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Decomposition of a Polynomial as a Sum-of-Squares of Polynomials and the S-Procedure

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Abstract— **This paper investigates links between the problem of determining a decomposition of a polynomial as a sum-of-squares of polynomials and the S-Procedure. We first show that the S-Procedure can be used to check whether a given polynomial is non-negative. Then, using mostly linear algebra arguments, we show that this non-negativity test leads to an affirmative answer if and only if such polynomial admits a decomposition as a sum-of-squares of polynomials.**

I. INTRODUCTION

Determining whether a real valued polynomial $p(x) : \mathbb{R}^n \to \mathbb{R}$ is non-negative for all values of $x \in R^n$ is, in general, an NP-hard problem. A sufficient condition for non-negativeness of a polynomial is the existence of a decomposition as a Sum-Of-Squares (SOS) of polynomials. When such decomposition exists, one says that $p(x)$ is SOS. The problem of checking whether $p(x)$ is SOS can be converted into a convex optimization problem, more specifically, into a semi-definite program [1], [2]. This observation has recently motivated several successful applications in the context of systems and control [3], [4], [5], [6], [7], [8], [9].

A technique known as the S-Procedure has found widespread use in systems and control for more than three decades [10], [11] (see also [12]). The S-Procedure provides sufficient conditions for the positivity of a set of quadratic forms, i.e., a set of polynomials forms of degree two, also in the form of a semi-definite program.

The result of this paper provides a link between SOS polynomials and the S-Procedure.

It is well known that the S-Procedure can be interpreted, in some cases, as a particular decomposition of a polynomial as a sum-of-squares of polynomials. Indeed, SOS of polynomials provide a nice and clean generalization of the S-Procedure for polynomials and polynomial forms of degree higher than two. It is also known that the S-Procedure will typically lead to results that can be conservative when used with sets of more than two quadratic forms.

In this paper we pose and answer the converse question: can the decomposition of a polynomial as a sum-of-squares of polynomials be interpreted as a particular case of the S-Procedure? Surprisingly, the answer is *yes*.

To obtain this result we first shown how the S-Procedure can be used to provide a sufficient condition for non-negativity of a given polynomial. We use a version of the S-Procedure where some quadratic forms are required to be identically null, and not only non-negative, as usual. We then use linear algebra to prove that a certain polynomial $p(x)$ is non-negative according to the S-Procedure if and only if $p(x)$ is SOS.

In addition, the S-Procedure also suggests some extra terms that could be incorporated in the positivity test while still preserving the underlying optimization problem in the form of a semi-definite program. We show that these refinements can provide no improvement with respect to a standard SOS decomposition.

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II. PRELIMINARIES

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

Definition 1 (Monomial): A *monomial* x^α is a product in the form $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The degree of the monomial x^α is $\alpha_1 + \cdots + \alpha_n$.

Definition 2 (Polynomial): The real valued function $p(x) : \mathbb{R}^n \to \mathbb{R}$ is a *polynomial* when it is a finite linear combination of monomials, i. e., $p(x) = \sum_\alpha b_\alpha x^\alpha$. The degree of the polynomial $p(x)$ is the largest degree of its monomials.

Definition 3 (Form): The polynomial $p(x)$ is a *form* of degree m if all its monomials have degree m.

Definition 4 (Polynomial vector): The vector of real valued function $r(x) : \mathbb{R}^n \to \mathbb{R}^m$ is a real polynomial vector when all its entries are real polynomial functions. The degree of the vector $r(x)$ is the largest degree of its polynomial entries.

Definition 5 (Linear Space): The set of polynomials (forms) of degree m is associated with a *linear vector space* of dimension $p_m = \binom{n+m}{m}$ ($\binom{n+m-1}{m}$).

