Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On Stability and L_2-gain for Switched Systems

Jun Zhao and David J. Hill

***Abstract*— This paper addresses the issues of stability and L_2-gain analysis for switched systems via multiple Lyapunov function methods. The proposed necessary and sufficient conditions enable derivation of improved stability tests, an L_2-gain characterization and a design method for state-dependent stabilizing switching laws.**

I. Introduction

In the last decade, considerable attention has been paid to switched systems. The main concern is the issue of stability [3], [10], [17], [20]. This issue is very difficult to deal with due to the hybrid nature of switched systems operation. A common *Lyapunov* function for all subsystems ensures asymptotic stablity under arbitrary switching laws [10]. Most switched systems in practice, however, do not possess a common *Lyapunov* function, yet they still may be asymptotically stable under some properly chosen switching law. A typical choice is given by the convex combination method [10]. The single Lyapunov function technique is obviously very restrictive. The multiple Lypunov function technique proposed by Peleties and DeCarlo [15] and later generalized by Branicky [1] and Hou, Michel and Ye [20] has proven to be a powerful and effective tool for finding such a switching law. The key point of these conditions is the nonincreasing requirement on any Lyapunov function over the "switched on" time sequence of the corresponding subsystem. This is usually difficult to satisfy and hard to check in general. In fact, in order to apply the multiple Lyapunov function methods, connecting adjacent Lyapunov functions at switching points is a commonly accepted strategy. This can be typically achieved by choosing the switching law according to the "min–switching" strategy of all Lyapunov functions [10].

On the other hand, L_2-gain analysis has been rarely addressed for switched systems. As an open problem, the L_2-gain that all linear subsystems share was put forward in [7]. A weighted L_2-gain was considered in [22] by using the dwell time concept. In this result, multiple Lyapunov functions are used without necessarily being connected to each other at switching points. In other words, "jumps" of adjacent Lyapunov functions at switching times may occur, but this results in a weaker attenuation property– an exponentially decayed weighted level. In order to have a standard form of attenuation level, which has been commonly accepted in the control area, multiple Lyapunov functions based on the "min–switching" strategy of all Lyapunov functions have been applied [19], [23]. As a result, the "jumps" are completely eliminated. Of course, this elimination of "jumps" is only possible when some strong assumptions are imposed. There seems a gap between maintaining a standard attenuation level and the use of multiple Lyapunov functions that are not necessarily connected at switching times.

This paper studies stability and L_2-gain for switched systems via multiple Lyapunov function methods. We give a necessary and sufficient condition for stability in terms of multiple Lyapunov functions. An algebraic condition and design method of state-dependent stabilizing switching laws are given. L_2-gain analysis and design are then explored.

II. Preliminaries

In this paper, we consider a switched system of the form:

$$\begin{array}{rcl} \dot{x} & = & f_\sigma(x, u_\sigma), \\ y & = & h_\sigma(x), \end{array} \tag{1}$$

where $\sigma : R_+ = [0, \infty) \rightarrow M = \{1, 2, \cdots, m\}$ is the switching signal, $x \in R^n$ is the state, u_i and $h_i(x)$ are the input vector and output vector of the i-th subsystem respectively. Further, $f_i(0, 0) = 0$, $h_i(0) = 0$. The switching signal σ can be characterized by the switching sequence:

$$\Sigma = \{x_0; (i_0, t_0), \cdots, (i_n, t_n), \cdots, | i_n \in M, n \in N\}, \tag{2}$$

in which t_0 is the initial time, x_0 is the initial state and N is the set of nonnegative integers. When $t \in [t_k, t_{k+1})$, $\sigma(t) = i_k$, that is, the i_k-th subsystem is activated. Let x_k denote $x(t_k)$. The solution $x(t)$ of the system (1) is assumed to exist and to be unique.

The switching sequence Σ is assumed to be minimal in the sense that $i_k \neq i_{k+1}$ for any k. For any $j \in M$, let

$$\Sigma \mid j = \{t_{j_1}, t_{j_1+1}, \cdots, t_{j_n}, t_{j_n+1}, \cdots, i_{j_q} = j, q \in N\}$$

be the sequence of switching times when the j-th subsystem is switched on or switched off.

For a given strictly increasing sequence of times $T = \{t_0, t_1, \cdots, t_n, \cdots, \}$, the interval completion $I(T)$ is the set $I(T) = \bigcup_{j \in N} [t_{2j}, t_{2j+1})$. Let $E(T)$ denote the even sequence of T: $E(T) = \{t_0, t_2, t_4, \cdots, \}$. Therefore, $E(\Sigma \mid j) = \{t_{j_1}, t_{j_2}, \cdots, t_{j_n}, \cdots, n \in N\}$ is the "switched on" times of the j-th subsystem.

This work was supported by the Hong Kong Research Grants Council under Project No. CityU 1232/02E, City University of Hong Hong under Project 9380026 and NSF of China under Grant No.60274009. The work was carried out when both authors were at City University of Hong Kong.

J.Zhao is with School of Information Science, Northeastern University Shenyang, 110004, P.R.China. zdongbo@pub.ln.cninfo.net

D.J.Hill is with Department of Information Engineering, Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia. David.Hill@anu.edu.au

0-7803-9568-9/05/$20.00 ©2005 IEEE

III. Stability

We first briefly review multiple Lyapunov function methods.

