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A New Approach to Spectral Factorization of a Class of Matrix-Valued Spectral Densities

Hendra I. Nurdin

Abstract— **In this paper we propose a new approach to spectral factorization of a class of matrix-valued spectral densities. Our results are based on a recent necessary and sufficient uniform log-integrability condition for the canonical spectral factorization mapping to be sequentially continuous. In particular, we derive a new set of easily verifiable sufficient conditions for uniform log-integrability to hold. The proposed approach does not require the spectral density to be coercive, and the class to which it is applicable is reasonably large as to include many spectral densities which are of interest in applications. We also present a new spectral factorization algorithm for scalar analytic spectral densities along with a numerical example.**

Index Terms— **Spectral factorization, rational covariance extension, second order stochastic processes, rational approximation.**

I. Introduction

It is known that a discrete-time second order wide sense stationary (WSS) stochastic process with a power spectral density (PSD), or simply a *spectral density*, satisfying a certain Szegö or Paley-Wiener condition can be modelled as the output of a discrete time causal linear time invariant system (i.e., a "shaping filter") driven by white noise [18]. If the spectral density is rational then determining a shaping filter is possible by spectral factorization of the spectral density, and there are practical algorithms to do this. In the case where the spectral density is non-rational, obtaining a spectral factor is much more difficult and explicit spectral factorization can only be done in special cases. Apart from deriving shaping filters to model WSS processes, spectral factorization plays an important role in the theory of optimal and robust control. A survey of spectral factorization methods for both rational and non-rational spectral densities is given in [17].

In this paper, we develop an approach to spectral factorization of non-rational spectral densities which is different from the methods in [17]. The approach is based on construction of a rational approximation of the spectral density and obtaining an approximate rational shaping filter by spectral factorization of the approximation. The question which arises is whether the approximate canonical spectral factor (i.e., the unique spectral factor which is positive at the origin) which is obtained in this way will be a good approximation of the true canonical spectral factor. This question is equivalent to asking whether the operation of taking canonical spectral factors is *continuous*. It has recently been shown that such an operation is *sequentially continuous*: Given a sequence of spectral densities which converge to a limiting spectral density (in the space of functions integrable on the unit circle) then their canonical spectral factors will also converge to that of the limiting spectral density if a uniform log-integrability assumption is satisfied [2] (for a related result, see also [7]).

Our approach takes advantage of this recent sequential continuity result of [2] to show that if a sequence of rational spectral densities converges to a limiting spectral density then under a set of mild and verifiable conditions we can guarantee uniform log-integrability and the convergence of the canonical spectral factors of the sequence to the canonical spectral factor of the limiting spectral density. Furthermore, with the new results we propose an algorithm for approximate spectral factorization of analytic matrix-valued spectral densities. The main advantage of the present approach, as we shall see, is that it provides a mechanism for mitigating the effect of zeros of the spectral density which are close to or on the unit circle. It is known that numerous algorithms which are based on the Schur method (see [17]) converge slowly when the spectral density has zeros close to or on the unit circle. This is due to slow decay of the so-called Schur parameters [8]. Consequently, an approximate rational spectral factor obtained via these algorithms can be of very high degree.

The paper is organized as follows. In Section II we introduce the main notation used throughout the paper and recall some definitions and results from the literature. Following that, in Section III we discuss a recent result on sequential continuity of the spectral factorization mapping. In Section IV we derive a new set of easily checkable sufficient conditions for uniform log-integrability of a sequence of spectral densities. In Sections V and VI we give the theoretical foundation of a new approach to spectral factorization and introduce a new algorithm. We also apply the algorithm to a numerical example. Finally, in Section VII we give the conclusions of this paper as well as directions for future research.

II. Mathematical Preliminaries

In this section we introduce the main notation which is used throughout the paper, and also recall some definitions and relevant results from the literature.

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The author is with the Department of Information Engineering, Research School of Information Sciences and Engineering (Building 115), Australian National University, Canberra ACT 0200, Australia. E-mail: Hendra.Nurdin@anu.edu.au

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- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{T}$ denote the set of real numbers, complex numbers, the open unit disc$= \{z \in \mathbb{C} : |z| < 1\}$ and the unit circle (the boundary of $\mathbb{D}$), respectively.
- A^* denotes the hermitian transpose of a matrix A.
- $\Re\{A\}$ denotes the real part of a complex matrix A.
- $\mathbb{N}$ denotes the set of natural numbers $1, 2, 3, \ldots$.
- $\log c$ denotes the natural logarithm of c.
- A *pseudopolynomial* is a matrix-valued function f of the form $f(z) = \sum_{i=-m}^{n} A_i z^i$, where $0 \leq m, n < \infty$ and $A_i \in \mathbb{C}^{l \times l}$ ($l \in \mathbb{N}$) for $i = -m, \ldots, n$.
- The $\|\cdot\|_p$ norm of a matrix $A \in \mathbb{C}^{m \times n}$ is defined as [2]:

$$\|A\|_p = \begin{cases} \left(\mathrm{tr}\{(A^*A)^{p/2}\}\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup_{v \in \mathbb{C}^n, \|v\| \leq 1} \|Av\| & \text{if } p = \infty. \end{cases}$$

