$$

under the integral-type constraint

$$J(\chi) = \int_\Omega \frac{1}{a(x)} |\nabla u(x)|^2\, dx \leq \gamma, \tag{2}$$

where $\gamma > 0$ is given. The integral $J(\chi)$ represents the rate at which energy is dissipated to heat in the composite material given by the design χ (the bigger this integral is, the more dissipation of energy and the less efficient the design is) so that (2) constraints the efficiency with which the conductivity a conducts the current load f through Ω.

A typical feature in optimal design problems like the one considered here is the lack of optimal solutions (see, for instance, [12]), so that relaxation is needed in order to understand the behavior of minimizing sequences. For our optimal design problem, relaxation has been analyzed in [10] using a suitable reformulation of the problem as a min-max variational problem amenable to relaxation. Recent papers about this subject are [1], [7], [11].

In this paper, we propose a different approach to analyze relaxation of this optimal design problem. We reformulate this one, in an equivalent way, as a genuine vector variational problem subject to an integral-type constraint, and then study relaxation for this new problem. As usual, relaxation for variational problems is carried out on two different levels: convexified problems in which we change the energy density by a suitable convex envelope of it, and generalized problems in which we enlarge the set of admissible functions to the set of Young measures generated by sequences of admissible functions for the original problem. Very recently this approach has been successfully used in other optimal design problems in conductivity (see [5], [16] for the two-dimensional case and [4], [6] for the three-dimensional situation).

Let us see how we reformulate the optimal design problem as a vector variational problem subject to an integral constraint. The state equation (1) is equivalent to the existence of a stream function $v \in H^1(\Omega)$ such that

$$a(x)\nabla u(x) + T\nabla v(x) = 0, \quad \text{a.e. } x \in \Omega, \tag{3}$$

where T is the counterclockwise rotation of angle $\frac{\pi}{2}$ (see [9]). Due to the fact that u verifies the boundary condition

$$a\nabla u \cdot n = f,$$

(3) implies that the tangential derivative of v is equal to the negative normal component of $a\nabla u$. Hence, up to an arbitrary constant, the boundary values of v are determined

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by indefinite integration along the boundary

$$v = v_0 = -\int f\, ds, \quad \text{on } \partial\Omega,$$

where $v_0 \in H^{\frac{1}{2}}(\partial\Omega)$.

Now we put the functions u and v together in a single field $U = (u, v)$ and consider the functions

$$W, V : \mathbb{M}^{2\times 2} \to \mathbb{R}^* = \mathbb{R} \cup \{+\infty\}$$

defined by

$$W(A) = \begin{cases} 0, & \text{if } A \in \Lambda_\alpha, \\ 1, & \text{if } A \in \Lambda_\beta \backslash \Lambda_\alpha, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$V(A) = \begin{cases} \frac{1}{\alpha}\left|A^{(1)}\right|^2, & \text{if } A \in \Lambda_\alpha, \\ \frac{1}{\beta}\left|A^{(1)}\right|^2, & \text{if } A \in \Lambda_\beta, \\ +\infty, & \text{otherwise,} \end{cases}$$

where, for $\delta > 0$

$$\Lambda_\delta = \left\{ A \in \mathbb{M}^{2\times 2} \ : \ \delta A^{(1)} + T A^{(2)} = 0 \right\}.$$

Here, $A^{(i)}$ stands for the i-row of the matrix A, $i = 1, 2$. Note that

$$\Lambda_\alpha \cap \Lambda_\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so there is no ambiguity in the definition of V. We must take into account that W and V are not Carathéodory functions since they take on the value $+\infty$ in a noncontinuous way; however, V is continuous where it is finite and W too, except at the origin.[1] This fact will be important in the study of relaxation in the next section.