A function $V \in C^1[R^n, R_+]$ with $V(0) = 0$ is called a Lyapunov-like function (see [1], [15]) for vector field f and the associated trajectory $x(t)$ over a strictly increasing sequence of times T if

(i) $\dot{V}(x(t)) \leq 0$ for all $t \in I(T)$,

(ii) V is monotonically nonincreasing on $E(T)$.

If for each j, V_j is a Lyapunov-like function for the j-th subsystem vector f_j and the associated trajectory over $T = \Sigma \mid j$, then the origin of the system (1) with $u_i \equiv 0$ is stable [1]. If in addition, for any $t_p, t_q \in E(\Sigma \mid j), p < q$,

$$V_{\sigma(t_q)}(x(t_{q+1})) - V_{\sigma(t_p)}(x(t_{p+1})) \leq -\rho \parallel x(t_{p+1}) \parallel^2, \quad (3)$$

holds for some constant $\rho > 0$, then global asymptotic stability follows [15].

The stability result has been generalized to hold for weak Lyapunov-like functions [20]:

A function $V \in C^1[R^n, R_+]$ with $V(0) = 0$ is called a weak Lyapunov-like function for vector field f and the associated trajectory $x(t)$ over a strictly increasing sequence of times T if

(i) there exists a function $\phi \in C(R_+, R_+)$ satisfying $\phi(0) = 0$, such that $V(x(t)) \leq \phi(V(x(t_{2j})))$ for all $t \in (t_{2j}, t_{2j+1})$ and all $j \in N$,

(ii) V is monotonically nonincreasing on $E(T)$.

In all the above-mentioned results using multiple Lyapunov functions, a fundamental common assumption is the nonincreasing condition of V on $E(T)$. This is obviously quite conservative. We will remove this restriction by defining more general weak Lyapunov-like functions.

Definition 3.1. V is called a generalized Lyapunov-like function if the condition (i) in the definition of weak Lyapunov-like functions holds.

In order to measure the change of a generalized Lyapunov-like function, we need the concept of class $\mathcal{GK}$ functions given below.

Definition 3.2. A function $\alpha : R_+ \to R_+$ is called a class $\mathcal{GK}$ function if it is increasing and right continuous at the origin and $\alpha(0) = 0$.

Class $\mathcal{GK}$ functions are generalization of class $\mathcal{K}$ functions. With the help of class $\mathcal{GK}$ functions, stability of switched systems via multiple Lyapunov functions can be characterized by the following trivial proposition.

Proposition 3.3. Consider the system (1) with $u_\sigma \equiv 0$. Suppose there exist continuous positive definite functions $V_i(x), i = 1, 2, \cdots, m$, all defined around the origin and $V_i(0) = 0$, such that $V_{i_k}(x(t_k)) \geq V_{i_k}(x(t))$ for $t \in [t_k, t_{k+1})$. Then, the origin of the system (1) is stable if and only if there exist a class $\mathcal{GK}$ function α satisfying

$$V_{i_k}(x(t_k)) \leq \alpha(\parallel x_0 \parallel), k \geq 0, \quad (4)$$

Though this proposition gives a necessary and sufficient condition for stability, it is almost useless because it can be used neither to test stability nor to guide the switching law design. The following theorem is the main result on stability via generalized Lyapunov-like functions. For simplicity, sometime we use $V_j(t)$ to denote $V_j(x(t))$.

Theorem 3.4. Suppose for each $i \in M$, there exists a generalized Lyapunov-like function $V_i(x)$ with respect to $f_i(x, 0)$ and the associated trajectory. Then,

(i) the origin of the system (1) with $u_\sigma \equiv 0$ is stable if and only if there exist class $\mathcal{GK}$ functions α_j satisfying

$$V_j(t_{j_{k+1}}) - V_j(t_{j_1}) \leq \alpha_j(\parallel x_0 \parallel), k \geq 1, j = 1, \cdots, m, \quad (5)$$

(ii) if all $V_i(x)$ are positive definite around the origin, then the origin of the system (1) is asymptotically stable if and only if (5) holds and there exists j, such that $V_j(t_{j_k}) \to 0$ as $k \to \infty$.

Proof. We first prove (i). Let ϕ_i be given by the generalized Lyapunov-like function $V_i(x)$ with respect to $f_i(x, 0)$ and the associated trajectory, that is,

$$V_{i_k}(x(t)) \leq \phi_{i_k}(V_{i_k}(x_k)), t_k \leq t < t_{k+1}.$$

Sufficiency. For any constants $c, c_1, c_2 > 0$, $c_1 \leq c_2$, let

$$B(c) = \{x| \parallel x \parallel \leq c\},$$

$$r_i(c_1, c_2) = \min_x \{V_i(x)| c_1 \leq \parallel x \parallel \leq c_2\}$$

and $r(c_1, c_2) = \min_i \{r_i(c_1, c_2)\}$. Now, for any $\epsilon > 0$, (5) enables us to choose $\lambda_0 > 0$, $\lambda_0 < \epsilon$, such that

$$\alpha_j(\parallel x_0 \parallel) < \frac{1}{2} r(\epsilon, \epsilon), k \geq 1, j = 1, 2, \cdots, m$$

whenever $x_0 \in B(\lambda_0)$. Since V_i and ϕ_i are continuous at the origin and $V_i(0) = 0, \phi_i(0) = 0$, it is always possible to select $\delta_1 > 0$, $\delta_1 < \lambda_0$, such that

$$V_i(x) + \phi_i(V_i(x)) < \frac{1}{2} r(\epsilon, \epsilon) \quad \text{when} \quad x \in B(\delta_1).$$

