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Optimality Zone Algorithms for Hybrid Systems Computation and Control: From Exponential to Linear Complexity

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***Abstract*— In [1], [2], [3], [4] necessary conditions were obtained for hybrid optimal control problems (HOCPs) which resulted in a general Hybrid Maximum Principle (HMP); further, in [4], [5], a class of efficient, provably convergent Hybrid Maximum Principle (HMP) algorithms were obtained based upon the HMP. In [3], [4] the notion of optimality zones (OZs) was introduced as a theoretical framework for the computation of optimal location (i.e. discrete state) sequence for HOCPs (i.e. discrete state sequences with the associated switching times and states). This paper presents the algorithm HMPZ which fully integrates the prior computation of the OZs into the HMP algorithms of [4], [5]. Adding (a) the computational investment in the construction of the OZs for a given HOCP, and (b) the complexity of (i) the computation of the optimal schedule, (ii) the optimal switching time and optimal switching state sequence, and (iii)the optimal continuous control input, yields a complexity estimate for the algorithm (HMPZ) which is linear (i.e. $O(L)$) in the number of switching times L; this is to be compared with the geometric (i.e. $O(|Q|^L)$) growth of a direct combinatoric search over the set of location sequence, where Q denotes the discrete state set of the hybrid system.**

I. INTRODUCTION

Over the last few years the notion of a hybrid control system with continuous and discrete states and dynamics has crystallized and various classes of optimal control problems for such systems have been formalized (see for example [3], [6], [2], [1], [7], [8], [9]). In particular, generalizing the standard Maximum Principle (MP), Sussmann [10] and Riedinger et al. [11], among other authors, have given versions of the Hybrid Maximum Principle (HMP) with indications of proof methods. An explicit theory for the two stage controlled switching optimal control problem was given by Tomiyama in [12] and a complete, rigorous treatment of the HMP is given in [13], [14] for the case of *a priori* fixed discrete state sequences. In [1], [2], [3], [4] a set of necessary conditions for hybrid optimal control problems (HOCPs) was derived which constitutes a general Hybrid Maximum Principle (HMP); based upon this, a class of efficient Hybrid Maximum Principle (HMP) algorithms has been constructed [5] and their convergence established. Next, in [3], [4] the notion of optimality zones (OZs) was introduced as a theoretical framework enabling the computation of optimal schedules (i.e. discrete state sequences with the associated switching times and states) for HOCPs.

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The contributions of this paper include: (i) the algorithm HMPZ which fully integrates the prior computation of the OZs into the HMP algorithms of [4], [5]; and (ii) computed examples of the application of HMPZ to a bilinear quadratic regulator HOCP, demonstrating the efficacy of HMPZ.

The computational complexity of HMPZ has two components: (a) complexity of the construction of the optimality zones for a given HOCP, which depends upon the cardinality of the discrete state set Q and the number of grid points $|G|$ but is independent of the number of switchings, and (b) the complexity of a single run of the HMP algorithm which is linear (i.e. $O(L)$) in the number of switchings L. This gives the overall complexity of HMPZ as $O(|G|^2 \cdot |Q|) + O(L)$; this is to be compared with the geometric (i.e. $O(|Q|^L)$) growth of a direct combinatoric search over the set of location sequences.

The notion of optimality zones must be distinguished from the so-called "switching regions" presented in [15], [16], [17]; switching regions partition the continuous state space of autonomous (steady state) hybrid systems whereas optimality zones partition the Cartesian product of the system's time and state space ($\mathbb{R}^1 \times \mathbb{R}^n$) with itself, that is to say, they partition $(\mathbb{R}^1 \times \mathbb{R}^n)^2$. As explained in Section III, these partitions are defined for any given finite horizon hybrid optimal control problem (HOCP).