It is easy to realize that the original optimal design problem is equivalent to the following variational problem:

$$\text{minimize} \int_\Omega W(\nabla U(x))\, dx \tag{4}$$

over the class of admissible functions

$$\mathcal{U} = \left\{ U \in H^1(\Omega; \mathbb{R}^2),\ U^{(2)} = v_0 \text{ on } \partial\Omega \right\}$$

subject to

$$\int_\Omega V(\nabla U(x))\, dx \le \gamma.$$

Due to this equivalence, the new problem does not have a solution, and therefore we are interested in characterization of minimizing sequences for it. To this end, we will analyze relaxation of this variational problem, proving, at first step, a subrelaxation result in terms of the appropriate convex envelope and Young measures generated by sequences of admissible gradients. Indeed, that convex envelope is defined in the following fashion. For a fixed $x \in \Omega$, $\rho > 0$, set

$$W^\sharp(A, \rho) = \inf_{\nu \in \mathcal{A}(A,\rho)} \int_{\mathbb{M}^{2\times 2}} W(F)\, d\nu(F), \tag{5}$$

[1] This fact is not a difficulty due to 0-1 law proved at [3, Theorem 1.3]. See section III for more details.

where $\mathcal{A}(A, \rho)$ is the set of homogeneous H^1 Young measures ν such that

$$\int_{\mathbb{M}^{2\times 2}} F\, d\nu(F) = A, \quad \int_{\mathbb{M}^{2\times 2}} V(F)\, d\nu(F) = \rho.$$

This will be carried out in section II.

As a second step, being the most important part of this paper and where the greatest emphasis is placed, we focus on the explicit computation of the envelope $W^\sharp$ and the optimal microstructures, that is to say, the minimizers of (5). Section III is devoted to that computation, whose conclusion is stated in the following theorem.

THEOREM 1: Let $\mathcal{B}$ be the set of pairs of matrices and real numbers (A, ρ) defined by the two inequalities

$$\left(\tfrac{\beta+\alpha}{\beta-\alpha}\right) \tfrac{\left|\alpha A^{(1)} + T A^{(2)}\right|^2}{\beta(\det A - \alpha^2 \rho)} \le 1 + \left(\tfrac{\beta+\alpha}{\beta-\alpha}\right) \tfrac{\left|\beta A^{(1)} + T A^{(2)}\right|^2}{\alpha(\det A - \beta^2 \rho)} \tag{6}$$

and

$$\alpha^2 \le \tfrac{\det A}{\rho} \le \beta^2. \tag{7}$$

The constrained envelope $W^\sharp$ is given by

$$W^\sharp(A, \rho) = \begin{cases} \left(\tfrac{\beta+\alpha}{\beta-\alpha}\right) \tfrac{\left|\alpha A^{(1)} + T A^{(2)}\right|^2}{\beta(\det A - \alpha^2\rho)} & \text{if } (A, \rho) \in \mathcal{B} \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, if (6) happens with equality, there exists a unique first order laminate,

$$\nu = (1 - t_0)\delta_{A_\alpha} + t_0 \delta_{A_\beta},$$

which is the optimal microstructure (that is, $W^\sharp(A, \rho) = \int_{\mathbb{M}^{2\times 2}} W(F)\, d\nu(F)$ and ν has barycenter A).

On the contrary, if (6) holds with strict inequality the optimal microstructures are second order laminates

$$\nu_{i,j} = (1 - \sigma_{i,j})\delta_{A_{\alpha,j}} + \sigma_{i,j}\left(\rho_{i,j}\delta_{A_\beta} + (1 - \rho_{i,j})\delta_{\bar{A}_{\alpha,i}}\right),$$

$i, j = 1, 2$, $i \ne j$, where

$$\sigma_{i,j} = \tfrac{t_0(r_j - r_i)}{t_0(1-r_i) - r_i(1-r_j)}, \quad \rho_{i,j} = \tfrac{t_0(1-r_i) - r_i(1-r_j)}{r_j - r_i}.$$

In both cases, $t_0 = \left(\tfrac{\beta+\alpha}{\beta-\alpha}\right) \tfrac{\left|\alpha A^{(1)} + T A^{(2)}\right|^2}{\beta(\det A - \alpha^2\rho)}$.

The values r_i, and the rest of the matrices will be defined in section III.

Note also that the function $W^\sharp$ is well-defined for (A, ρ) such that $\det A = \alpha^2 \rho$, because in that case, $A \in \Lambda_\alpha$. In this situation, $W(A, \rho) = 0$ and the laminate is $\nu = \delta_A$. In the same way, if $A \in \Lambda_\beta$, $W(A, \rho) = 1$ and the optimal is attained at the same laminate.