- μ denotes the Lebesque measure on $\mathbb{T}$.
- $\mathcal{L}^p_{m \times n}$, $1 \leq p \leq \infty$, denotes the space of measurable functions mapping from $\mathbb{T}$ to $\mathbb{C}^{m \times n}$ with a finite $\|\cdot\|_p$ norm defined by:

$$\|f\|_p = \begin{cases} \left(\frac{1}{2\pi}\int_{\mathbb{T}} \|f(z)\|_p^p \mu(dz)\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \operatorname{ess\,sup}_{z \in \mathbb{T}} \|f(z)\|_\infty & \text{if } p = \infty \end{cases}$$

  If $n = 1$, we write $\mathcal{L}^p_{m \times n}$ simply as $\mathcal{L}^p_m$.
- $\mathcal{H}^p_{m \times n}$, $1 \leq p \leq \infty$, denotes the subspace of functions in $\mathcal{L}^p_{m \times n}$ having an analytic continuation from $\mathbb{T}$ to $\mathbb{D}$. If $n = 1$, we write $\mathcal{H}^p_{m \times n}$ simply as $\mathcal{H}^p_m$.
- H_* denotes the *parahermitian conjugate* of H: $H_*(z) = H(\bar{z}^{-1})^*$.

If H is a rational element of $\mathcal{H}^p_{n \times n}$ or $\mathcal{L}^p_{n \times n}$ then the degree of H, denoted by $\deg(H)$, is defined to be the McMillan degree of H. Let $\mathcal{P}_n$ denote the linear space of $\mathbb{C}^n-$valued trigonometric polynomials on $\mathbb{T}$. It is well-known that this space is dense in $\mathcal{L}^p_n$ for all $p \in [1, \infty)$. In a similar fashion we define the linear space $\mathcal{P}^+_n$ to be the set of $\mathbb{C}^n-$valued polynomials on $\mathbb{C}$. We may view $\mathcal{P}^+_n$ as a linear subspace of $\mathcal{P}_n$. With $\mathcal{P}^+_n$ being properly defined we are in a position to introduce the notion of *outer functions* and *spectral densities*. A function $\rho \in \mathcal{H}^2_{n \times n}$ is said to be *outer* if $\overline{\rho \mathcal{P}^+_n} = \mathcal{H}^2_n$, i.e., the set of products $\rho \mathcal{P}^+_n$ is dense in $\mathcal{H}^2_n$ [2]. In the special case where $n = 1$ (the scalar case) and ρ is a rational function, it is known that ρ is outer if and only if all its zeros and poles lie in $\mathbb{D}^c$.

A function W mapping from $\mathbb{T}$ to $\mathbb{C}^{n \times n}$ is a *spectral density* if 1) it is in $\mathcal{L}^1_{n \times n}$, and 2) there exists an outer function $H \in \mathcal{H}^2_{n \times n}$ such that $W(e^{i\theta}) = H(e^{i\theta})^* H(e^{i\theta})$. Note that the definition implies that $W^* = W$ and W is non-negative definite a.e. on $\mathbb{T}$. The function H is called a *spectral factor* of W. A spectral factor is not unique since one spectral factor can be obtained from another by (right) multiplication with an arbitrary constant unitary matrix of the corresponding dimension. However, a spectral factor can be made unique if a condition is imposed on its value at the origin. We call the *unique* spectral factor which is positive definite at the origin the *canonical spectral factor* (CSF). Furthermore, we say that a spectral density W is *rational* if each element W_{ij} is of the form $W_{ij}(e^{i\theta}) = \frac{P_{ij}(e^{i\theta})}{Q_{ij}(e^{i\theta})}$ for some scalar pseudopolynomials P_{ij} and Q_{ij}. We have the following well-known characterization of spectral densities:

Theorem 1: A non-negative definite function $W \in \mathcal{L}^1_{n \times n}$ is a spectral density if and only if

$$\int_{\mathbb{T}} |\log \det W(z)| \, \mu(dz) < \infty. \tag{1}$$

The above condition is known as the Szegö condition in the scalar case and the Paley-Wiener condition in the matrix case [18]. It was also independently derived by Helson and Lowdenslager [10].

III. Sequential Continuity of the Spectral Factorization Mapping

Let W be a spectral density and let $\Phi(W)$ denote its unique CSF. Then the mapping $\Phi : W \mapsto \Phi(W)$ is called the *spectral factorization mapping*. It was recently shown in [2] that the mapping Φ is *sequentially continuous*, that is

Theorem 2: Let W be a spectral density, and let $\{W_r\}_{r \in \mathbb{N}}$ be a sequence of spectral densities such that $W_r \to W$ in $\mathcal{L}^1_{n \times n}$ as $r \to \infty$. Then the following are equivalent:

1) The sequence $\{\log \det W_r\}_{r \in \mathbb{N}}$ is *uniformly integrable*.
2) $\Phi(W_r) \to \Phi(W)$ as $r \to \infty$.