Definition 6 (Basis vector): A polynomial vector $s_m(x) \in \mathbb{R}^{p_m}$ of degree m is a *basis vector* if it spans the entire set of polynomials (forms) of degree m and there exists no vector $b \in \mathbb{R}^{p_m}$, $b \neq 0$, such that $b^T s_m(x) = 0$ for all $x \in \mathbb{R}^n$.

Definition 7 (Representation): Given a basis vector $s_m(x)$, any real polynomial (form) $p(x)$ of degree m have a unique *representation* $p(x) = b^T s_m(x)$.

The operator $\mathrm{vec}(X) : \mathbb{R}^{p\times p} \to \mathbb{R}^{p(p+1)/2}$ take the entries of a given symmetric matrix into a vector. For a symmetric matrix $X \in \mathbb{R}^{p\times p}$ and a vector $y \in \mathbb{R}^p$, the operator $\mathrm{vec}(yy^T)$ is such that $y^T X y = \mathrm{vec}(X)^T \mathrm{vec}(yy^T)$.

A. The S-Procedure

The S-Procedure [10], [11], in its most usual form, is a technique that provides sufficient conditions for which

$$q_0(x) \geq 0, \quad \forall x \in \mathbb{R}^p, \quad q_j(x) \geq 0, \quad j = 1, \ldots, l, \quad (1)$$

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where $q_j(x) : \mathbb{R}^p \to \mathbb{R}$, $j = 0, \ldots, l$, are quadratic functions, i. e., polynomials of degree two. In this paper we consider the slightly more general problem of determining sufficient conditions for

$$q_0(x) \geq 0, \quad \forall x \in \mathbb{R}^p, \quad q_i(x) = 0, \quad i = 1, \ldots, r, \\ q_j(x) \geq 0, \quad j = 1, \ldots, l. \quad (2)$$

Lemma 1 (S-Procedure): Let $q_0(x)$, $q_i(x)$, $i = 1, \ldots, r$, and $q_j(x)$, $j = 1, \ldots, l$, be quadratic functions of $x \in \mathbb{R}^p$. If there exist $\tau_i \in \mathbb{R}$, $i = 1, \ldots, r$, and $\gamma_j \in \mathbb{R}$, $j = 1, \ldots, l$, such that

$$q_0(x) - \sum_{i=1}^{r} \tau_i q_i(x) - \sum_{j=1}^{l} \gamma_j q_j(x) \geq 0, \quad \forall x \in \mathbb{R}^p, \\ \gamma_j \geq 0, \quad j = 1, \ldots, l. \quad (3)$$

then $q_0(x) \geq 0$ for all $x \in \mathbb{R}^p$ such that $q_i(x) = 0$, $i = 1, \ldots, r$, and $q_j(x) \geq 0$, $j = 1, \ldots, l$.

If $q(x)$ is a *quadratic form*, then $q(x)$ admits a representation as $q(x) = x^T Q x$ for some symmetric matrix $Q \in \mathbb{R}^{p \times p}$. Therefore, when $q_0(x)$, $q_i(x)$, $i = 1, \ldots, r$, and $q_j(x)$, $j = 1, \ldots, l$, are quadratic forms, condition (3) is equivalent to the semi-definite constraint

$$Q_0 - \sum_{i=1}^{r} \tau_i Q_i - \sum_{j=1}^{l} \gamma_j Q_j \succeq 0, \\ \gamma_j \geq 0, \quad j = 1, \ldots, l. \quad (4)$$

The existence of a solution to the above inequality relative to a certain specified precision $\epsilon > 0$ can be verified in polynomial time by solving a convex semi-definite program [13].