Before going into relaxation a word should be said about the envelope $W^\sharp$. Due to the fact that functions W, V are not Carathéodory functions, the infimum in (5) defined over homogeneous gradient Young measures could be different (less than or equal to) the corresponding infimum defined over gradients, so that being rigorous we should say that $W^\sharp$ is a constrained semiconvex envelope instead of a constrained quasiconvexification. Anyhow, working with $W^\sharp$ is enough to have a relaxation result in our setting, and we are even able to give a full and explicit computation of it. Therefore, with a little abuse of language, we will refer to that function as constrained quasiconvexification.

II. Relaxation

A general analysis of relaxation of variational problems under integral constraints has been carried out in [14] where densities W and V are Carathéodory functions. That analysis has been applied in that paper to obtain a subrelaxation (in the sense that the infimum of the relaxed problem is less than or equal to the original one) for general structural design problems in which also the fact of having non-Carathédory densities happen. We could apply those results to directly obtain a subrelaxation; however, we will go into the analysis of relaxation of problem (4) in a slightly different way. We get to improve the subrelaxation result, although we do not get a complete relaxation result; we will deal with this question at the end of the section. We would like to emphasize that the more important and interesting result of this paper is Theorem 1 rather than the results shown in this section, which are mainly of a technical nature (although of considerable theoretical interest). Any reader not particularly interested in the questions analyzed here can pass over this section, keeping in mind the conclusion of Theorem 2, and read directly to section III.

Proposition 1: The infimum

$$m^\sharp = \inf_{U\in\mathcal{U}\ t\in\mathcal{T}} \int_\Omega W^\sharp(\nabla U(x), t(x))\,dx$$

where $\mathcal{T} = \left\{t \in L^\infty(\Omega) : \|t\|_\infty \leq M,\ \int_\Omega t(x)\,dx \leq \gamma\right\}$, is attained.

Proof: Let us call

$$I^\sharp(U,t) = \int_\Omega W^\sharp(\nabla U(x), t(x))\,dx.$$

This functional is coercive because of any function taking values on the set where $W^\sharp$ is finite is bounded. This property was proved in [5] in the context of another optimal design problem in conductivity, and the proof uses the essential fact that Sobolev functions with gradients taking values on the support of function $W^\sharp$ are solutions of linear elliptic PDEs with uniformly bounded coefficients.

To apply the direct method we need to prove that the functional $I^\sharp$ is also weak and lower semicontinuous. To this end, it is enough to show that $W^\sharp$ verifies the following jointly convex property:

$$W^\sharp(A,\theta) \leq \frac{1}{|\Omega|}\int_\Omega W^\sharp(A + \nabla V(y), \theta + t(y))\,dy,$$

for all $(A,\theta) \in \mathbb{M}^{2\times 2}\times\mathbb{R}$, $V \in H_0^1(\Omega;\mathbb{R}^2)$ and $t \in L^\infty(\Omega)$ with vanishing mean-value (cf. [8]).

Let $V \in H_0^1(\Omega;\mathbb{R}^2)$ and $t \in L^\infty(\Omega)$ such that $\int_\Omega t(y)\,dy = 0$. For a.e. $y \in \Omega$, we can find a minimizing probability measure $\nu^y \in \mathcal{A}(A+\nabla V(y), \theta + t(y))$ such that

$$W^\sharp(A + \nabla V(y), \theta + t(y)) = \int_{\mathbb{M}^{2\times 2}} W(F)\,d\nu^y(F).$$

Proving that such a minimizing measure exists, there is an elementary fact of functional analysis due to the functional to minimize linear over measures, and the set $\mathcal{A}(A+\nabla V(y), \theta + t(y))$ is closed convex and consequently compact in the weak-$\star$ topology on the Radon measures' space.

Let us prove that the family of probability measures $\nu = \{\nu^y\}_{y\in\Omega}$ is an H^1 Young measure. To this end, we use [13, Theorem 8.16]) which gives the following sufficient conditions for being a gradient Young measure: in our case, $\nu = \{\nu^y\}_{y\in\Omega}$ is a H^1-Young measure if it verifies

(a) $\nabla U(y) = \int_{\mathbb{M}^{2\times 2}} F\,d\nu^y(F)$ for some $U \in H^1$;
(b) $\int_{\mathbb{M}^{2\times 2}} \varphi(F)\,d\nu^y(F) \geq \varphi(\nabla U(y))$ for any quasiconvex function φ;
(c) $\int_\Omega \int_{\mathbb{M}^{2\times 2}} |F|^2\,d\nu^y(F)\,dy < \infty$.