Thus,

$$V_i(x) + \phi_i(V_i(x)) + \alpha_i(\parallel x_0 \parallel) < r(\epsilon, \epsilon), \ \forall x, x_0 \in B(\delta_1).$$

For this $\delta_1 > 0$, we use the same procedure to choose $\delta_2 > 0$, $\delta_2 < \delta_1$ such that

$$V_i(x) + \phi_i(V_i(x)) + \alpha_i(\parallel x_0 \parallel) < r(\delta_1, \epsilon)$$

when $x, x_0 \in B(\delta_2)$. Continuing this procedure up to $2m$ steps, we finally have

$$\epsilon = \delta_0 > \delta_1 > \delta_2 > \cdots > \delta_{2m} > 0$$

with the property that for $p = 1, \cdots, 2m - 1$, and $\forall i$

$$\begin{array}{rl} & V_i(x) + \phi_i(V_i(x)) + \alpha_i(\parallel x_0 \parallel) \\ < & r(\delta_p, \delta_{p-1}), \text{if } x, x_0 \in B(\delta_{p+1}), \\ & V_i(x) + \phi_i(V_i(x)) + \alpha_i(\parallel x_0 \parallel) \\ < & r(\delta_0, \delta_0), \text{if } x, x_0 \in B(\delta_1). \end{array} \quad (6)$$

For any $k \geq 0$, let R_k be the number of the different elements of the set $\{i_0, i_1, \cdots, i_k\} \bigcap \{1, 2, \cdots, m\}$, which is actually the total number of different subsystems that have ever been activated for some time on $[t_0, t_{k+1})$. $\{R_k, k = 0, 1, \cdots,\}$ is obviously a nondecreasing sequence bounded by m.

We can know by (6) that if $x_k = x(t_k) \in B(\delta_q)$ for some k and q, then $x(t) \in B(\delta_{q-1}), t \in [t_k, t_{k+1}]$. In fact, generalized Lyapunov-like function V_{i_k} gives that

$$V_{i_k}(x(t)) \le \phi_{i_k}(V_{i_k}(x(t_k))) < r(\delta_{q-1}, \delta_{q-2}).$$

By induction with respect to $k \ge 1$, we can show the following claim:

(a). If $R_k = R_{k-1} + 1$, then

$$\begin{aligned} & V_{i_k}(x_k) + \phi_{i_k}(V_{i_k}(x_k)) + \alpha_{i_k}(\| x_0 \|) \\ < \;\; & r(\delta_{2m-2R_k+2}, \delta_{2m-2R_k+1}), \end{aligned} \tag{7}$$

and $x(t) \in B(\delta_{2m-2R_k+2}), t \in [t_k, t_{k+1}]$.

(b). If $R_k = R_{k-1}$, then

$$\begin{aligned} & V_{i_k}(x_k) + \phi_{i_k}(V_{i_k}(x_k)) + \alpha_{i_k}(\| x_0 \|) \\ < \;\; & r(\delta_{2m-2R_k+1}, \delta_{2m-2R_k}), \end{aligned} \tag{8}$$

and $x(t) \in B(\delta_{2m-2R_k+1}), t \in [t_k, t_{k+1}]$. Therefore, $x(t) \in B(\epsilon)$ for any $t \in [0, \infty)$ if $x_0 \in B(\delta_{2m})$ and thus stability follows.

The necessity follows directly by choosing

$$\alpha_j(s) = \sup_{k \ge 1, \|x_0\| \le s} \left\{0, \;\; V_j(t_{j_{k+1}}) - V_j(t_{j_1})\right\}.$$

The proof of (ii) is straightforward.

Remark 3.5. If we knew $V_j(t_{j_1})$ could be set "small enough" by letting the initial state $x(t_0)$ be close to the origin, Theorem 3.4 would be trivial. However, we have no apriori knowledge that $V_j(t_{j_1})$ can be set "small enough" because the switching law can be arbitrary: time dependent, state dependent, or both, or even determined by a hybrid logic-based controller. Consequently, we have no idea about when and how the j-the subsystem is activated for the first time. The meaning of Theorem 3.4 is that stability is ensured as long as the change of V_j between any "switched on" time and the first activate time is bounded by a class $\mathcal{GK}$ function, regardless of where $V_j(t_{j_1})$ is. This is not true for the general case of non-generalized Lyapunov-like functions. In fact, one can easily construct an example where for some j, $V_j(t_{j_1})$ can be arbitrarily large though x_0 is set arbitrarily close to the origin. Thus, stability is lost.

Remark 3.6. (5) can be equivalently rewritten as

$$\sum_{q=1}^{k} \left(V_j(t_{j_{q+1}}) - V_j(t_{j_q})\right) < \alpha_j(\| x_0 \|). \tag{9}$$

Note that $V_j(t_{j_{q+1}}) - V_j(t_{j_q})$ stands for the change of $V_j(x)$ at the adjacent "switched on" times, (9) means that V_j is allowed to grow on $E(\Sigma \mid j)$ but the total growth should be bounded from above by a class $\mathcal{GK}$ function. As a special case, when the well-known Branicky's "non-increasing" condition $V_j(t_{j_{q+1}}) - V_j(t_{j_q}) \le 0$ holds, (9) is automatically satisfied with $\alpha_j = 0$.