Recall that a family of scalar random variables $\{X_\gamma \mid \gamma \in \Gamma\}$ parametrized by a non-empty set Γ on a measurable space $(\Omega, \mathcal{F})$ with measure M is said to be *uniformly integrable* if:

$$\lim_{\alpha \to \infty} \sup_{\gamma \in \Gamma} \int_{\{\omega \in \Omega \mid |X_\gamma(\omega)| > \alpha\}} |X_\gamma(\omega)| M(d\omega) = 0.$$

Remark 3: We shall refer to the condition in Point 1 of Theorem 2 as *uniform log-integrability*.

The last theorem is important since it provides a justification for the altenative two-step procedure discussed in the introduction of constructing a good approximant of a given spectral density and taking the CSF of the approximant as an approximation of the true CSF, if the uniform log-integrability condition is satisfied. Several conditions which are equivalent to uniform log-integrability are given in [2, Proposition 4.2]. However, these conditions are general and do not indicate how one may construct a uniformly log-integrable sequence $\{W_r\}_{r \in \mathbb{N}}$ which converges to W in $\mathcal{L}^1_{n \times n}$. For this reason, we shall shortly develop some explicit sufficient conditions for uniform log-integrability.

IV. A Sufficient and Verifiable Set of Conditions for Uniform Log-Integrability

In this section we shall derive a new set of conditions on the sequence of convergent spectral densities *and* the limiting spectral density which ensures that the uniform log-integrability condition of Theorem 2 is satisfied. First, we assume the following:

A1. $\sup_{z \in \mathbb{T}} \|W(z)\|_1 < \infty$.
A2. $\sup_{z \in \mathbb{T}} \|W_r(z)\|_1 < \infty$ for all $r \in \mathbb{N}$ large enough.
A3. The sequence $\{W_r\}_{r \in \mathbb{N}}$ converges in $\mathcal{L}^1_{n \times n}$ to W as $r \to \infty$.

Note that from a practical point of view, Assumption A1 is not too restrictive. A majority of, if not all, spectral densities that are encountered in applications are of the bounded type. For $\alpha \geq 0$, define:

$$\begin{aligned} A_r(\alpha) &= \{z \in \mathbb{T} \mid |\log \det W_r(z)| > \alpha\}, \\ A_{r+}(\alpha) &= \{z \in \mathbb{T} \mid \det W_r(z) > e^{\alpha}\}, \\ A_{r-}(\alpha) &= \{z \in \mathbb{T} \mid \det W_r(z) < e^{-\alpha}\}, \end{aligned}$$

and note that $A_{r+}(\alpha) \cap A_{r-}(\alpha) = \phi$ and $A_r(\alpha) = A_{r+}(\alpha) \cup A_{r-}(\alpha)$. Then we have the following inequality:

$$\begin{aligned} &\sup_{r\in\mathbb{N}} \int_{A_r(\alpha)} |\log \det W_r(z)| \mu(dz) \\ &\quad \leq \sup_{r\in\mathbb{N}} \int_{A_{r+}(\alpha)} \log \det W_r(z) \mu(dz) \\ &\qquad + \sup_{r\in\mathbb{N}} \int_{A_{r-}(\alpha)} -\log \det W_r(z) \mu(dz). \end{aligned} \tag{2}$$

Before proceeding further, note the following matrix inequality:

Lemma 4: For any non-negative definite matrix $A \in \mathbb{C}^{n\times n}$, $\log \det A \leq \|A\|_1$

Proof: Note that the result is trivial if A is singular, since in this case we have $\log \det A = -\infty$. Therefore we assume that A is *positive definite*. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the singular values of A, with $\sigma_1 \geq \sigma_2 \geq \ldots > 0$. Since A is positive definite, we have that $\det(A^*) = \det(A)$ and

$$\log \det A = \frac{1}{2} \log \det(AA^*) = \frac{1}{2} \log \left(\prod_{k=1}^{n} \sigma_k^2 \right) = \sum_{k=1}^{n} \log \sigma_k.$$

On the other hand, we also have that $\|A\|_1 = \mathrm{tr}((AA^*)^{\frac{1}{2}}) = \sum_{k=1}^{n} \sigma_k$ and the result follows since $\log \sigma_k \leq \sigma_k$ for $k = 1, \ldots, n$. ■

Now we can show the following result:

Lemma 5: Under Assumptions A1, A2, and A3:

$$\lim_{\alpha\to\infty} \sup_{r\in\mathbb{N}} \int_{A_{r+}(\alpha)} \log \det W_r(z) \mu(dz) = 0.$$

Proof: See See [12], [13]. ■

Let us impose three further assumptions on the sequence $\{W_r\}_{r\in\mathbb{N}}$:

A4. $W_r(e^{i\theta})$ is a piecewise continuous function of θ for each $r \in \mathbb{N}$.

A5. Let Z_a denote the set of all points $z \in \mathbb{T}$ for which there exist a sequence of increasing integers $r_1, r_2, \ldots$ and a corresponding convergent sequence $\{z_1, z_2, \ldots\} \subset \mathbb{T}$ such that $\lim_{l\to\infty} \det W_{r_l}(z_l) = 0$ and $\lim_{l\to\infty} z_l = z$. Then the cardinality of Z_a is finite.