B. Sum-of-Squares of Polynomials

A sufficient condition for a polynomial $p(x)$ of degree $2m$ to be non-negative is the existence of a decomposition as a sum-of-squares of polynomials. That is, $p(x)$ is non-negative if there exists polynomials $q_i(x)$ of degree m such that

$$p(x) = \sum_k q_k(x)^2. \quad (5)$$

Representing

$$p(x) = b^T s_{2m}(x), \quad (6)$$
$$q_k(x) = q_k^T s_m(x), \qquad \forall k, \quad (7)$$

where $s_{2m}(x) \in \mathbb{R}^{p_{2m}}$ and $s_m(x) \in \mathbb{R}^{p_m}$ are basis vectors for the space of all polynomials of degree $2m$ and m, respectively, we can rewrite (5) in the form

$$b^T s_{2m}(x) = p(x) = \sum_k q_k(x)^2 \\ = \sum_k s_m(x)^T q_k q_k^T s_m(x) = s_m(x)^T Q s_m(x), \quad (8)$$

where $Q := \sum_k q_k q_k^T$. It is clear that $Q \in \mathbb{R}^{p_m \times p_m}$ must be a symmetric positive semi-definite matrix. The following lemma characterizes the existence of SOS decompositions [1], [2].

Lemma 2 (SOS): The polynomial $p(x)$ of degree $2m$ admits a decomposition as a sum-of-squares of polynomials, in which case we say $p(x)$ is SOS, if and only if there exists a positive semi-definite matrix $Q \in \mathbb{R}^{p_m \times p_m}$ such that $b^T s_{2m}(x) = s_m(x)^T Q s_m(x)$ for all $x \in \mathbb{R}^n$, where $s_{2m}(x) \in \mathbb{R}^{p_{2m}}$ and $s_m(x) \in \mathbb{R}^{p_m}$ are basis vectors for the space of all polynomials of degree $2m$ and m, respectively.

The condition that (8) must hold for any $x \in \mathbb{R}^n$ can be converted into a linear constraint on Q, which means that the verification of the condition in Lemma 2 can be done by solving a semi-definite program. Indeed, consider the polynomial vector $\text{vec}(s_m(x) s_m(x)^T)$ of degree $2m$. It certainly admits a representation in the form

$$\text{vec}(s_m(x) s_m(x)^T) = A s_{2m}(x), \quad (9)$$

for some constant matrix $A \in \mathbb{R}^{p_m(p_m+1)/2 \times p_{2m}}$ (more on that later). Therefore, equation (8) can be rewritten as

$$s_m(x)^T Q s_m(x) = \text{vec}(Q)^T \text{vec}(s_m(x) s_m(x)^T) \\ = \text{vec}(Q)^T A s_{2m}(x) = b^T s_{2m}(x), \quad (10)$$

from which it is clear that $p(x)$ is SOS whenever the next semi-definite problem has some solution.

Problem 1: Find $Q \succeq 0$ such that $A^T \text{vec}(Q) = b$.

C. Illustrative Example

Consider $n = 1$ and $m = 2$ and the polynomial

$$p(x) = 5 + 2x - 3x^2 + x^4$$

of degree $2m$. Consider the monomial basis

$$s_m(x) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix}^T, \\ s_{2m}(x) = \begin{pmatrix} 1 & x & x^2 & x^3 & x^4 \end{pmatrix}^T,$$

Compute

$$\text{vec}(s_m(x) s_m(x)^T) = \begin{pmatrix} 1 & x^2 & x^4 & 2x & 2x^2 & 2x^3 \end{pmatrix}^T,$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}, \qquad b = \begin{pmatrix} 5 \\ 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 1 reads as: find $Q \in \mathbb{R}^{3 \times 3}$ such that $Q \succeq 0$, and $A^T \text{vec}(Q) = b$. For this example, one can verify that

$$Q = \begin{bmatrix} 5 & 1 & -9/5 \\ 1 & 3/5 & 0 \\ -9/5 & 0 & 1 \end{bmatrix},$$

is a possible solution to Problem 1.

III. Auxiliary Linear Algebra Results

In this section we derive some auxiliary linear algebra results that will be used later in this paper.