Example 3.7. Consider the switched linear system

$$\dot{x} = A_\sigma x \tag{10}$$

with three subsystems:

$$A_1 = \begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & -3 \\ \frac{1}{3} & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $q_j(t)$ be the total number of times that the j-th subsystem is activated before time t. The switching law is given by:

$$\sigma(t) = \begin{cases} 1, & \text{if } x(t) \text{ is in the first, or second} \\ & \text{or third quadrants,} \\ 2, & \text{if } q_2(t) \le q_3(t) + 1 \\ & \text{and } x(t) \text{ is in the fourth quadrant,} \\ 3, & \text{if } q_2(t) > q_3(t) + 1 \\ & \text{and } x(t) \text{ is in the fourth quadrant.} \end{cases}$$

Choose generalized Lyapunov-like functions as:

$$V_1(x) = \frac{1}{2}x_1^2 + 2x_2^2, V_2(x) = \frac{1}{3}x_1^2 + 3x_2^2, V_3(x) = x_1^2 + x_2^2.$$

A straightforward calculation shows that all conditions of Theorem 3.4 are satisfied and thus asymptotic stability follows. The system trajectory is depicted in Fig.1.

Sometimes, instead of finding a class $\mathcal{GK}$ function, it might be convenient to check uniform convergence of certain series as shown in the following.

Theorem 3.8. Suppose for each $i \in M$, there exists generalized Lyapunov-like function $V_i(x)$ with respect to $f_i(x, 0)$ and the associated trajectory. Then, the origin of the system (1) with $u_\sigma \equiv 0$ is stable if any of the following conditions is satisfied:

(A) the series $\sum_{p=1}^{\infty} \max\left\{0, V_j(t_{j_{p+1}}) - V_j(t_{j_p})\right\}$ convergent uniformly with respect to the initial state x_0 in a neighborhood of the origin;

(B) the series $\sum_{p=1}^{\infty} \left(V_j(t_{j_{p+1}}) - V_j(t_{j_p})\right)$ convergent uniformly with respect to the initial state x_0 in a neighborhood of the origin;

(C) there exists a $\mathcal{GK}$ class function α such that $\sum_{p=0}^{k} \left(V_{i_p+1}(t_{p+1}) - V_{i_p}(t_{p+1})\right) \le \alpha(\| x_0 \|)$, or a little stronger, the series $\sum_{p=0}^{\infty} \left(V_{i_p+1}(t_{p+1}) - V_{i_p}(t_{p+1})\right)$ is convergent uniformly with respect to the initial state x_0 in a neighborhood of the origin;

(D) there exists a class $\mathcal{GK}$ function $\alpha_j(\cdot)$ such that for any $k > 1$

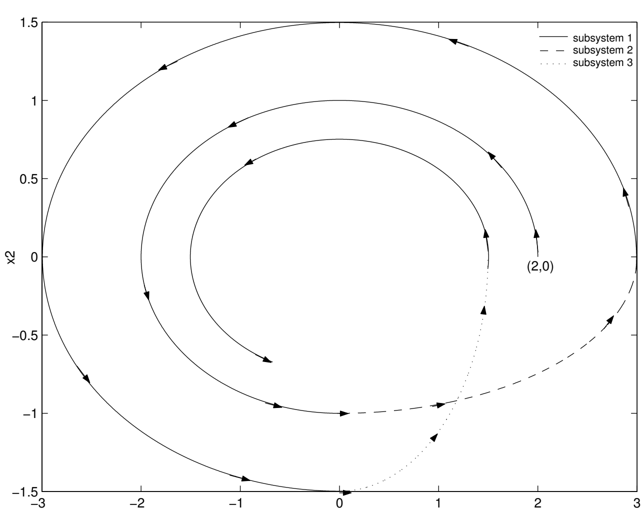

Fig. 1. Trajectory of the switched system (10)

$$V_j(t_{j_{k+1}})-V_j(t_{j_1})=\sum_{p=1}^{k}\big(V_j(t_{j_{p+1}})-V_j(t_{j_p})\big)\leq \alpha_j(\| x_0 \|).$$

Proof. It is easy to prove the implications: (A)⇒(B) ⇒(D) and (C)⇒(D).

Remark 3.9. If V_{i_k} and $V_{i_{k+1}}$ are connected at t_{k+1}, i.e. $V_{i_k}(t_{k+1})=V_{i_{k+1}}(t_{k+1})$, which is suggested by the well known "min–switching" switching law (see, for example, [10])

$$\sigma(t)=arg\ min\{V_i(x(t)), i=1,2,\cdots,m\},$$

the condition (C) is automatically satisfied. The condition (C) gives us considerable freedom in the switching law design, i.e. rather than following the "min–switching" law.

Next, we discuss how to design a switching law to achieve stability with the help of the necessary and sufficient condition given in Theorem 3.4 and 3.8.