A6. Let Z_r denote the set of all zeros of W_r on $\mathbb{T}$. There exist positive constants M_1, M_2, Δ_1 and Δ_2 such that for any $r \in \mathbb{N}$ and any $\theta_{0,r} \in (-\pi, \pi]$ such that $e^{i\theta_{0,r}} \in Z_r \cup Z_a$, the inequality:

$$\det W_r(e^{i\theta}) \geq M_1 |\theta - \theta_{0,r}|^{M_2}, \tag{3}$$

holds for all $\theta \in [\theta_{0,r} - \Delta_1, \theta_{0,r} + \Delta_2] \cap (-\pi, \pi]$.

Remark 6: Assumption A6 implies that the cardinality of Z_r is uniformly bounded for all r.

We have the following result:

Lemma 7: Under Assumptions A4, A5, and A6:

$$\lim_{\alpha\to\infty} \sup_{r\in\mathbb{N}} \int_{A_{r-}(\alpha)} -\log \det W_r(z) \mu(dz) = 0.$$

Proof: See [12], [13]. ■

A direct consequence of Lemma 5 and Lemma 7 is the following theorem, which can be considered to be the central result of this paper:

Theorem 8: Under Assumptions A1 through to A6:

$$\lim_{\alpha\to\infty} \sup_{r\in\mathbb{N}} \int_{A_r(\alpha)} |\log \det W_r(z)| \mu(dz) = 0.$$

In other words, under Assumptions A1 through to A6, the sequence $\{\log \det W_r\}_{r\in\mathbb{N}}$ *is uniformly integrable*.

Proof: Follows directly from Lemma 5 and Lemma 7 by taking the limit $\alpha\to\infty$ on both sides of inequality (2). ■

Remark 9: Assumptions A1 to A6 do not require W to be non-coercive.

The following result is then immediate:

Corollary 10: Let W be a spectral density satisfying Assumption A1 and suppose $\det W$ has a finite number of zeros on $\mathbb{T}$. If $\{W_r\}_{r\geq 1}$ is a sequence of piecewise continuous spectral densities such that $\lim_{r\to\infty} \sup_{z\in\mathbb{T}} \|W(z) - W_r(z)\|_1 = 0$ then $\lim_{r\to\infty} \|\Phi(W) - \Phi(W_r)\|_2 = 0$.

The corollary is a simple, but useful result which allows for W to be non-coercive. We shall exploit this result in a later section.

V. Construction of Convergent Rational Spectral Densities with Converging Canonical Spectral Factors

In this section we give the main ideas for the construction of a sequence of rational spectral densities with CSF's converging to the true CSF. Let W satisfy Assumption A1 and let $\{W_r\}_{r\in\mathbb{N}}$ be a sequence of rational spectral densities (see the definition in Section 2) having no poles on $\mathbb{T}$. Let us define

$$\begin{aligned} c_k &= \frac{1}{2\pi} \int_{\mathbb{T}} W(z) z^{-k} \mu(dz) \quad k = 0, 1, \ldots \quad &(4) \\ c_{k,r} &= \frac{1}{2\pi} \int_{\mathbb{T}} W_r(z) z^{-k} \mu(dz) \quad k = 0, 1, \ldots. \quad &(5) \end{aligned}$$

The sequences $\{c_k\}_{k\in\mathbb{N}}$ and $\{c_{k,r}\}_{k\in\mathbb{N}}$ is the unique covariance sequence associated with W and W_r, respectively. By the Riemann-Lebesque Lemma, $c_k \to 0$ as $k \to \infty$. The rationality and non-negativeness of W_r imply that $c_{k,r}$ has the form:

$$c_{k,r} = C_r A_r^k B_r + \sum_{m=0}^{m_r} D_{m,r} \Delta(k - m), \tag{6}$$

where A_r, B_r, C_r, and $D_{0,r}, D_{1,r}, \ldots, D_{m_r,r}$ are $n \times n$ matrices with A_r having eigenvalues in $\mathbb{D}$, (A_r, B_r, C_r) is a minimal realization, and $\Delta(m) = \begin{cases} 1 & \text{if m=0} \\ 0 & \text{otherwise} \end{cases}$. The central idea of our construction is to require the sequence $\{W_r\}_{r\in\mathbb{N}}$ to satisfy

$$\deg \Phi(W_r) \le n d_r, \tag{7a}$$

$$c_{k,r} = c_k \quad \text{for} \quad k = 0, 1, \ldots, d_r, \tag{7b}$$

where $\{d_r\}_{r\in\mathbb{N}}$ is an increasing sequence of positive integers. That a sequence $\{W_r\}_{r\in\mathbb{N}}$ satisfying (7) always exists and can be computed is the content of the theory of *rational covariance extension with degree constraint* [8], [5], [4], [9], [14], [15]. By plugging in the infinite series expansion of W_r in the definition of $\|W - W_r\|_1$, we obtain:

$$\begin{aligned}
&\int_{\mathbb{T}} \|W(z) - W_r(z)\|_1 \mu(dz), \\
&= \int_{\mathbb{T}} \left\| W(z) - \sum_{k=0}^{\infty} \Re\{c_{k,r} z^k\} \right\|_1 \mu(dz), \\
&\le \int_{\mathbb{T}} \left\| W(z) - \sum_{k=0}^{d_r} \Re\{c_k z^k\} \right\|_1 \mu(dz) \\
&\quad + \Bigg\| \int_{\mathbb{T}} \Re\{C_r A_r^{d_r+1}(I - A_r z)^{-1} B_r \\
&\qquad + I_{\{d_r \le m_r - 1\}}(d_r) \sum_{m=d_r+1}^{m_r} D_{m,r} z^m\} \,\mu(dz) \Bigg\|_1, \\
&\le \int_{\mathbb{T}} \left\| W(z) - \sum_{k=0}^{d_r} \Re\{c_k z^k\} \right\|_1 \mu(dz) + \mathrm{R}(W_r, d_r) \qquad (8)
\end{aligned}$$

where $I_A(x)$ is the indicator function for the set A and

$$\mathrm{R}(W_r, d_r) = \left\| \Re\{C_r A_r^{d_r+1} \int_{\mathbb{T}} (I - A_r z)^{-1} \mu(dz) B_r\} \right\|_1.$$

Since W satisfies Assumption A1, the Fourier series of W converges to W in $\mathcal{L}^2_{n\times n}$, hence also in $\mathcal{L}^1_{n\times n}$. Therefore, the first term on the right hand side of (8) goes to 0 as r tends to ∞ . It now follows that the left hand side will go to zero if $\lim_{r\to\infty} \mathrm{R}(W_r, d_r) = 0$. Thus:

Theorem 11: Let the spectral density W satisfy Assumption A1. Let $\{W_r\}_{r\in\mathbb{N}}$ be a sequence of rational spectral densities satisfying Assumptions A2, A4, A5, A6 and the interpolation constraints of (7). If $\lim_{r\to\infty} \mathrm{R}(W_r, d_r) = 0$ then Assumption A3 holds and $\lim_{r\to\infty} \|\Phi(W) - \Phi(W_r)\|_2 = 0$.

Naturally, the requirement on $\mathrm{R}(W_r, d_r)$ may not be satisfied by an arbitrary sequence $\{W_r\}_{r\ge1}$ satisfying the constraints (7). However, it is reasonable to expect, at least at an intuitive level, that there could be "many" sequences which satisfy it if the spectral density W is not too "irregular" (indeed, we see later in Corollary 13 a particular instance where this is true). However, for applications and other practical purposes, we would like to be able to explicitly construct W_r such that $\Phi(W_r) \approx \Phi(W)$. Fortunately, when W is continuous we can do more. In fact, it is possible to explicitly construct the approximating sequence $W_1, W_2, \ldots$. To this end, we say that a matrix-valued function f defined on $\mathbb{T}$ is Lipschitz if $\|f(e^{i\theta}) - f(e^{i\psi})\|_1 \le K|\theta - \psi| \ \forall \theta, \psi \in (-\pi, \pi]$ for some positive constant K. Note that a Lipschitz function is continuous, but the converse is false. We start with the scalar results by observing that a scalar spectral density W which is continuous on $\mathbb{T}$ can be represented as $W = \frac{U}{V}$ where $U = WV$, and V is any positive definite scalar Lipschitz spectral density. We may then show:

Theorem 12: Let $W = \frac{U}{V}$ be a continuous scalar spectral density with U continuous, and V Lipschitz and positive definite. If $\{U_r\}_{r\ge1}$ is a sequence of non-negative definite pseudopolynomials converging uniformly to U then there exists a unique sequence $\{V_r\}_{r\ge1}$ of non-negative pseudopolynomials such that 1) $\{V_r\}_{r\ge1}$ and $\{W_r\}_{r\ge1}$ (with $W_r = U_r(V_r)^{-1}$) converge uniformly to V and W, respectively, and 2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta} W_r(e^{i\theta}) d\theta = c_k$ for $k = 0, 1, \ldots, d_r$.

For a proof of the above theorem, see [12]. An important consequence of the theorem and Corollary 10 is the following:

Corollary 13: Suppose that W is a continuous scalar spectral density with a finite number of zeros on $\mathbb{T}$. If $\{W_r\}_{r\ge1}$ is a sequence of rational scalar spectral densities as defined in Theorem 12 then $\lim_{r\to\infty} \|\Phi(W_r) - \Phi(W)\|_2 = 0$.

Remark 14: Note that the corollary gives us an instance where the hypotheses of Theorem 11 are satisfied.