Lemma 3: Matrix $A \in \mathbb{R}^{s\times t}$, where $s = p_m(p_m+1)/2$, $t = p_{2m}$, which has been implicitly defined by (9), has a non-empty left null-space and is full column-rank for any integer n, and m, except $n = m = 1$.

Proof: First note that the matrix A is such that

$$\frac{1}{2}p_m(p_m+1) = \frac{1}{2}\left[\binom{n+m}{m}^2 + \binom{n+m}{m}\right] > \binom{n+2m}{2m} = p_{2m}, \quad \forall n \geq 1, \quad m \geq 1, \quad (11)$$

except for $n = m = 1$, when the left and right sides are both equal to three. This implies that A is a *tall* matrix, therefore, always has a non-empty left null-space.

Assume now that A is not full column-rank. In this case, there exists $v \in \mathbb{R}^{p_{2m}}$ such that $Av = 0$ with $v \neq 0$. Now pick an appropriate matrix $V \in \mathbb{R}^{p_{2m}\times(p_{2m}-1)}$ that makes the following square matrix

$$T := \begin{bmatrix} v & V \end{bmatrix} \in \mathbb{R}^{p_{2m}\times p_{2m}}$$

non-singular. Clearly, the polynomial vector

$$\bar{s}_{2m}(x) := T^{-1}s_{2m}(x)$$

is also a basis.

Now consider the polynomial $u(x) = e_1^T\bar{s}_{2m}(x) \neq 0$, where $e_1 \in \mathbb{R}^{p_{2m}}$ is the vector $e_1 := (1, 0, \ldots, 0)^T$. Recall that $\mathrm{vec}(s_m(x)s_m(x)^T)$ spans the entire space of polynomials of degree $2m$ (for instance, if $s_m(x)$ is a basis of monomials, this conclusion is immediate), to conclude that there exists $c \in \mathbb{R}^{p_{2m}}$ such that

$$c^T \mathrm{vec}(s_m(x)s_m(x)^T) = u(x) = e_1^T\bar{s}_{2m}(x)$$

Now observe that

$$c^T \mathrm{vec}(s_m(x)s_m(x)^T) = c^TAs_{2m}(x) = c^TAT\bar{s}_{2m}(x) = \begin{pmatrix} 0 \\ V^TA^Tc \end{pmatrix}^T \bar{s}_{2m}(x).$$

This establishes a contradiction since

$$e_1 \neq \begin{pmatrix} 0 \\ V^TA^Tc \end{pmatrix},$$

which implies that $u(x)$ has two distinct realizations on the basis $\bar{s}_{2m}(x)$. Therefore, A must be full column-rank. ■

The next corollary follows immediately from the previous result.

Corollary 1: There exists a full row-rank matrix $C \in \mathbb{R}^{r\times s}$, where $r = p_m(p_m+1)/2 - p_{2m}$, $s = p_m(p_m+1)/2$, such that $CA = 0$.

IV. Testing when Polynomials are Non-Negative by the S-Procedure

We now return to the original question of non-negativeness of the polynomial $p(x)$ of degree $2m$. We want to use the S-Procedure, Lemma 1, so our first task is to write a set of quadratic relations in the form (2). That is, to restate the original problem as a test of whether

$$p(x) = s_m(x)^TQs_m(x) \geq 0, \quad \forall x \in \mathbb{R}^n. \quad (12)$$

where $p(x)$ is a given polynomial of degree $2m$. Note that we do not require that Q be necessarily positive semi-definite!

It is straightforward to find *some* Q such that $p(x) = s_m(x)^TQs_m(x)$. Clearly, Q is not unique except for $n = m = 1$ (recall that $\mathrm{vec}(s_m(x)s_m(x)^T)$ is not a basis.) If $Q \succeq 0$ then $p(x)$ is SOS. Indeed, a solution to Problem 1, if one exists, provides such positive semi-definite Q.