(a) holds by definition of the set $\mathcal{A}(A,\rho)$, actually $U = V + A\cdot y$ in this case; (b) is true because each ν^y is a Young measure, as any term of a Young measure is a homogeneous Young measure itself, and consequently it verifies Jensen's inequality for any quasiconvex function (see [13]), and (c) holds because of the facts that supp$(\nu^y) \subset \Delta$, a.e. $y \in \Omega$, and that the quadratic growth conditions on the function V are finite at any time.

Finally, the average measure of ν, $\bar\nu$ is an homogeneous H^1 Young measure belonging to $\mathcal{A}(A,\theta)$ and such that

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega W^\sharp(A+\nabla V(y), \theta + t(y))\,dy &= \\ \int_\Omega\int_{\mathbb{M}^{2\times 2}} W(F)\,d\nu^y(F)\,dy &= \\ \int_{\mathbb{M}^{2\times 2}} W(F)\,d\bar\nu(F) &\geq W^\sharp(A,\theta).\end{aligned}$$

The averaging measure procedure is a standard technique when dealing with Young measures and can be checked at [13]. ■

Proposition 2: If $\mathcal{A}$ stands for the set of H^1 Young measures such that there exists $U \in H^1(\Omega;\mathbb{R}^2)$ with

$$\nabla U(x) = \int_{\mathbb{M}^{2\times 2}} F\,d\nu^x(F), \quad U^{(2)} = v_0, \text{ on } \partial\Omega,$$

then the infimum

$$\tilde m = \inf\left\{\int_\Omega\int_{\mathbb{M}^{2\times 2}} W(F)\,d\nu^x(F)\,dx : \nu = \{\nu^x\}_{x\in\Omega}\in\mathcal{A},\ \int_\Omega\int_{\mathbb{M}^{2\times 2}} V(F)\,d\nu^x(F)\,dx \leq \gamma\right\}$$

is attained.

The proof is standard and is based on the minimization of a linear functional on a closed convex set; see [13]. The following is a subrelaxation result.

Theorem 2: Under above conditions, if we put

$$m = \inf_{U\in\mathcal{U}_\gamma}\int_\Omega W(\nabla U(x))\,dx$$

where $\mathcal{U}_\gamma = \{U \in \mathcal{U},\ \int_\Omega V(\nabla U(x))\,dx \leq \gamma\}$, then

$$\tilde m \leq m^\sharp \leq m.$$

Proof: The inequality

$$m^\sharp \leq m$$

is trivial as a consequence of the definition of the constrained semiconvex envelope $W^\sharp$. Let us prove the other inequality. Let $U \in H^1(\Omega;\mathbb{R}^2)$, $t \in L^\infty(\Omega)$ be such that

$$U^{(2)} = v_0, \quad \text{on } \partial\Omega, \quad \int_\Omega W^\sharp(\nabla U(x), t(x))\, dx < +\infty,$$

$$\int_\Omega t(x)\, dx \le \gamma, \text{ and } \|t\|_{L^\infty(\Omega)} \le M.$$

As was shown in the proof of Proposition 1, for a.e. $x \in \Omega$ there exists an homogeneous H^1 Young measure $\nu^x \in \mathcal{A}(\nabla U(x), t(x))$ such that

$$W^\sharp(\nabla U(x), t(x)) = \int_{\mathbb{M}^{2\times 2}} W(F)\, d\nu^x(F)$$

and the family of probability measures $\nu = \{\nu^x\}_{x\in\Omega}$ is an H^1 Young measure. Moreover, ν verifies

$$\int_\Omega \int_{\mathbb{M}^{2\times 2}} W(F)\, d\nu^x(F)\, dx = \int_\Omega W^\sharp(\nabla U(x), t(x))\, dx,$$

and

$$\int_\Omega \int_{\mathbb{M}^{2\times 2}} V(F)\, d\nu^x(F)\, dx = \int_\Omega t(x)\, dx \le \gamma.$$