For a matrix-valued spectral density W, the situation is slightly more complicated. If W is a matrix-valued Lipschitz spectral density then we write $W = (W^{-1})^{-1} = \det(W)\mathrm{adj}(W)^{-1}$, where $\mathrm{adj}(W)$ denotes the adjoint of W. Define $U = P \det W$ and $V = P\mathrm{adj}(W)$ for any arbitrary scalar spectral density P which is Lipschitz and positive definite. Then U is a scalar Lipschitz spectral density while V is a matrix-valued Lipschitz spectral density. The representation $W = UV^{-1}$ can be viewed as the matricial counterpart of the scalar fractional representation, but with the important difference that $V(z)$ can be non-invertible for some $z \in \mathbb{T}$. If W is positive definite then so is V and the analysis used in deriving Theorem 12 for the scalar case can be adapted to the matrix case with the theory developed in [3]. In particular, we obtain:

Theorem 15: Let $W = UV^{-1}$ be a matrix-valued Lipschitz spectral density which is positive definite on $\mathbb{T}$, with $U = P \det W$ and $V = P\mathrm{adj}(W)$ for some positive definite scalar Lipschitz spectral density P. If $\{U_r\}_{r\ge1}$ is a sequence of non-negative definite pseudopolynomials converging uniformly to U then there exists a unique sequence $\{V_r\}_{r\ge1}$ of positive definite pseudopolynomials such that: 1) $\{V_r\}_{r\ge1}$ and $\{W_r\}_{r\ge1}$ (with $W_r = U_r(V_r)^{-1}$) converge uniformly to V and W, respectively, and 2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta} W_r(e^{i\theta}) d\theta = c_k$ for $k = 0, 1, \ldots, d_r$.

From which it follows:

Corollary 16: Let W and $\{W_r\}_{r\in\mathbb{N}}$ be as defined in Theorem 15. Then $\lim_{r\to\infty} \|\Phi(W_r) - \Phi(W)\|_2 = 0$.

It seems likely that Theorem 15 and Corollary 16 can be extended to the case where U has zeros on $\mathbb{T}$. However, to do this, we must allow some "spectral zeros" (see [3]) of W_r to be on $\mathbb{T}$ while the current theory precludes this case. Work on an extension of the theory is currently in progress.

Corollary 10 and the development of this section provide us with a general strategy for computing an approximate spectral factor for W satisfying the hypotheses of Theorem 12 or Theorem 15: We may use some approximation scheme, such as the Fejér sum (see [11] for details) or the innovative technique described in [16], to construct U_r, V_r or W_r (resp., $\Phi(U_r)$, $\Phi(V_r)$, or $\Phi(W_r)$) as approximations of U, V or W (resp., $\Phi(U)$, $\Phi(V)$ or $\Phi(W)$). Notice that if some approximant U_r of U has been obtained, it is possible to directly compute $\Phi(V_r)$ by the method of [6] by imposing the interpolation constraints (7) on $U_r(V_r)^{-1}$ (this is, in fact, the main idea of the proof of Theorem 12). Thus, we see that, in particular, we may reduce the approximate spectral factorization problem to that of spectral factorization of at most two non-negative definite pseudopolynomials.

VI. A Spectral Factorization Algorithm

At the end of the last section we have seen that the approximate spectral factorization problem can be reduced to spectral factorization of non-negative pseudopolynomials. However, it is known that spectral factorization is difficult when the given pseudopolynomial has zeros close to or on $\mathbb{T}$. To partially alleviate this problem, we propose a new spectral factorization algorithm for scalar spectral densities which are analytic on the unit circle (thus, it includes all non-negative pseudopolynomials and rational functions). It is based on Corollary 13 and a simple heuristic for sequentially estimating spectral zeros of the given spectral density. We state this algorithm below followed by a discussion of the intuition and steps involved.

Spectral factorization algorithm

Given: A scalar spectral density $W \in \mathcal{L}^1_{n\times n}$ analytic on $\mathbb{T}$ (i.e., $\sum_{k=0}^{\infty}\|c_k\|_1 < \infty$), desired accuracy $\epsilon > 0$ and maximum iteration $r_{\max}$.

Initialize: Normalize W so that c_0=1. Let $\lambda_1, \ldots, \lambda_L \in (-\pi, \pi]$ be the local minima of $W(e^{i\cdot})$ satisfying $0 \le W(e^{i\lambda_l}) \le 0.2$ (this includes all zeros of $W(e^{i\cdot})$). Define $m_l = \min\{k \in \mathbb{N} \mid D^k_\theta W(e^{i\lambda_l}) \neq 0\}$ (note: $D^m_x = \frac{d^m}{dx^m}$) and set $z_l = \max\{0, r_l\} e^{i\lambda_l}$ with $r_l = 1 - \left(\frac{W(e^{i\lambda_l})}{D^{m_l}_\theta W(e^{i\lambda_l})}\right)^{\frac{1}{m_l}}$ for $l = 1, \ldots, L$. Set r=1, d_0=L, and $\eta_0 = \prod_{l=1}^{L}(z - z_l)^{\frac{m_l}{2}}$. Compute the outer polynomial R_0 such that $W_0 = R_{0*}^{-1}|\eta_0|^2 R_0^{-1}$ satisfies (7).