In order to use the S-Procedure we define the quadratic form

$$q_0(y) := y^TQ_0y, \qquad y \in \mathbb{R}^{p_m}, \quad (13)$$

where $Q_0 \in \mathbb{R}^{p_m\times p_m}$ is *any* symmetric matrix (not necessarily positive semi-definite) such that

$$q_0(s_m(x)) = p(x), \qquad \forall x \in \mathbb{R}^n. \quad (14)$$

We now look for relationships in the original problem that would provide equalities and inequalities that can be formulated as quadratic functions of y when $y = s_m(x)$.

We first consider the equalities. We use Corollary 1 and multiply both sides of the relation (9) by a full row-rank constant matrix C such that $CA = 0$ on the left to obtain

$$C\,\mathrm{vec}(s_m(x)s_m(x)^T) = 0, \qquad \forall x \in \mathbb{R}^n. \quad (15)$$

Then we compute $r = p_m(p_m+1)/2 - p_{2m}$ constant matrices $Q_i \in \mathbb{R}^{p_m\times p_m}$ such that

$$\mathrm{vec}(Q_i) = C_i^T, \quad i = 1, \ldots, r, \quad (16)$$

where C_i denotes the ith row of matrix C. For all matrices Q_i so constructed we define the quadratic forms

$$q_i(y) := y^TQ_iy, \quad i = 1, \ldots, r. \quad (17)$$

It follows that

$$q_i(s_m(x)) = 0, \qquad \forall x \in \mathbb{R}^n, \quad i = 1, \ldots, r. \quad (18)$$

Now for the inequalities, we observe that some entries of $\mathrm{vec}(s_m(x)s_m(x)^T)$ are non-negative for all values of $x \in \mathbb{R}^n$. For example, when $s_m(x)$ is a basis of monomials, these non-negative terms include all diagonals and some off-diagonal terms of the matrix $s_m(x)s_m(x)^T$, the ones which contain only monomials x^α where all exponents α_i, $i = 1, \ldots, n$, are even. We represent these inequality constraints generically as

$$D\,\mathrm{vec}(s_m(x)s_m(x)^T) \geq 0, \qquad \forall x \in \mathbb{R}^n, \quad (19)$$

for some constant matrix $D \in \mathbb{R}^{l\times p_m(p_m+1)/2}$. We will provide more details on the structure of D in the next sections. As before, we compute l matrices $Q_j \in \mathbb{R}^{p_m \times p_m}$ such that

$$\text{vec}(Q_j) = D_j^T, \quad j = 1, \ldots, l, \tag{20}$$

where D_j denotes the jth row of matrix D. We then define

$$q_j(y) := y^T Q_j y, \quad j = 1, \ldots, l, \tag{21}$$

such that

$$q_j(s_m(x)) \geq 0, \qquad \forall\, x \in \mathbb{R}^n, \quad j = 1, \ldots, l. \tag{22}$$

The following lemma provides a sufficient condition for non-negativeness of polynomials.

Lemma 4: Let $q_0(y)$, $q_i(y)$, $i = 1, \ldots, r$, and $q_j(y)$, $j = 1, \ldots, l$, be quadratic functions constructed to satisfy (14), (18) and (22). If there exist $\tau_i \in \mathbb{R}$, $i = 1, \ldots, r$, and $\gamma_j \in \mathbb{R}$, $j = 1, \ldots, l$ such that

$$q_0(y) - \sum_{i=1}^{r} \tau_i q_i(y) - \sum_{i=1}^{l} \gamma_j q_j(y) \geq 0, \quad \forall y \in \mathbb{R}^{p_m},$$
$$\gamma_j \geq 0, \quad i = 1, \ldots, l, \tag{23}$$

then the polynomial $p(x)$ of degree $2m$ is non-negative for all $x \in \mathbb{R}^n$.

Condition (23) can be verified by solving the following semi-definite program.