This finishes the proof. ■

In order to obtain a full relaxation result it would be enough to prove the equality

$$m = \tilde{m}. \tag{8}$$

This fact would be true if given any Young measure with support contained in the set where W is finite, Λ, and verifying the integral constrained, there exists a generating sequence of gradients such that it verifies all the admissibility constraints for the problem, and, further, those gradients take values on Λ. Otherwise, it could happen that, for an optimal Young measure there is no admissible generating sequence taking values on Λ and therefore the inequality $\tilde{m} \le m$ would be strict. That is to say, there is a gap between the value of the two infima. From the point of view of applications that is not really important because it just means that small errors in the designs would improve the cost, as is extensively discussed in [15]. In our case the authors have not been able to prove equality (8). Concretely, the difficulty is the following: given a Young measure with support contained on Δ and satisfying the integral constraint, we are able to find a generating sequence for the Young measure taking values on Δ, but it is not clear how to modify a generating sequence in order to make sure that such a constraint is verified.

The following result is an improvement of the subrelaxation result in the sense pointed out above.

THEOREM 3: For any family of measures $\nu = \{\nu^x\}_{x\in\Omega}$ belonging to $\mathcal{A}$ and such that

$$\mathrm{supp}(\nu^x) \subset \Lambda, \quad \text{a.e. } x \in \Omega,$$

there exists a sequence of gradients $\{\nabla U_j\}$ generating ν and verifying that, for any j,

(i) $U_j \in H^1(\Omega;\mathbb{R}^2)$, $\quad U_j^{(2)} = v_0$,

(ii) $\left\{|\nabla U_j|^2\right\}$ is equi-integrable,

(iii) $\nabla U_j(x) \in \Lambda$, $\quad$ a.e. $x \in \Omega$,

The proof is of technical nature and we will skip out here. A complete proof can be found in [2].

III. Proof of Theorem 1

This section is devoted to the computation of the constrained envelope defined by

$$W^\sharp(A,\rho) = \inf_{\nu\in\mathcal{A}(A,\rho)} \int_{\mathbb{M}} W(F)\, d\nu(F).$$

The proof follows along the lines of the computations in [17]. For the sake of clarity, we divide the proof in various steps.

Step 1

Let us consider ν a homogeneous H^1 Young measure with barycenter A. To avoid singular cases we first assume $A \notin \Lambda$.

By definition of W and V, we can restrict our attention to all admissible measures ν with support in Λ. That is,

$$\nu = (1-t)\nu_\alpha + t\nu_\beta, \quad t \in (0,1),$$

with

$$\mathrm{supp}(\nu_\alpha) \subset \Lambda_\alpha, \quad \mathrm{supp}(\nu_\beta) \subset \Lambda_\beta.$$

This decomposition would not be well defined when the null matrix belongs to $\mathrm{supp}(\nu)$. However this situation cannot happen, as it was proved in [3, Theorem 1.3]. There, it was shown that if null matrix belongs to $\mathrm{supp}(\nu)$, then $\nu = \delta_0$, and consequently it does not have barycenter $A \neq 0$.[2]

Let us start studying some properties of first and second moments of such admissible measures. Considering the respective first moments of ν_α and ν_β,

$$A_\alpha = \int_{\Lambda_\alpha} F\, d\nu_\alpha(F), \quad A_\beta = \int_{\Lambda_\beta} F\, d\nu_\beta(F).$$

It is clear that $A = (1-t)A_\alpha + tA_\beta$, with $A_\alpha \in \Lambda_\alpha$ and $A_\beta \in \Lambda_\beta$. Then, using the definition of Λ_δ, we can write

$$A_\alpha = \begin{pmatrix} z \\ \alpha T z \end{pmatrix}, \quad A_\beta = \begin{pmatrix} w \\ \beta T w \end{pmatrix}, \tag{9}$$

and therefore, it is an easy computation to obtain

$$\begin{aligned} z &= \tfrac{1}{(1-t)(\beta-\alpha)} \left(\beta A^{(1)} + T A^{(2)}\right), \\ w &= \tfrac{-1}{t(\beta-\alpha)} \left(\alpha A^{(1)} + T A^{(2)}\right). \end{aligned} \tag{10}$$