Step 1: Select θ_r such that $\theta_r = \arg\max_{\theta\in(-\pi,\pi]} \left(W_{r-1}(e^{i\theta}) - W(e^{i\theta})\right)$, and:

1) If W is symmetric (i.e., $W(e^{-i\theta}) = W(e^{i\theta})$) or $\theta_r \notin \{0, \pi\}$, set $\mathrm{R} = 1 - \left(\frac{W(e^{i\theta_r})}{W_{r-1}(e^{i\theta_r})}\right)^{\frac{1}{2n}}$, $z_r = \mathrm{R}e^{i\theta_r}$, $d_r = d_{r-1} + 2$, and $\eta_r = \eta_{r-1}(z - z_r)(z - z_r^*)$, otherwise
2) Set R to be the smallest positive solution of the quartic equation: $|1-R|^2|1-Re^{-i2\theta_r}|^2 - \left(\frac{W(e^{i\theta_r})}{W(e^{i\theta_r})}\right)^{\frac{1}{n}} = 0$, and set $z_r = \mathrm{R}e^{i\theta}$, $d_r = d_{r-1} + 1$, and $\eta_r = \eta_{r-1}(z - z_r)$.

Step 2: Compute the outer polynomial R_r such that $W_r = R_{r*}^{-1}|\eta_r|^2 R_r^{-1}$ satisfies (7).

Step 3 Compute $e = \frac{1}{2}\|W - W_r\|_1 + \frac{1}{2}\|\Phi(W) - \Phi(W_r)\|_2$. If $e > \epsilon$ and $r \le r_{\max}$, set $r = r + 1$ and return to Step 1.

End $\sqrt{c_0}\,\eta_{r*}(R_r)^{-1}$ is the approximate CSF.

Computation of the polynomial R_r, $r = 0, 1, 2, \ldots$, is given in [6]. The analysis in [14], [15] implies that when R_r is positive definite then the continuation method of [6] is applicable to the case where η_r has zeros on $\mathbb{T}$. The main idea of the algorithm is to find a sequence $z_1, z_2, \ldots \in \overline{\mathbb{D}}$ such that $W_r = U_r V_r^{-1}$ satisfies (7) and $W_r \to W$ in $\mathcal{L}^\infty$, where $U_r(z) = \Pi_{k=1}^{d_r}(z - z_k)(z - z_k)_*$. By Theorem 12, it then follows that $\Phi(W_r) \to \Phi(W)$ in $\mathcal{H}^2$. This idea works as follows. Since $W(e^{i\theta})$ is analytic it has a continuation $W(z)$ to an open annulus of the complex plane containing $\mathbb{T}$. Moreover, if $D^m_\theta W(e^{i\lambda}) = 0$ for m=1,…,l then also $W^{(m)}(e^{i\lambda}) = W^{(m)}(z)\big|_{z=e^{i\lambda}} = 0$ for the same values of m, and $W^{(l+1)}(e^{i\lambda}) = (-ie^{-i\lambda})^{l+1} D^{l+1}_\theta W(e^{i\lambda})$ (where $W^{(m)}$ denotes the m^{th} derivative of the analytic continuation of W). Since λ_l is a local minimum, we have that $D^{m_l}_\theta W(e^{i\lambda_l}) > 0$. We first take care of points $\theta \in (-\pi, \pi]$ for which $W(e^{i\theta}) \approx 0$. They are characterized as points λ_l, $l = 1, \ldots, L$, which are local minima of $W(e^{i\cdot})$ satisfying $0 \le W(e^{i\lambda_l}) \le 0.2$ (the value 0.2 has been chosen subjectively based on the consideration that it is not "too small" and not "too large"). The Taylor series expansion of $W(z)$ about $e^{i\lambda_l}$ gives $W(z) \approx W(e^{i\lambda_l}) + (-ie^{-i\theta})^{m_l} D^{m_l}_\theta W(e^{i\lambda_l})(z - e^{i\lambda_l})^{m_l}$ for z sufficiently close to $e^{i\lambda_l}$. To estimate a zero of $W(z)$ about $e^{i\lambda_l}$, we set $W(z)$=0 to get $|z - e^{i\lambda_l}| \approx \left(\frac{W(e^{i\lambda_l})}{D^{m_l}_\theta W(e^{i\lambda_l})}\right)^{\frac{1}{m_l}}$. Assuming the form $z_l = r_l e^{i\theta}$ with $0 \le r_l \le 1$ for our zero estimate, we obtain $|1 - r_l| = \left(\frac{W(e^{i\lambda_l})}{D^{m_l}_\theta W(e^{i\lambda_l})}\right)^{\frac{1}{m_l}}$. Thus we choose $r_l = 1 - \left(\frac{W(e^{i\lambda_l})}{D^{m_l}_\theta W(e^{i\lambda_l})}\right)^{\frac{1}{m_l}}$ and set $z_l = \max\{0, r_l\}e^{i\lambda_l}$ (hence automatically $z_l = e^{i\lambda_l}$ if $W(e^{i\lambda_l}) = 0$). Points $z \in \mathbb{T}$ for which $W(z) \approx 0$ are *critical* since, as argued in [8], in the case of Schur and other interpolation based spectral factorization methods, the presence of such points slows convergence down significantly due to slow decay of the Schur parameters. In our algorithm, we reduce the influence of these points by suitably placing a zero of η_0 in their vicinity. Continuing on to Step 1, for each r (including $r = 0$) we have that $\int_{-\pi}^{\pi}\left(W_r(e^{i\theta}) - W(e^{i\theta})\right)d\theta$=0. If $W_r - W$ is not identically zero (for which the algorithm then terminates), it can be shown, using the mean value theorem of calculus, that $\exists\theta$ such that $W_r(e^{i\theta}) - W(e^{i\theta}) > 0$. Since a zero of W_r can decrease the magnitude of W_r in certain regions of $\mathbb{T}$, the main idea now is to try to reduce the excess (or overshoot) of W_r over W at a point θ_r for which the excess is largest or almost largest. If W is not symmetric or $\theta_r \in \{0, \pi\}$ then we place a zero at $z_r = \mathrm{R}e^{i\theta_r}$ (with $0 < \mathrm{R} < 1$ so that $z_r \in \mathbb{D}$) such that $\frac{W_{r-1}(z)|z - z_r|^2}{W(z)}\Big|_{z=e^{i\theta_r}} = \frac{W_{r-1}(e^{i\theta_r})}{W(e^{i\theta_r})}(1-\mathrm{R})^2$=1. From the last equality we obtain the required value of R for Step 1. In case W is symmetric and $\theta_r \notin \{0, \pi\}$, we must place two zeros at z_r and z_r^* to ensure W_r is also symmetric. By a procedure similar to the symmetric case, we find that a quartic equation $|1 - R|^2|1 - Re^{-i2\theta_r}|^2 - \left(\frac{W_{r-1}(e^{i\theta_r})}{W(e^{i\theta_r})}\right)$=0 must be solved for R and a real solution satisfying $0 < R < 1$ is chosen. It is easy to see, since $\frac{W_{r-1}(e^{i\theta_r})}{W(e^{i\theta_r})} < 1$, that the quartic solution always has such a solution.