II. HYBRID OPTIMAL CONTROL THEORY

In this paper we consider hybrid systems which in each location are governed by non-linear dynamics of the form

$$\mathbb{H}: \qquad \dot{x}_q = f_q(x_q, u), \quad q \in Q \triangleq \{1, 2, \ldots, |Q|\}.$$

At a controlled discrete state transition at an instant t, $t \in [t_0, t_f]$, the piecewise constant, right continuous, Q valued, discrete state (component) trajectory satisfies

$$\mathbb{H}: \qquad q(t-) = q_i \in Q, \quad q(t) = q_j \in Q, \quad q_i \neq q_j.$$

On the other hand, this transition relation is satisfied at an autonomous transition at the instant t if the continuous state (component) arrives at the switching manifold m_{q_i,q_j} (see below) at the instant t under the q_i dynamics $f_{q_i}(x, u)$ and then passes through it.

In this paper no constraints are imposed on the dynamics of the discrete state transition, but in [4], [5] the controlled transitions satisfy the Q-dependent dynamics of the form $q_j = \Gamma(q_i, \sigma_{ij})$, where σ_{ij} is a partially defined discrete input.

The reader is referred to [4], [5] for the complete presentation of the formulation, hypotheses and specification of hybrid dynamical systems of the form $\mathbb{H}$.

Consider the initial time t_0, final time $t_f < \infty$, initial hybrid state $h_0 = (q_0, x_0)$, and an upper bound on the number of switchings $\bar{L} \leq \infty$. Let $S_L = ((t_0, q_0), (t_1, q_1), \ldots, (t_L, q_L))$ be a hybrid switching sequence and let $I_L \,\underline{\Delta}\, (S_L, u)$, $u \in \mathcal{U}$, where $\mathcal{U} = \mathcal{U}^0$ or $\mathcal{U} = \mathcal{U}^{\text{cpt}}$, $\bar{L} \leq \infty$, be a hybrid input trajectory which (subject to the assumptions of [4], [5]) results in a (necessarily unique) hybrid execution and is such that $L \leq \bar{L}$ controlled and autonomous switchings occur on the time interval $[t_0, T(I_L)]$, where $T(I_L) \geq t_f$. Here the set of *admissible input control functions* is $\mathcal{U}^0 \,\underline{\Delta}\, \mathcal{U}(\overset{o}{U}, L_\infty([0, T_*)))$, (respectively $\mathcal{U}^{\text{cpt}} \,\underline{\Delta}\, \mathcal{U}(U^{\text{cpt}}, L_\infty([0, T_*)))$), the set of all bounded measurable functions on some interval $[0, T_*)$, $T_* \leq \infty$, taking values in the open bounded set $\overset{o}{U}$ (respectively the compact set U^{cpt}). Further let the collection of such inputs be denoted $\{I_L\}$. We define the hybrid cost function as:

$$J: \quad J(t_0, t_f, h_0; I_L, \bar{L}, \mathcal{U}) \,\underline{\Delta}\, \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s))\, ds + g(x_{q_L}(t_f)),$$

where for $i = 0, 1, \ldots, L$,

$$\begin{aligned}
&\dot{x}_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), \quad a.e.\, t \in [t_i, t_{i+1}), \\
&u(t) \in U \subset \mathbb{R}^n, \text{ where in case } \mathcal{U} = \mathcal{U}^{\text{cpt}},\, U = \bar{\overset{o}{U}}, \\
&u(\cdot) \in L_\infty(U), \\
&h_0 = (q_0, x_{q_0}(t_0)) = (q_0, x_0), \\
&x_{q_{i+1}}(t_{i+1}) = \lim_{t \uparrow t_{i+1}} x_{q_i}(t), \text{ and} \\
&t_{L+1} = t_f < \infty, \qquad L \leq \bar{L} \leq \infty.
\end{aligned}$$

Definition 2.1: ([4], [5], Hybrid Optimal Control Problem (HOCP)) Given a hybrid system $\mathbb{H}$, loss functions $\{l_q,\, q \in Q\}$, initial and final times, t_0, t_f, the initial hybrid state $h_0 = (q_0, x_0)$, and an upper bound on the number of switchings $\bar{L} \leq \infty$, the *hybrid optimal control problem* (HOCP($t_0, t_f, x_0, \bar{L}, \mathcal{U}$)), where $\mathcal{U} = \mathcal{U}^0$ or $\mathcal{U} = \mathcal{U}^{\text{cpt}}$, is to find the infimum $J^0(t_0, t_f, h_0, \bar{L}, \mathcal{U})$ of the hybrid cost function $J(t_0, t_f, h_0; I_L, \bar{L}, \mathcal{U})$ over the family of input trajectories $\{I_L\}$.