Theorem 3.10. Suppose that we have positive definite smooth functions $V_i(x)$ with $V_i(0)=0$, functions $\beta_{ij}(x)\leq 0$, $\mu_{ij}(x), i,j=1,2,\cdots,m$ with $\mu_{ij}(0)=0$ and $\mu_{ii}(x)=0$, such that

$$\begin{aligned}&\frac{\partial V_i}{\partial x}f_i(x,0)+\sum_{j=1}^{m}\beta_{ij}(x)(V_i(x)-V_j(x)+\mu_{ij}(x))\\&\leq 0, i=1,2,\cdots,m\end{aligned} \tag{11}$$

$$\frac{\partial \mu_{ij}}{\partial x}f_i(x,0)\leq 0, i,j=1,2,\cdots,m \tag{12}$$

and

$$\mu_{ij}(x)+\mu_{jk}(x)\leq \min\{0,\mu_{ik}(x)\},\quad \forall i,j,k. \tag{13}$$

Then, there exists a state-dependent switching law under which the origin of the system (1) with $u_\sigma\equiv 0$ is stable. Moreover, if the inequalities in (11) hold strictly for $x\neq 0$, asymptotic stability is assured.

Proof. First of all, for any integers $i_1,i_2,\cdots,i_q\in\{1,2,\cdots,m\}$, it can be easily derived from (13) that

$$\mu_{i_1i_2}(x)+\mu_{i_2i_3}(x)+\cdots+\mu_{i_{q-1}i_q}(x)+\mu_{i_qi_1}(x)\leq 0. \tag{14}$$

Let

$$\Omega_i=\{x\mid V_i(x)-V_j(x)+\mu_{ij}(x)\leq 0, j=1,2,\cdots,m\} \tag{15}$$

and

$$\widetilde{\Omega}_{ij}=\{x\mid V_i(x)-V_j(x)+\mu_{ij}(x)=0\}, j\neq i. \tag{16}$$

Note that $\widetilde{\Omega}_i=\bigcup_{j=1,j\neq i}^{m}\widetilde{\Omega}_{ij}$ contains the boundary of Ω_i. Also, $\bigcup_{i=1}^{m}\Omega_i=R^n$ must hold because otherwise for some $x\in R^n$, we have a sequence $i_1,i_2,\cdots,i_q, i_k\neq i_{k+1}, k=1,2\cdots,q$ and i_{q+1} is considered as i_1, such that

$$V_{i_k}(x)-V_{i_{k+1}}(x)+\mu_{i_ki_{k+1}}(x)>0. \tag{17}$$

Taking sum over k and noticing (14) yields

$$\sum_{k=1}^{q}\big(V_{i_k}(x)-V_{i_{k+1}}(x)+\mu_{i_ki_{k+1}}\big)=\sum_{k=1}^{q}\mu_{i_ki_{k+1}}(x)\leq 0,$$

which contradicts (17).

The sets Ω_i have the property that if $x\in\Omega_i\bigcap\widetilde{\Omega}_{ij}$ for some i,j and $x\in R^n$, , then $x\in\Omega_j$. In fact, $x\in\Omega_i\bigcap\widetilde{\Omega}_{ij}$ means that $V_i(x)-V_k(x)+\mu_{ik}(x)\leq 0$ for any k and $V_i(x)-V_j(x)+\mu_{ij}(x)=0$. Thus, $V_j(x)=V_i(x)+\mu_{ij}(x)$. This in turn gives

$$\begin{aligned}&V_j(x)-V_k(x)+\mu_{jk}(x)\\=\ &V_i(x)-V_k(x)+\mu_{ij}(x)+\mu_{jk}(x)\\\leq\ &V_i(x)-V_k(x)+\mu_{ik}(x)\leq 0,\end{aligned}$$

in which (13) was used to derive the inequality.

Now, we design the switching law as follows.

$$\begin{aligned}&\sigma(t)=i\ \text{ if }\ \sigma(t^-)=i\ \text{ and }\ x(t)\in\ \text{int}\ \Omega_i,\\&\sigma(t)=j\ \text{ if }\ \sigma(t^-)=i\ \text{ and }\ x(t)\in\widetilde{\Omega}_{ij}\ .\end{aligned} \tag{18}$$

Thus, once the trajectory enters Ω_i it will remain in Ω_i until it hits the boundary in $\widetilde{\Omega}_{ij}$ and then enters Ω_j. In other words, switching only takes place on $\widetilde{\Omega}_{ij}$ for some j. Recall that $\beta_{ij}(x)\leq 0$, (11) implies that on Ω_i, it holds that

$$\frac{\partial V_i}{\partial x}f_i(x,0)\leq 0, i=1,2,\cdots,m,$$

which together with (12) tell us that $V_{i_k}(x(t))$ and $\mu_{i_kj}(x(t))$ are decreasing on $[t_k,t_{k+1})$.

For $k\geq 0$, according to the switching law (18), at each switching time we have

$$V_{i_{k+1}}(t_{k+1})-V_{i_k}(t_{k+1})=\mu_{i_ki_{k+1}}(x(t_{k+1})). \tag{19}$$

Thus,

$$\begin{aligned}&V_{i_{k+1}}(t_{k+1})-V_{i_k}(t_{k+1})+V_{i_{k+2}}(t_{k+2})-V_{i_{k+1}}(t_{k+2})\\&=\mu_{i_ki_{k+1}}(x(t_{k+1}))+\mu_{i_{k+1}i_{k+2}}(x(t_{k+2}))\\&\leq\mu_{i_ki_{k+1}}(x(t_{k+1}))+\mu_{i_{k+1}i_{k+2}}(x(t_{k+1}))\leq 0.\end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}&\sum_{p=0}^{k}\big(V_{i_{p+1}}(t_{p+1})-V_{i_p}(t_{p+1})\big)\\\leq\ &\begin{cases}0, & \text{if } k \text{ is odd}\\ \mu_{i_0i_1}(x(t_1))\leq\mu_{i_0i_1}(x_0), & \text{if } k \text{ is even.}\end{cases}\end{aligned} \tag{21}$$

Choose $\alpha(s)=\max_{\|x\|\leq s}\{|\mu_{ij}(x)|, 1\leq i,j\leq m\}$, the result follows immediately from (C) in Theorem 3.8. Moreover, if the inequalities in (11) hold strictly for $x\neq 0$, asymptotic stability follows from the standard argument of Lyapunov theory.