Problem 2: Find τ_i, $i = 1, \ldots, r$, and γ_j, $j = 1, \ldots, l$ such that $Q_0 - \sum_{i=1}^{r} \tau_i Q_i - \sum_{i=1}^{l} \gamma_j Q_j \succeq 0$, and $\gamma_j \geq 0$, $j = 1, \ldots, l$.

Illustrative examples can be found in the next sections.

V. Putting the Pieces Together

In this section we establish a link between Problems 1 and 2, that is, the relationship between polynomials that admit an SOS decomposition, as in Lemma 2, and polynomials that are non-negative according to Lemma 4.

A. Positivity Without Inequalities

The next theorem is the key result of this section.

Theorem 1: Let a polynomial $p(x)$ of degree $2m$, vector b and matrices A, Q_0, Q_j, $j = 1, \ldots, r$, constructed to satisfy (6), (9), (14) and (18), respectively, be given. The following statements are equivalent:

a) $p(x)$ is SOS.

b) There exists $Q \in \mathbb{R}^{p_m \times p_m}$ such that

$$Q \succeq 0, \quad A^T \text{vec}(Q) = b. \tag{24}$$

c) There exists τ_i, $i = 1, \ldots, r$, such that

$$Q_0 - \sum_{i=1}^{r} \tau_i Q_i \succeq 0. \tag{25}$$

Furthermore, if the above conditions are satisfied, then conditions (24) are satisfied by $Q = Q_0 - \sum_{i=1}^{r} \tau_i Q_i$.

Proof: The equivalence of a) and b) is given by Lemma 2. We now prove the equivalence of b) and c). Use Lemma 3 and Corollary 1 to show that all solutions to the linear equation $A^T \text{vec}(Q) = b$ are given by

$$\text{vec}(Q) = \text{vec}(Q_0) + C^T z,$$

where $z \in \mathbb{R}^r$, $r = p_m(p_m+1)/2 - p_{2m}$ and $Q_0 \in \mathbb{R}^{p_m \times p_m}$ is any symmetric matrix such that

$$A^T \text{vec}(Q_0) = b.$$

In other words, any vector such that its associated quadratic form satisfies (14). Therefore, since Q_i, $i = 1, \ldots, r$ are defined as in (16), we have immediately the correspondence $z_i = \tau_i$, $i = 1, \ldots, r$, proving that conditions b) and c) are equivalent. ■

Theorem 1 states that the non-negativity test derived using the S-Procedure necessarily returns true when the tested polynomial is SOS. However, the variables γ_j, which are associated with inequalities in the S-Procedure, are, apparently, extra degrees of freedom. These variables are present in Lemma 4 but not in Lemma 2. The impact of these extra inequalities are discussed in the next sections, after the following illustrative example.

B. Illustrative Example

For the same example in Section II-C we have

$$Q_0 = \begin{bmatrix} 5 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \not\succeq 0$$

as one possible choice. Matrix C is given by

$$C = \begin{bmatrix} 0 & -2 & 0 & 0 & 1 & 0 \end{bmatrix},$$

where $r = 1$, and the symmetric matrix associated with the first and only row of C is given by

$$Q_{i=1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

One can verify that for $\tau = 9/5$, $\min_i \lambda_i(Q_0 + \tau Q_1) \approx 0.013 > 0$. Note that the matrix Q that has been given in the Section II-C is exactly $Q = Q_0 - (9/5)Q_1$, as allowed by Theorem 1.