If a hybrid input trajectory I_{L^0} exists which realizes $J^0(t_0, t_f, h_0, \bar{L}, \mathcal{U})$ it is called a *hybrid optimal control* for the HOCP($t_0, t_f, x_0, \bar{L}, \mathcal{U}$). □

In Theorem 2.1 we state the necessary conditions upon which the algorithms of this paper are based; the theorem is stated for the cases where the optimal control takes values in the open bounded set $\overset{o}{U}$, or in the compact set U^{cpt}, respectively. The reader is referred to [4] and the associated paper [5] for a complete exposition of the HMP necessary conditions, and the definition of the small time tubular fountain (STTF) condition which is crucial for the proof of the necessary conditions related to autonomous switchings.

Theorem 2.1: ([4], [5]) Consider a hybrid system $\mathbb{H}$ and the HOCP($t_0, t_f, x_0, \bar{L}, \mathcal{U}^{\text{cpt}}$), where $\mathcal{U} = \mathcal{U}^0$ or $\mathcal{U} = \mathcal{U}^{\text{cpt}}$, and define

$$\begin{aligned}
H_q(x, \lambda, u) &= \lambda^T f_q(x, u) + l(x, u), \\
&x, \lambda \in \mathbb{R}^n, \quad u \in \overset{o}{U}, \quad q \in Q.
\end{aligned}$$

1) Let $J^0(t_0, t_f, h_0, \mathcal{U}) = \inf_{\{I_{L(L)}\}} J^0(t_0, t_f, h_0, I_L, \bar{L}, \mathcal{U})$ be realized at $I^0_{L^0}$, (x^0, q^0).
2) Let $I^0_{L^0}$ have L_c controlled switchings and L_a autonomous switchings, and let $L_a + L_c = L^0$.
3) Let $t_1, t_2, \ldots, t_{L^0}$, denote the switching times along the optimal trajectory (x^0, q^0).
4) Assume that x^0 meets $\tilde{m} = \bigcup \tilde{m}^k_{p,q}$, the collection of switching manifold subcomponents, transversally and does not meet $\partial\tilde{m}^{k_i}_{p,q} \bigcap \partial\tilde{m}^{k_j}_{p,q}$ for any k_i, k_j, p, q.
5) Assume that either (a) $\bar{L} < \infty$ and $L^0 = L_a + L_c + 2 \leq \bar{L}$, or (b) $\bar{L} = \infty$ and $L^0 < \infty$.

Then

(i) There exists a (continuous to the right), piecewise absolutely continuous adjoint process λ^0 satisfying

$$\dot{\lambda}^0 = -\frac{\partial H_{q^0(t)}}{\partial x}(x^0, \lambda^0, u^0), \quad t \in (t_j, t_{j+1}), \quad j \in \{0, 1, 2, \ldots, L^0\}, \tag{1}$$

where $t_{L^0+1} = t_f$ and where the following boundary value conditions hold with $\lambda^0(t_0)$ free:

(a) $\lambda^0(t_f) = \nabla_x g(x^0(t_f))$.

(b) If t_j is a controlled switching time, then

$$\lambda^0(t_j-) \equiv \lambda^0(t_j) = \lambda^0(t_j+), \quad j \in \{0, 1, 2, \ldots, L^0\}. \tag{2}$$

(c) If t_j is an autonomous switching time satisfying $m_{j,j+1}(x(t_j)) = 0$, then

$$\begin{aligned}
\lambda^0(t_j-) &\equiv \lambda^0(t_j) \\
&= \lambda^0(t_j+) + p_j \nabla_x m_{j,j+1}\big|_{t=t_j},
\end{aligned} \tag{3}$$

for some $p_j \in \mathbb{R}$.