Although the algorithm is intuitively appealing, at this stage we do not have a theoretical guarantee of its convergence; this will be studied in the future. A practical application of the algorithm is given in the following example.

Example 17: Consider the non-coercive spectral density $W(e^{i\theta})=\frac{2+\cos\theta-2\cos 2\theta}{24.1-18.9\cos\theta+2\cos 2\theta}$ which has a zero at $z=-1$. The exact CSF of W is given by $\Phi(W)(z)=-\sqrt{10}\frac{(z-2)(z+1)}{(z-4)(z-5)}$. Applying the algorithm by setting $\epsilon = 10^{-4}$ gives $\eta_0 = z+1$ and results in convergence curves as shown in Fig. 1. The algorithm returns a real approximate CSF of degree 10 and the final value of e is 4.1266×10^{-5}. In this case, the algorithm terminates after a few iterations.

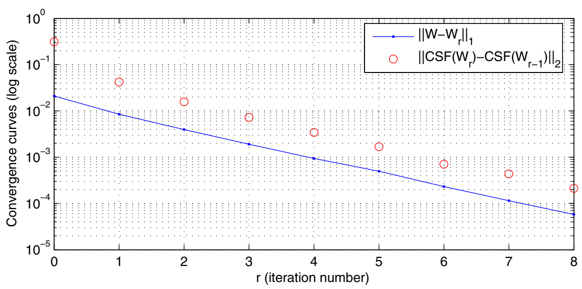

Fig. 1. Convergence curves

In case W has thin and sharp "spectral line"-like peaks then the algorithm will perform poorly. This is because such a peak indicates a (non-cancelling) pole and zero close to each other and to the unit circle, while the zero is not included in η_0. However, it is possible to remedy the situation. Let H be a scalar notch filter with narrow stop bands around frequencies corresponding to the peaks. Letting $P = H_* H$, we apply the algorithm to $W' = WP$ to obtain an approximate CSF, say A. Then $\Phi(W) \approx \frac{1}{\Phi(P)} A$.

VII. Conclusions and Further Research

In this paper we have made three primary contributions. First and foremost, we have derived a set of sufficient, easy to verify conditions for uniform log-integrability of a sequence of matrix-valued spectral densities and the convergence of the canonical spectral factors of the sequence to the canonical spectral factor of the limiting spectral density. Secondly, we state some theoretical results on the existence of approximating sequences of rational spectral densities. Finally, we proposed a new algorithm for spectral factorization of scalar analytic matrix-valued spectral densities based on a heuristic estimation of spectral zeros. The performance of the new algorithm is illustrated in a numerical example. An important subject for future research is convergence analysis of the algorithm. It would also be interesting to investigate how to relax the assumption of continuity of W and uniform convergence of W_r to W in some of the results of Section VI. Development of improved heuristics for estimating the spectral zeros could also be another theme of future research.

The results and algorithm of this paper may find application in areas of science and engineering in which spectral factorization plays a prominent role, or in which signals with non-rational power spectra is a central theme (such as control of aircraft subject to windgust, adaptive optics, and laser scintillation [1] to name a few), or for computing approximate solutions of algebraic Riccati equations (AREs) in optimal control of linear, distributed parameter systems.

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