Remark 3.11. As adopted in most existing literature on state dependent switching strategies, the switching law designed here neglects up to a set of measure zero where no switching signal is specified. In particular, no specific index j is chosen when more than one indexes j's satisfy (18). This can be easily fixed, for example, by the method in [10].

Remark 3.12. For the switched linear system

$$\dot{x} = A_\sigma x + B_\sigma u_\sigma, \tag{22}$$

we may look for quadratic functions $V_i(x) = x^T P_i x$, $\mu_{ij}(x) = x^T Q_{ij} x$ and constants β_{ij}. Thus, (11), (12) and (13) become respectively the following matrix inequalities

$$P_i A_i + A_i^T P_i + \sum_{j=1}^{m} \beta_{ij}(P_i - P_j + Q_{ij}) \le 0, i, j = 1, 2, \cdots, m, \tag{23}$$

$$Q_{ij} A_i + A_i^T Q_{ij} \le 0, i, j = 1, 2, \cdots, m \tag{24}$$

$$Q_{ij} + Q_{jk} - Q_{ik} \le 0, \forall i, j, k, \tag{25}$$

$$Q_{ij} + Q_{jk} \le 0, \forall i, j, k. \tag{26}$$

In particular, when $Q_{ij} = 0, \forall i, j$, all (24), (25) and (26) disappear and (23) is the well known result in [10] and the switching law given by (18) degenerates exactly into the "min-switching" strategy: $\sigma(t) = \sigma(x(t)) = \arg\min\{V_i(x(t)), i = 1, 2, \cdots, m\}$.

IV. L_2-Gain

We first give the descriptions of L_2-gain for switched systems.

Definition 4.1. The system (1) has weak L_2-gain γ under the switching law Σ if there exist positive definite continuous functions $V_1(x), V_2(x), \cdots, V_m(x)$ with $V_i(0) = 0$, and class $\mathcal{GK}$ functions α_j such that for $j = 1, 2, \cdots, m, k = 1, 2, \cdots$, and $\forall u_i$ satisfying $\int_{t_0}^{\infty} u^T_{\sigma(t)}(t) u_{\sigma(t)}(t) dt < \infty$, we have

$$\text{(i)} \quad \begin{aligned} & V_{i_k}(x(t)) - V_{i_k}(x(s)) \\ & \le \int_s^t (\gamma^2 \parallel u_{i_k}(\tau) \parallel^2 - \parallel h_{i_k}(\tau) \parallel^2) d\tau, \\ & t_k \le s \le t < t_{k+1}. \end{aligned} \tag{27}$$

(ii) When $u_i = 0$,

$$\begin{array}{rcl} Q_j(x_0) & = & \sum_{k=1}^{p} \left(V_j(x(t_{j_{k+1}})) - V_j(x(t_{j_k+1})) \right) \\ & \le & \alpha_j(\parallel x_0 \parallel), \forall p. \end{array} \tag{28}$$

If, in addition,

$$\int_0^T (\gamma^2 \parallel u_{\sigma(t)}(t) \parallel^2 - \parallel h_{\sigma(t)}(t) \parallel^2) dt \ge 0 \tag{29}$$

holds for any $T > 0$ when $x(0) = 0$, the system (1) is said to have strong L_2-gain γ.

In the case that the switched system (1) has only one subsystem, Condition (ii) is automatically satisfied, and weak L_2-gain and strong L_2-gain merges into the usual L_2-gain.

Remark 4.2. In Definition 4.1, (27) is the usual dissipative inequality with the supply rate function $\gamma^2 \parallel u_{i_k} \parallel^2 - \parallel h_{i_k} \parallel^2$ for the i_k-th subsystem when being activated, and V_{i_k} is the associated storage function. It is worthwhile noticing that though the j-th subsystem is inactivated on the time interval $[t_{j_k+1}, \ t_{j_{k+1}})$, the "energy" $V_j(x)$ changes from $V_j(x(t_{j_k+1}))$ to $V_j(x(t_{j_{k+1}}))$ because all subsystems share the same state variable. Condition (ii) indicates that the total changed "energy" of the j-th subsystem, when inactivated, is uniformly bounded.

Remark 4.3. As a special case, when the "sequence nonincreasing condition" [10], [15] $V_j(x(t_{j_{k+1}})) - V_j(x(t_{j_k+1})) \le 0, j = 1, 2, \cdots, m$, is satisfied, which is commonly used in the switched systems literature, Condition (ii) is automatically satisfied.