(ii) The Hamiltonian minimization conditions are satisfied, i.e.

(a)

$$H_{q^0(t)}(x^0(t), \lambda^0(t), u^0(t)) \leq H_{q^0(t)}(x^0(t), \lambda^0(t), v), \quad \text{a.e. } t \in [t_j, t_{j+1}),\, \forall v \in U,\, j \in \{0, 1, 2, \ldots, L^0\}. \tag{4}$$

(b1) If $\bar{L} < \infty$ and $L^0(\bar{L}) = L^0_a + L^0_c + 2 \leq \bar{L}$, then

$$H_{q^0(t)}(x^0(t), \lambda^0(t), u^0(t)) \leq H_q(x^0(t), \lambda^0(t), u^0(t)), \quad \text{a.e. } t \in [t_j, t_{j+1}),\, j \in \{0, 1, 2, \ldots, L^0\},\, \forall q \in Q. \tag{5}$$

(b2) If $\bar{L} = \infty$ and $L^0(\bar{L}) < \infty$, then

$$H_{q^0(t)}(x^0(t), \lambda^0(t), u^0(t)) \leq H_q(x^0(t), \lambda^0(t), u^0(t)), \quad \text{a.e. } t \in [t_j, t_{j+1}),\, j \in \{0, 1, 2, \ldots, L^0\},\, \forall q \in Q. \tag{6}$$

(iii) If $\mathcal{U} = \mathcal{U}^{\mathrm{cpt}}$ then the following *Hamiltonian continuity* condition holds at a controlled switching time $t = t_j$

$$\begin{aligned} H(t_j-) &\equiv H_{q^0(t_j-)}(t_j-) = H_{q^0(t_j-)}(t_j) \\ &= H_{q^0(t_j)}(t_j) = H_{q^0(t_j+)}(t_j+) \equiv H(t_j+), \\ &\qquad j \in \{1, 2, \ldots, L^0\}. \end{aligned}$$

(iv) If $\mathcal{U} = \mathcal{U}^0$ and if almost every continuous state x on the optimal trajectory $x^0(\cdot)$ is a small time tubular fountain with respect to $x^0(\cdot)$ then the following *Hamiltonian continuity* condition holds at a (controlled or autonomous) switching time $t = t_j$

$$\begin{aligned} H(t_j-) &\equiv H_{q^0(t_j-)}(t_j-) = H_{q^0(t_j-)}(t_j) \\ &= H_{q^0(t_j)}(t_j) = H_{q^0(t_j+)}(t_j+) \equiv H(t_j+), \\ &\qquad j \in \{1, 2, \ldots, L^0\}. \end{aligned}$$

□

Throughout this paper the HOCPs under consideration do not have any autonomous switchings defined, in other words the collection of autonomous switching manifold sets is empty.

III. Optimality Zones, Location Sequences and the HMPZ Algorithm

A. Fundamental Implications of DPP for Optimal Location Sequences

1) DP Principle: Along an optimal hybrid execution $(I^0_{L^0}, x^0)$ the Dynamic Programming Principle implies that the part of the hybrid input $I^0_{L^0}$ (and correspondingly the hybrid trajectory (q^0, x^0)) from the j-th switching time and state pair to the $j+1$-st switching time and state pair, $(t^0_s, x^0_s)_j \to (t^0_s, x^0_s)_{j+1}$, is optimal. Hence, in particular, q^0_j must be an optimal location for the trajectory from $(t^0_s, x^0_s)_j$ to $(t^0_s, x^0_s)_{j+1}$.

2) Non-hybrid Optimal Control Problem: For each $q = q((t^0_s, x^0_s)_j, (t^0_s, x^0_s)_{j+1}) \in Q$ the optimization above is a standard (non-hybrid) optimal control problem which is not linked to an analogous optimization over any other interval.