C. Positivity with Diagonal Inequalities

To evaluate the effect of the extra variables γ_j in Lemma 4, we look at the structure of these inequality constraints. Note that all diagonal entries of the matrix $s_m(x)s_m(x)^T$ satisfy inequalities in the form (19). In this case, the associated matrices Q_j will have the particular form

$$Q_j = e_j e_j^T \geq 0, \quad j = 1, \ldots, p_m, \tag{26}$$

where e_j is a vector with all entries equal to zero except for the jth entry, which is equal to one. It is not hard to see that we can therefore set all coefficients $\gamma_j \geq 0$ associated with these inequalities to be equal to zero in Problem 2. In fact, for any $\bar{Q} \succeq 0$, if there exists no τ_i, $i = 1, \ldots, r$, such that inequality (25) is feasible, then there should exist no τ_i, $i = 1, \ldots, r$, and $\gamma > 0$ such that

$$Q_0 - \sum_{i=1}^{r} \tau_i Q_i \succeq \gamma \bar{Q} \succeq 0 \tag{27}$$

either. The conclusion is that the inequalities associated with diagonal entries of the matrix $s_m(x)s_m(x)^T$ can not help to improve the positivity test of Lemma 4.

Nevertheless, matrix $s_m(x)s_m(x)^T$ might have positive entries which are not on the diagonal. This is easier to see (and enumerate) in the case of monomial basis, as illustrated by the next example.

D. Illustrative Example

For the same example in Section II-C we have all possible inequalities in the form (19) represented by matrix

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Only the last row of this matrix is not associated with a diagonal entry, which we use to define matrix

$$Q_{j=1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Note that identifying such off-diagonal inequalities may be much more complicated when the bases are not monomial basis. For this same example, the choice of basis

$$s_m(x) = \begin{pmatrix}(1-x) & x & x^2\end{pmatrix}^T,$$

provides

$$\mathrm{vec}(s_m(x)s_m(x)^T) = \begin{pmatrix} 1-2x+x^2 \\ x^2 \\ x^4 \\ 2(x-x^2) \\ 2x(x-x^2) \\ 2x^3 \end{pmatrix}.$$

The only positive constraint associated with an off-diagonal term is given by the choice of

$$D = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

E. Positivity with Off-Diagonal Inequalities

Assume, without loss of generality, that $s_m(x)$ is a monomial basis. All monomial x^α for which the exponents α_i, $i = 1, \ldots, n$, are even must necessarily appear on the diagonal of the symmetric matrix $S_m(x) := s_m(x)s_m(x)^T$. Suppose that for some exponent $\bar{\alpha}$ with even entries, the monomial $x^{\bar{\alpha}}$ appears on some diagonal entry of matrix $S_m(x)$, say $S_m(x)_{(t,t)}$, and on the off-diagonal entries $S_m(x)_{(u,v)}$ and $S_m(x)_{(v,u)}$. We have already seen that the matrix

$$Q_t := e_t e_t^T \succeq 0 \tag{28}$$

is positive semi-definite and, therefore, cannot help improving the result of Theorem 1. However, the matrix

$$\begin{aligned} Q_{u,v} &:= e_u e_v^T + e_v e_u^T \\ &= \frac{1}{2}(e_u+e_v)(e_u+e_v)^T - \frac{1}{2}(e_u-e_v)(e_u-e_v)^T \end{aligned} \tag{29}$$

is not positive semi-definite, and there is a chance that it might help. Unfortunately, that is not the case, as we show below.

Denote by c_t and $c_{u,v}$ vectors in $\mathbb{R}^{p_m(p_m+1)/2}$ such that

$$\mathrm{vec}(Q_t) = c_t, \qquad \mathrm{vec}(Q_{u,v}) = c_{u,v}. \tag{30}$$

Clearly

$$(2c_t - c_{u,v})^T \,\mathrm{vec}(s_m(x)s_m(x)^T) = 0, \tag{31}$$

In other words, $2c_t - c_{u,v}$ is necessarily a linear combination of the rows of matrix C, given in Corollary 1. That is, there exists a vector $\beta \in \mathbb{R}^{p_m(p_m+1)/2 - p_{2m}}$ such that

$$2c_t - c_{u,v} = C^T\beta. \tag{32}$$

This equation can be rewritten as

$$Q_{u,v} = 2Q_t - \sum_{i=1}^{r} \beta_i Q_i. \tag{33}$$

We use the above expression to analyze the inequality

$$Q_0 - \sum_{i=1}^{r} \tau_i Q_i \succeq \gamma Q_{u,v} = 2\gamma Q_t - \sum_{i=1}^{r} \gamma\beta_i Q_i. \tag{34}$$