Remark 4.4. It is easy to see that the system (1) has strong L_2-gain γ if Condition (i) and (ii) in Definition 4.1 hold and the function

$$V(t) = \left\{ \begin{array}{l} V_{i_0}(x(t)), t \in [t_0, t_1) \\ V_{i_k}(x(t)) - \sum_{j=1}^{k} (V_{i_j}(x(t_j)) - V_{i_{j-1}}(x(t_j))), \\ t \in [t_k, t_{k+1}), k = 1, 2, \cdots \end{array} \right. \tag{30}$$

is nonnegative.

For an affine switched system

$$\begin{array}{rcl} \dot{x} & = & f_\sigma(x) + g_\sigma(x) u_\sigma, \\ y & = & h_\sigma(x), \end{array} \tag{31}$$

and smooth V-functions and state-dependent switching law: $\sigma(x) = i$, when $x \in \Omega_i, \bigcup_{i=1}^{m} \Omega_i = R^n, int\ \Omega_i \bigcap int\ \Omega_j = \emptyset, i \ne j$, weak L_2-gain can be characterized by "local" Hamilton-Jacobi inequalities.

$$\frac{\partial V_i}{\partial x} f_i + \frac{1}{2\gamma^2} \frac{\partial V_i}{\partial x} g_i g_i^T \frac{\partial^T V_i}{\partial x} + \frac{1}{2} h_i^T h_i \le 0, x \in \Omega_i, \tag{32}$$

and (28) with $u_i = 0$.

We now consider how to achieve L_2-gain by design of state-dependent switching laws. The key idea here is adopt the strategy in dealing with stability developed in Theorem 3.10.

Theorem 4.5. Consider the switched system (31). Suppose that we have positive definite functions $V_i(x)$ with $V_i(0) = 0$, functions $\beta_{ij}(x) \le 0$, $\mu_{ij}(x)$ with $\mu_{ij}(0) = 0$ and $\mu_{ii}(x) = 0$, such that

$$\begin{array}{l} \frac{\partial V_i}{\partial x} f_i + \frac{1}{2\gamma^2} \frac{\partial V_i}{\partial x} g_i g_i^T \frac{\partial^T V_i}{\partial x} + \frac{1}{2} h_i^T h_i + \\ \sum_{j=1}^{m} \beta_{ij}(x)(V_i(x) - V_j(x) + \mu_{ij}(x)) \le 0, \\ i, j = 1, 2, \cdots, m \end{array} \tag{33}$$

$$\frac{\partial \mu_{ij}}{\partial x} f_i(x) \le 0, i, j = 1, 2, \cdots, m \tag{34}$$

and

$$\mu_{ij}(x) + \mu_{jk}(x) \le \min\{0, \mu_{ik}(x)\}, \quad \forall i, j, k. \tag{35}$$

Then, under the state-dependent switching law given by (18) the system (31) has weak L_2-gain. If in addition,

$$\frac{\partial \mu_{ij}}{\partial x} g_i = 0, \forall i, j \tag{36}$$

then, system (31) has strong L_2-gain.

Proof. Similar to the proof of Theorem 3.10.

Remark 4.6. For switched linear system

$$\begin{array}{rcl} \dot{x} & = & A_{\sigma(t)}x + B_{\sigma(t)}u_{\sigma(t)} \\ y & = & C_{\sigma(t)}x \end{array} \tag{37}$$

and quadratic Lyapunov functions $V_i(x) = \frac{1}{2}x^T P_i x$ with positive definite matrices P_i, we need to solve the following matrix inequalities

$$\begin{array}{l} P_iA_i + A_i^TP_i + \frac{1}{\gamma^2}P_iB_iB_i^TP_i + C_i^TC_i \\ + \quad \sum_{j=1}^{m}\beta_{ij}(P_i - P_j + Q_{ij}) \le 0, i, j = 1, 2, \cdots, m, \end{array} \tag{38}$$

$$Q_{ij}A_i + A_i^TQ_{ij} \le 0, i, j = 1, 2, \cdots, m \tag{39}$$

$$Q_{ij} + Q_{jk} - Q_{ik} \le 0, \forall i, j, k, \tag{40}$$

$$Q_{ij} + Q_{jk} \le 0, \forall i, j, k. \tag{41}$$

If we want strong L_2-gain, Q_{ij} needs to satisfy $Q_{ij}B_i = 0, \forall i, j$.

As L_2-gain of non-switched systems gives stability, L_2-gain of switched systems is also expected to imply stability. This will be shown in the following.

Theorem 4.7. If the system (1) has weak L_2-gain γ under the switching law Σ, then, the origin of the system (1) with $u_i = 0$ is stable.

Proof. Applying Theorem 3.4 concludes the proof.

In terms of asymptotic stability, many types of conditions can be imposed. Here, we consider some conditions of LaSalle's type. To this end, we need some kind of observability property.

Definition 4.8. A system

$$\begin{array}{l} \dot{x} = f(x), \\ y = h(x) \end{array} \tag{42}$$

is called asymptotically detectable if for any $\epsilon > 0$, there exists $\delta > 0$, such that when $\| y(t+s) \| < \delta$ holds for some $t \ge 0$, $\Delta > 0$ and $0 \le s \le \Delta$, we have $\| x(t) \| < \epsilon$.

Remark 4.9. This asymptotic detectability is a weaker version of small-time norm observability [6].