3) $|Q|$ Complexity Search: For each time and state pair $\{(t^0_s, x^0_s)_j,\ (t^0_s, x^0_s)_{j+1}\}$ the set-up cost of a search over a set Q to find the optimal $q^0((t^0_s, x^0_s)_j, (t^0_s, x^0_s)_{j+1})$ is proportional to $|Q|$ and is not linked to an analogous search over any other interval.

B. Variations in Switching Time and State and Local Optimality w.r.t. Discrete Location

1) Local Optimality for fixed $q \in Q$: In [4] we show that under certain assumptions the value function $v(t, x, q)$ of HOCP is bounded and continuous in (t, x) for each $q \in Q$. For simplicity, consider the case where we have two locations, $Q = \{1, 2\}$, and two controlled switchings at (t_{s_1}, x_{s_1}) and (t_{s_2}, x_{s_2}) with $t_{s_2} > t_{s_1}$. Further assume that over the interval $[t_{s_1}, t_{s_2}]$ the optimal cost $J^0_1((t_{s_1}, x_{s_1}), (t_{s_2}, x_{s_2}))$ of a trajectory from x_{s_1} to x_{s_2} in location 1 is strictly smaller than the corresponding cost $J^0_2((t_{s_1}, x_{s_1}), (t_{s_2}, x_{s_2}))$ in location 2. Hence by the continuity of each J^0_i, $i = 1, 2$, in $((t_{s_1}, x_{s_1}), (t_{s_2}, x_{s_2}))$, there is a neighbourhood $N^1_{((t_{s_1}, x_{s_1}), (t_{s_2}, x_{s_2}))}$ of $((t_{s_1}, x_{s_1}), (t_{s_2}, x_{s_2}))$ such that for any $((t'_{s_1}, x'_{s_1}), (t'_{s_2}, x'_{s_2})) \in N^1_{((t_{s_1}, x_{s_1}), (t_{s_2}, x_{s_2}))}$ the optimality of location 1 is preserved.

2) Specification of OZs: This preservation of optimality of location 1 w.r.t. the perturbations of $((t_{s_1}, x_{s_1}), (t_{s_2}, x_{s_2}))$ gives rise to the notion of (the set of) optimality zones (OZs). The optimality zone corresponding to a location q^o is a subset of the Cartesian product $(\mathbb{R}^1 \times \mathbb{R}^n)^2$ of the time and space region $\mathbb{R}^1 \times \mathbb{R}^n$ with itself which corresponds to the distinct optimal location q^o. The formal definition of OZs is given in III-E.

C. Basic Idea of the (Linear in L) HMPZ Algorithm

1) Discretization of Space-Time: For simplicity and for the purpose of estimation of computational complexity assume that Γ is a rectangular region in $\mathbb{R}^{n+1}$:

$$\Gamma \,\underline{\Delta}\, [t_0, t_f] \times [x^1_i, x^1_f] \times \cdots \times [x^n_i, x^n_f]$$

Let a grid G on Γ be defined as follows. The time interval $[t_0, t_f] \in \mathbb{R}$ is divided into N_0 uniform subintervals and let $\delta_0 \,\underline{\Delta}\, (t_f - t_0)/N_0$. For each point $t_0 + \delta_0 k$, $k = 0, 1, \ldots, N_0$, let each edge of Γ be divided into N_i uniform subintervals and let $\delta_i \,\underline{\Delta}\, (x^i_f - x^i_0)/N_i$, $i = 1, 2, \ldots, n$. Then

$$G \,\underline{\Delta}\, \{t_0, \ldots, t_f\} \times \left(\times^n_{k=1} \{x^k_i, \ldots, x^k_f\} \right).$$

2) Set-up Computation: We shall adopt the name $PREP(G)$ for an algorithm performing the following calculation: find the optimal location $q^o = q^o((t_r, x_r), (t_s, x_s)) \in Q$, $((t_r, x_r), (t_s, x_s)) \in G$, $t_0 \le t_r < t_s \le t_f$, for all such strictly ordered t_r, t_s on the lattice points of the grid G with $|G|$ elements, where the envelope of G is assumed to contain the optimal trajectory $(x^o(t); t_0 \le t \le t_f)$.