Now, let us define the second moments

$$x_\alpha = \int_{\Lambda_\alpha} \left|F^{(1)}\right|^2 d\nu_\alpha(F), \quad x_\beta = \int_{\Lambda_\beta} \left|F^{(1)}\right|^2 d\nu_\beta(F).$$

From Jensen's inequality follows

$$x_\alpha = \int_{\Lambda_\alpha} |F^{(1)}|^2\, d\nu_\alpha(F) \ge \left| \int_{\Lambda_\alpha} F^{(1)}\, d\nu_\alpha(F) \right|^2 = |z|^2,$$

$$\text{and similarly} \quad x_\beta \ge |w|^2. \tag{11}$$

[2]In the same way, gradients of admissible functions cannot take zero value.

On the other hand, using the weak continuity of the determinant and the facts that ν can be generated by a sequence satisfying the hypotheses of Theorem 3, and that if $F \in \Lambda_\alpha$ then $\det F = \alpha|F^{(1)}|^2$, and $F \in \Lambda_\beta$ implies $\det F = \beta|F^{(1)}|^2$, it is obtained that

$$\det A = (1-t)\alpha x_\alpha + t\beta x_\beta. \tag{12}$$

We now consider the integral constraint. By definition of V we have

$$\rho = \int_{\mathbb{M}} V(F)\, d\nu(F) = \frac{(1-t)}{\alpha} x_\alpha + \frac{t}{\beta} x_\beta, \tag{13}$$

Therefore, from (12) and (13), the second moments of all admissible measures $\nu \in \mathcal{A}(A,\rho)$ such that $\nu = (1-t)\nu_\alpha + t\nu_\beta$, $t \in (0,1)$, have to verify

$$x_\alpha = \tfrac{\alpha(\det A - \beta^2\rho)}{(1-t)(\alpha^2-\beta^2)}, \qquad x_\beta = \tfrac{\beta(\det A - \alpha^2\rho)}{t(\beta^2-\alpha^2)}. \tag{14}$$

Step 2

Once we have explicit expressions of x_α and x_β, (11) reads as

$$\begin{aligned} \tfrac{\alpha(\det A - \beta^2\rho)}{(1-t)(\alpha^2-\beta^2)} &\geq \tfrac{1}{(1-t)^2(\beta-\alpha)^2}\left|\beta A^{(1)} + TA^{(2)}\right|^2, \\ \tfrac{\beta(\det A - \alpha^2\rho)}{t(\beta^2-\alpha^2)} &\geq \tfrac{1}{t^2(\beta-\alpha)^2}\left|\alpha A^{(1)} + TA^{(2)}\right|^2. \end{aligned} \tag{15}$$

First, note that $\det A - \beta^2\rho \leq 0$ and $\det A - \alpha^2\rho \geq 0$ (otherwise, first and second inequalities have no sense, respectively) and consequently (7) holds.

On the other hand, if $\det A = \beta^2\rho$ or $\det A = \alpha^2\rho$, then $x_\alpha = 0$ or $x_\beta = 0$, and therefore $A \in \Lambda_\beta$ or $A \in \Lambda_\alpha$, respectively, but we have previously assumed that $A \notin \Lambda$. Then, for $A \notin \Lambda$ we can assume strict inequality in (7).

Therefore, making some easy computations, first inequality writes as

$$t \leq 1 + \left(\tfrac{\beta+\alpha}{\beta-\alpha}\right) \tfrac{\left|\beta A^{(1)} + TA^{(2)}\right|^2}{\alpha(\det A - \beta^2\rho)}, \tag{16}$$

and the second one is

$$t \geq \left(\tfrac{\beta+\alpha}{\beta-\alpha}\right) \tfrac{\left|\alpha A^{(1)} + TA^{(2)}\right|^2}{\beta(\det A - \alpha^2\rho)}, \tag{17}$$

which implies (6). As a consequence, for all admissible $\nu = (1-t)\nu_\alpha + t\nu_\beta \in \mathcal{A}(A,\rho)$, (A,ρ) has to verify the inequalities given in the statement of Theorem 1.