That is,

$$Q_0 - \sum_{i=1}^{r} (\tau_i - \gamma\beta_i) Q_i \succeq 2\gamma Q_t. \tag{35}$$

Using the arguments of the previous section, it becomes clear that the above inequality has a feasible solution if and only if there exist $\tilde{\tau}_i := \tau_i - \gamma\beta_i$, $i = 1, \ldots, r$, such that

$$Q_0 - \sum_{i=1}^{r} \tilde{\tau}_i Q_i \succeq 0. \tag{36}$$

This inequality, however, is the same as condition c) of Theorem 1.

Since this analysis can be repeated for all inequalities in the form (22), the conclusion is that Theorem 1 is the best result for testing polynomial non-negativity that can be achieved by Lemma 4.

F. Illustrative Example

For the same example in Section II-C we have that

$$Q_{u,v} = Q_{j=1}, \qquad Q_t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$Q_{j=1} = 2Q_t + Q_{i=1}.$$

Indeed

$$\begin{aligned} &\tau_i Q_{i=1} + \gamma Q_{j=1} \\ &\quad = \tau \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &\quad = (\tau+\gamma) \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} + 2\gamma \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad = -(\tau+\gamma) Q_{i=1} + 2\gamma Q_t. \end{aligned}$$

VI. DISCUSSION

Theorem 1 establishes a close link between the S-Procedure and SOS polynomials. Item c) of Theorem 1 corresponds exactly to what was called "implicit representation" in [2]. As discussed in this reference, solving the semi-definite program associated with this implicit representation is more efficient when n and m are large. Our results provides an interesting interpretation to the implicit representation along with a constructive way to build the kernel matrices Q_i, $i = 0, \ldots, r$. Indeed, it is easier to build such matrices than to build the much larger matrix A, associated with the "explicit representation". For monomial basis, matrices A and Q_i, $i = 0, \ldots, r$, can be built "by inspection". It is also interesting that the kernel matrices Q_i, $i = 0, \ldots, r$, obtained in such way are sparse matrices, such as A. This is in contrast with kernel matrices obtained after factoring matrix A, which can lead to dense kernel matrices.

After having proved Theorem 1, one can interpret the fact that all $\gamma_j \geq 0$, $j = 1, \ldots, l$, are of no use in Lemma 4 as a consequence of the fact that the polynomials $q_j(s_m(x))$ are sums-of-squares. As so, then each $q_j(s_m(x))$, must admit a representation in the form $q_j(s_m(x)) = s_m(x)^T Q_j s_m(x)$, where $Q_j \succeq 0$. The arguments in Sections V-E and V-E simply provide constructions for such representations. One could then think of a better use of Lemma 4, where the quadratic forms $q_j(y)$, $j = 1, \ldots, l$, are non-negative polynomials which are not SOS. For instance, to test non-negativity of forms of degree $2m$ on $n = m$ variables one could think of setting $q_j(s_m(x))$, $j = 1, \ldots, l$, to be all possible variations of the Motzkin polynomial [14].

Another use for the results of this paper is in trying to extrapolate the many years of experience with the S-Procedure to the SOS decomposition problem. It is common sense that the S-Procedure is a useful tool, yet it typically produces conservative results in many applications. Therefore, one could say that we should expect that testing non-negativity of polynomials with the S-Procedure or, equivalently, using SOS decompositions, should also inherit some of this "conservativeness", despite the many successful applications reported for instance in [1], [2], [3]. This qualitative statement is in agreement with the recently reported results of [15], which shows, quantitatively, that if one considers families of polynomials with some fixed degree m greater than two, there are significantly more non-negative polynomials than sums of squares as the number of variables n grows.

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