3) HMP with OZ Data; Conceptual Algorithm: Let the execution of the basic HOCP algorithm HMP (see [4], [5]) be modified so that, after an iterative shift of the vector of switching time and state pairs $(t_s, x_s)^{[k]}$ to $(t_s, x_s)^{[k+1]}$ in $\mathbb{R}^{L(n+1)}$, the location $q^{[k+1]}_j$ on the interval $[t^{[k+1]}_j, t^{[k+1]}_{j+1})$ is chosen so as to be optimal among all trajectories from $x^{[k+1]}_{t_{s_j}}$ to $x^{[k+1]}_{t_{s_{j+1}}}$ (such a location is generated by $PREP(G)$). Upon incrementing k to $k+1$ the HMPZ algorithm repeats its basic HMP operation if the halting rule of HMP has not been satisfied.

D. Optimality and Complexity of HMPZ

Based upon the conceptual specification of Algorithm HMPZ above, and invoking the DP Principle III-A together with the global convergence analysis (subject to the associated conditions) of the Algorithm HMP in [4], [5], it may be shown ([18]) that if HMPZ halts at some $(x^H, u^H, q^H) \equiv (u^H, q^H)$ then neither (i) a change in q^H with the given $\{x^H_s, t^H_s\}$, nor (ii) a change in u^H for the given q^H can strictly decrease the cost J.

The algorithm $PREP(G)$ solves one standard (i.e. non-hybrid) optimal control problem for each pair of points in the grid G, for each location $q \in Q$. Hence the computational

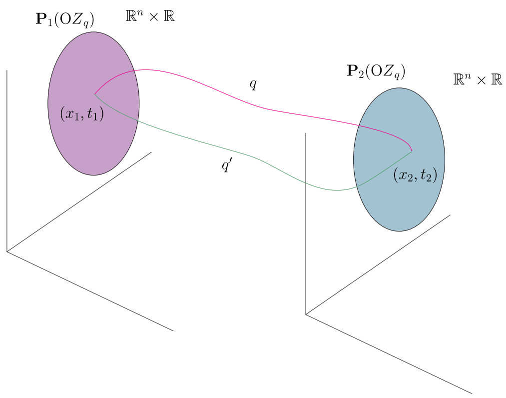

Fig. 1. Optimality zones.

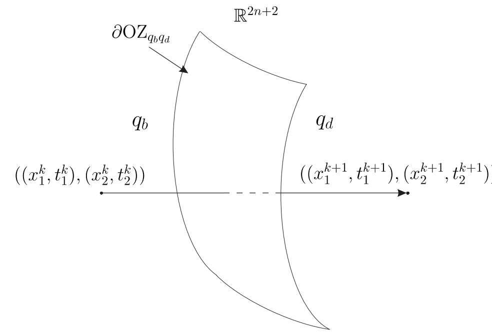

Fig. 2. An iteration of the Algorithm HMPZ: ∂OZ crossing.

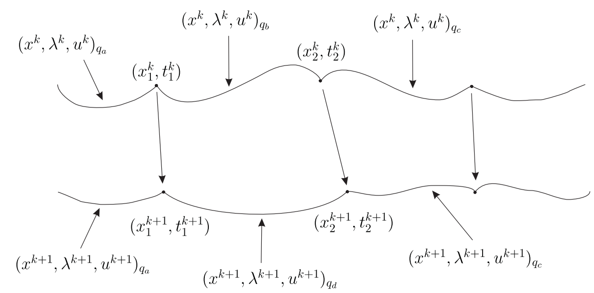

Fig. 3. An iteration of the Algorithm HMPZ: switching time and switching state update.

cost of the determination of the optimality zones for HOCP by use of $PREP(G)$ in $\mathbb{R}^{2(n+1)}$ is $O(|G|^2 \cdot |Q|)$ which is independent of the number of switchings L. The HMPZ algorithm resulting from the enhancement of HMP with $PREP$ computes (i) the optimal continuous variables and controls, and (ii) the optimal discrete location sequence with an overall complexity cost of $O(|G|^2 \cdot |Q|) + O(L)$, where $O(L)$ corresponds to the complexity of a single run of the HMP algorithm.