Theorem 4.10. If the system (1) has weak L_2-gain γ under the switching law Σ and moreover, if $V_i(x), i = 1, 2 \cdots, m$ are globally defined positive definite radially unbounded functions, and there exists j with $\lim_{k\to\infty}(t_{j_k+1} - t_{j_k}) \ne 0$ and the corresponding subsystem is asymptotically detectable, then, the origin of the system (1) with $u_i = 0$ is globally asymptotically stable.

Proof. Similar to [23].

V. Concluding Remarks

We have given a necessary and sufficient condition for stability of switched systems in terms of multiple generalized Lyapunov-like functions. This condition tells us how much the corresponding Lyapunov function is allowed to grow on the "switched on" time sequence without violating stability. Using this condition we do not need worry when and how each subsystem is activated for the first time.

The L_2-gain description and analysis proposed are based on the consideration of change of value of associated V-functions when being inactivated. This change represents a kind of energy exchange from an activated subsystem to an inactivated one. The boundedness requirement of such energy exchange is natural and reasonable in order to maintain stability.

References

[1] M.S.Branicky, "Multiple *Lyapunov* functions and other analysis tools for switched and hybrid systems," IEEE Trans. Automat. Contr., vol.43, pp.475-482, 1998.

[2] F.A.Cuzzola, M.Morari, "An LMI approach for H_∞ analysis and control of discrete-time piecewise affine systems," Int. J. Control, vol.75, pp.1293-1301, 2002.

[3] R.A.Decarlo, M.S.Branicky, S.Pettersson and B.Lennartson, "Perspectives and results on the stability and stabilization of hybrid systems," Proceedings of the IEEE, vol.88, pp.1069-1082, 2000.

[4] W.M.Haddad and V.Chellaboina, "Dissipativity theory and stability of feedback interconnections for hybrid dynamical systems," Mathematical Problems in Engineering, vol.7, pp.299-335, 2001.

[5] J.W.Helton and M.R.James, Extending H_∞ Control to Nonlinear Systems, SIAM, 1999.

[6] J.P.Hespanha, D.Liberzon D.Angeli and E.D.Sontag, Nonlinear Norm Observability Notions and Stability of Switched Systems, IEEE Trans.Automat.Contr., vol.50, 2005, pp.154-168.

[7] J.P.Hespanha and A.S.Morse, "L_2-induced gains of switched linear systems," in Open problems in mathematical systems theory and control, ed. V.D.Blondel, E.D.Sontag, M.Vidyasagar, and J.C.Willems, Springer, 1999, pp.45-47.

[8] J.P.Hespanha, "Root-mean-square gains of switched linear systems," IEEE Trans. Automat. Contr., Vol.48, pp.2040-2045, 2003.

[9] D.J.Hill and P.J.Moylan, "Dissipative dynamical systems: Basic input-output and state properties," J. Franklin Inst., vol.309, pp.327-357, 1980.

[10] D.Liberzon, Switching in Systems and Control, Birkhauser, Boston, 2003.

[11] H.Lin and P.J.Antsaklis, "Disturbance attenuation properties for discrete-time uncertain linear switched systems," in Proc. 42nd IEEE Conf.on Decision and Control, 2003, 5289-5294.

[12] R.Lozano, B.Brogliato, O.Egeland and B.Maschke, Dissipative Systems Analysis and Control, Springer, 2000.

[13] G.Michaletzky and L.Gerencser, "BIBO stability of linear switching systems," IEEE Trans. Automat. Contr., vol.47, pp.1895-1898, 2002.

[14] K.S.Narendra, O.A.Driollet, M.Feiler and K.George, "Adaptive control using multiple models, switching and tuning," Int. J. Adapt. Control Signal Process., vol.17, pp.87-102, 2003.

[15] P.Peleties and R.DeCarlo, "Asymptotic stability of m-switched systems using Lyapunov-like functions," in Proc. American Control Conf., Boston, MA, 1991, pp.1679-1684.

[16] A.Y.Pogromsky, M. Jirstrand and P.Spangeus, "On stability and passivity of a class of hybrid systems," in Proc. 37th IEEE Conf.on Decision and Control, Tempa, Florida, 1998, pp.3705-3710.

[17] Z.Sun and S.S.Ge, Switched Linear Systems – Control and Design, Springer,2004.

[18] A.J.van der Schaft, L_2 Gain and Passivity Techniques in Nonlinear Control, Springer,2000.

[19] D.Xie, L.Wang, F.Hao and G.Xie, "LMI approach to L_2-gain analysis and control synthesis of uncertain switched systems," IEE Proceedings on Contr. Theory Appl., vol.151,pp.21-28,2004.

[20] H.Ye, A.N.Michel and L.Hou, "Stability theory for hybrid dynamical systems," IEEE Trans. Automat. Contr., Vol.43, pp.461-474, 1998.

[21] M.Zefran, F.Bullo and M.Stein, "A notion of passivity for hybrid systems," in Proc. 40th IEEE Conf.on Decision and Control, 2001, pp.768-773.

[22] G.Zhai, B.Ho, K.Yasuda and A.N.Michel, "Disturbance attenuation properties of time-controlled switched systems," J. Franklin Institute, vol.338, pp.765-779, 2001.

[23] J.Zhao and D.J.Hill, "Hybrid H_∞ control based on multiple Lyapunov functions," Proc. of the 6th IFAC Symposium on Nonlinear Control Systems, pp.567-572, 2004.