Step 3

Obtaining the value of $W^\sharp$ is a direct consequence of the definition of W and (16)–(17). If (A,ρ) satisfies (6) and (7),

$$\begin{aligned} W^\sharp(A,\rho) &= \inf_{\nu\in\mathcal{A}(A,\rho)} \int_{\mathbb{M}} W(F)\, d\nu(F) = \\ &\inf_{\nu\in\mathcal{A}(A,\rho)} (1-t)\int_{\Lambda_\alpha} W(F)\, d\nu_\alpha(F) + t\int_{\Lambda_\beta} W(F)\, d\nu_\beta(F) \\ &= \inf\{t : \nu \in \mathcal{A}(A,\rho)\} = \left(\tfrac{\beta+\alpha}{\beta-\alpha}\right) \tfrac{\left|\alpha A^{(1)} + TA^{(2)}\right|^2}{\beta(\det A - \alpha^2\rho)} \end{aligned}$$

for $A \notin \Lambda$.

Step 4

For the construction of the laminates, we will follow the idea presented in [17], [16]. First, we observe that for

$$t_0 = \left(\tfrac{\beta+\alpha}{\beta-\alpha}\right) \tfrac{\left|\alpha A^{(1)} + TA^{(2)}\right|^2}{\beta(\det A - \alpha^2\rho)}, \tag{18}$$

substituting in (10) and (14), we see that

$$x_\beta = \tfrac{\beta^2(\det A - \alpha^2\rho)^2}{\left|\alpha A^{(1)} + TA^{(2)}\right|^2(\beta+\alpha)^2} = |w|^2,$$

so that,

$$\int_{\Lambda_\beta} \left|F^{(1)}\right|^2 \, d\nu_\beta(F) = |w|^2 = \left|A^{(1)}_\beta\right|^2,$$

and due to the strict convexity of the integrand $\nu_\beta = \delta_{A_\beta}$.

On the other hand, if we determine the rank-one directions going through A with extreme points on Λ_α and Λ_β, we must look for $A^\alpha \in \Lambda_\alpha$, $A^\beta \in \Lambda_\beta$, and $r \in [0,1]$ such that

$$A = (1-r)A^\alpha + rA^\beta \text{ and } \det(A^\alpha - A^\beta) = 0. \tag{19}$$

After some manipulations, this condition implies that A and r have to satisfy

$$\begin{aligned} &\tfrac{\alpha}{(1-r)^2(\beta-\alpha)^2}\left|\beta A^{(1)} + TA^{(2)}\right|^2 + \\ &\qquad \tfrac{\beta}{r^2(\beta-\alpha)^2}\left|\alpha A^{(1)} + TA^{(2)}\right|^2 + \\ &\tfrac{\alpha+\beta}{r(1-r)(\beta-\alpha)^2}\left(\alpha A^{(1)} + TA^{(2)}\right)\cdot\left(\beta A^{(1)} + TA^{(2)}\right) = 0. \end{aligned} \tag{20}$$

Therefore, there will be such rank-one directions if A is such that the above expression has solutions for $r \in [0,1]$. We assert that if $(A,\rho) \in \mathcal{B}$, then there exists $r \in [0,1]$ satisfying (20).

Let us prove our claim. It is clear that condition $(A,\rho) \in \mathcal{B}$ implies (15). That is to say, $x_\alpha \geq |z|^2$ and $x_\beta \geq |w|^2$. Then, from (12) we deduce that

$$\det A \geq (1-t)\alpha|z|^2 + t\beta|w|^2,$$

which can be written as

$$\begin{aligned} t^2(\beta-\alpha)^2\det A + t\Bigg(\alpha\left|\beta A^{(1)} - TA^{(2)}\right|^2 . \\ -\beta\left|\alpha A^{(1)} + TA^{(2)}\right|^2 - (\beta-\alpha)^2\det A\Bigg) \\ + \beta\left|\alpha A^{(1)} + TA^{(2)}\right|^2 \leq 0. \end{aligned}$$

Now, if $P_A(t)$ stands for the second degree polynomial on the left-hand side, then $(A,\rho) \in \mathcal{B}$ implies $P_A(t)$ has its roots in $[0,1]$ (note that P_A is an upward parabola where $P_A(0)$ and $P_A(1)$ are positive and $t \in (0,1)$). Making some easy computations it turns out that (20) writes as $P_A(r) = 0$. That is, $(A,\rho) \in \mathcal{B}$ implies that there exist as many rank-one directions going through A with extreme points in Λ_α and Λ_β as roots in equation $P_A(r) = 0$.