E. Definition of Optimality Zones

Under the assumptions generating the class of hybrid systems $\mathbb{H}$ (and the associated HOCP) the value function $J^0((t_1, x_1), (t_2, x_2), q)$ of HOCP is bounded and continuous in $((t_1, x_1), (t_2, x_2))$ for each $q \in Q$ (see [4]). So it is possible to define a region Z_q of points $((t_1, x_1), (t_2, x_2))$ in the space $(\mathbb{R} \times \mathbb{R}^n)^2$ for which a specific location $q \in Q$ corresponds to the optimal hybrid system trajectory starting at (t_1, x_1) and terminating at (t_2, x_2).

We adopt the convention that if (t_2, x_2) is not accessible from (t_1, x_1) and similarly if (t_1, x_1) is not co-accessible to $(t_2, x_2))$ when the system $\mathbb{H}$ is in the discrete state $q \in Q$ then $J_q((t_1, x_1), (t_2, x_2)) = \infty$.

Definition 3.1: For $t_0 \leq t_1 < t_2 \leq t_f$, the *optimality zone* OZ_q, corresponding to the discrete state $q \in Q$, is given by

$$\begin{aligned} Z_q \quad \underline{\Delta} \quad & \{((t_1, x_1), (t_2, x_2)) \in ((t_0, t_f) \times \mathbb{R}^n)^2 : \\ & J_q^0((t_1, x_1), (t_2, x_2)) \leq \\ & J_{q'}^0((t_1, x_1), (t_2, x_2)),\ t_1 < t_2,\ \forall q' \in Q\}. \end{aligned}$$

□

Figure 1 shows the projections $\mathbf{P_1}(OZ_q)$ and $\mathbf{P_2}(OZ_q)$ of the optimality zone OZ_q on (t_1, x_1) and (t_2, x_2) spaces respectively.

IV. Single Pass Schedule Optimization: The Algorithm Class HMPZ

Let $z_i \ \underline{\Delta}\ (t_i, x_i)$ and let $OZ : \mathbb{R}^{(n+1)L} \to Q^{L+1}$ be such that $OZ(\{z_i\}_{i=0}^L) = \{q_i\}_{i=1}^{L+1}$, i.e. for a given HOCP, the function OZ takes a sequence of time and state pairs and returns a sequence of locations from the precomputed Optimality Zones look-up database. Notice that the initial and final time-state are not passed to the OZ as they are part of the specification of HOCP.

Let $HMP : \mathbb{R}^{(n+1)L} \times Q^{L+1} \to \mathbb{R}^{(n+1)L}$ be such that for a given HOCP it performs the switching time and switching state update step of the Algorithm HMP of [4], [5].

Also, let $SC : \mathbb{R}^{2(n+1)L} \times Q^{2(L+1)} \times \mathbb{Z}_+ \to \mathbb{R}_+$ be a function which, for a given HOCP, computes a quantity to be compared to the stopping condition tolerance $\epsilon > 0$ of the Algorithm HMP of [4], [5].

Then the ALgorithm HMPZ may be specified as follows:

1. Initialization: Fix $0 < \epsilon \ll 1$. Set the iteration counter $k = 0$. Let $\{z_i\}_{i=1}^L \ \underline{\Delta}\ \{(t_i, x_i)\}_{i=1}^L$ be initial switching time and state pairs satisfying $t_0 < t_1 < t_2 < \cdots < t_L < t_f$.
2. $\{q_i\}_1^{L+1} \leftarrow OZ(\{z_i\}_1^L)$.
3. $\{z_i\}_{i=0}^L \leftarrow HMP\left(\{z_i\}_{i=0}^L, \{q_i\}_{i=1}^L\right)$.
4. If $SC\left(\{z_i\}_{i=0}^L, \{q_i\}_{i=1}^L; k, k-1\right) \leq \epsilon$ then STOP; else $k \leftarrow k+1$, go to Step 2.