Namely, if (6) happens with equality, then it is easy to realize that $x_\alpha = |z|^2$ and $x_\beta = |w|^2$. Therefore the laminate has to be $\nu = (1-t_0)\delta_{A_\alpha} + t_0\delta_{A_\beta}$ for t_0 given in (18). Note that A_α and A_β are only dependent on t_0 and A, so the laminate is unique. That is, $P_A(r)$ has only one solution.

On the contrary, if (6) is a strict inequality, then there are two solutions of $P_A(r) = 0$, denoted by r_i, $i = 1, 2$, and therefore two rank-one directions going through A with extreme points in Λ_α and Λ_β. If we denote by $A_{\alpha,i}$ and $A_{\beta,i}$, $i = 1, 2$ the extreme points in Λ_α and Λ_β, respectively, we can construct second order laminates in the following way.

Let us consider $\bar{A}_{\alpha,i} = A_\beta + \zeta_i(A_{\alpha,i} - A_{\beta,i})$ such that $\bar{A}_{\alpha,i} \in \Lambda_\alpha$. We have to adjust the parameter ζ_i conveniently. Making some easy computations,

$$A_\beta + \zeta_i(A_{\alpha,i} - A_{\beta,i}) = \left(\begin{array}{c} w + \zeta(z_i - w_i) \\ \alpha T \left(\frac{\beta}{\alpha} w + \frac{\zeta_i}{\alpha}(\alpha z_i - \beta w_i) \right) \end{array} \right),$$

where z_i and w_i are the corresponding vectors for $A_{\alpha,i}$ and $A_{\beta,i}$ in the same way as (10).

This matrix will be in Λ_α if and only if

$$w + \zeta_i(z_i - w_i) = \left(\frac{\beta}{\alpha} w + \frac{\zeta_i}{\alpha}(\alpha z_i - \beta w_i) \right).$$

This implies $w - \zeta_i w_i = 0$, and thus $\zeta_i = \frac{r_i}{t_0}$. Then,

$$\bar{A}_{\alpha,i} = \left(\begin{array}{c} \bar{z}_i \\ \alpha T \bar{z}_i \end{array} \right)$$

for $\bar{z}_i = \frac{r_i}{t_0(1-r_i)(\beta-\alpha)} \left(\beta A^{(1)} + T A^{(2)} \right)$, and t given in (18).

In this situation, the laminates

$$\nu_{i,j} = (1 - \sigma_{i,j})\delta_{A_{\alpha,j}} + \sigma_{i,j} \left(\rho_{i,j}\delta_{A_\beta} + (1 - \rho_{i,j})\delta_{\bar{A}_{\alpha,i}} \right),$$

$i, j = 1, 2$, $i \neq j$, where

$$\sigma_{i,j} = \tfrac{t_0(r_j - r_i)}{t_0(1-r_i) - r_i(1-r_j)}, \quad \rho_{i,j} = \tfrac{t_0(1-r_i) - r_i(1-r_j)}{r_j - r_i},$$

where t_0 is given in (18), are optimal microstructures,[3] and obviously

$$\int_{\mathbb{M}} W(F)\, d\nu_{i,j}(F) = \left(\frac{\beta + \alpha}{\beta - \alpha} \right) \frac{\left| \alpha A^{(1)} + T A^{(2)} \right|^2}{\beta(\det A - \alpha^2 \rho)},$$

$i, j = 1, 2$, $i \neq j$.

Step 5

Finally, let us see what happens if $A \in \Lambda$. For instance, we assume $A \in \Lambda_\alpha$. Then $A_\alpha = A$, $A_\beta = 0$ and (12), (13) imply $\det A = \alpha^2 \rho$, and as a consequence $t_0 = 0$. That is $W^\sharp(A, \rho) = 0$. In the same way, if $A \in \Lambda_\beta$, $\det A = \beta^2 \rho$ and $W^\sharp(A, \rho) = 1$. This finishes the proof.

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[3]Note that the optimal microstructure has sense with respect to the original problem. The optimum attains in a laminate with as little mass on Λ_β as possible (that is, with less account of expensive material).

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