Figures 2 and 3 show a typical iteration of the Algorithm HMPZ where an OZ boundary crossing takes place.

V. The Hybrid Bilinear Quadratic Regulator (BLQR) Problem

In the set of computational experiments applying HMPZ to a hybrid system whose discrete state set consists of the two locations corresponding to the bilinear dynamics:

$$\dot{x} = x + xu, \quad \dot{x} = -x + xu,$$

with the cost function:

$$J(u) = \frac{1}{2}\int_0^3 u^2(s)\,ds,$$

the program $PREP$ was first applied to the product time-space $(R^{1+1})^2$ and this generated the OZ region data which was stored in the main program look-up table. The internal zone boundary between the OZs corresponding to the middle location being $q = 1$ or $q = 2$ is shown in Figures 4 and 5.

In this particular case it is possible to obtain closed form expressions for the optimal cost of transferring the system from (t_1, x_1) to (t_2, x_2) under the two dynamics $(i = 1, 2)$:

$$J_i((t_1, x_1), (t_2, x_2)) = \begin{cases} \frac{1}{2}\left[1 + \frac{(-1)^i}{t_2 - t_1}\log\left(\frac{x_2}{x_1}\right)\right]^2 (t_2 - t_1), & \\ \qquad\qquad\qquad \text{if } t_1 \neq t_2 \wedge x_1 x_2 > 0 \\ 0, \qquad \text{if } t_1 = t_2 \wedge x_1 = x_2 \wedge x_1 x_2 > 0 \\ \infty, \qquad \text{if } (t_1 = t_2 \wedge x_1 \neq x_2) \vee (x_1 x_2 \leq 0). \end{cases}$$

The interesting case is that of $t_1 \neq t_2$ and in this case equating the costs corresponding to the two distinct dynamics gives:

$$\left[1 - \frac{1}{t_2 - t_1}\log\left(\frac{x_2}{x_1}\right)\right]^2 - \left[1 + \frac{1}{t_2 - t_1}\log\left(\frac{x_2}{x_1}\right)\right]^2 = 0.$$

Hence

$$\frac{1}{t_2 - t_1}\log\left(\frac{x_2}{x_1}\right) = 0,$$

so

$$x_1 = x_2.$$

Hence the switching surface is given in the (t_1, t_2, x_1, x_2)-space by:

$$\partial\text{OZ} = \{(t_1, t_2, x_1, x_2) \in \mathbb{R}^4 : x_1 = x_2\},$$

which agrees with the computational experiments.

Example 5.1: For the subsequent implementation of HMPZ the initial location sequence was arbitrarily chosen to be $(1, 1, 1)$. The initial values of t_{s_1} and t_{s_2} were chosen to be 0.5 and 1.5, and the associated values of the switching states were $x(t_{s_1}) = 1$ and $x(t_{s_2}) = 1.5$. After just one iteration of HMP the second switching time and state pair passed through an OZ boundary resulting in the location sequence transition from $(q_1, q_2, q_3) = (1, 1, 1)$ to $(q_1, q_2, q_3) = (1, 1, 2)$ as shown in the second line of Table V. Subsequently, at iteration number 3 the two switching time and state pairs passed through an OZ boundary resulting in the location sequence change from $(q_1, q_2, q_3) = (1, 1, 2)$ to $(q_1, q_2, q_3) = (1, 2, 2)$ as shown in the third line of Table V. Finally, after six iterations the algorithm corresponding to the location sequence $(1, 2, 2)$ converged giving the optimal cost 0.16674.

The computational time for $PREP$ in this experiment was 7231 seconds (about two hours) for the case where space-time constraints were taken to be $0 < t_{s_1} < 2$, $t_{s_1} < t_{s_2} < 2$, $1 < x_{s_1} < 2$, $1 < x_{s_2} < 2$, and 121 seconds for the second case where they were taken to be $0.4 < t_{s_1} < 0.8$, $1.2 < t_{s_2} < 1.7$, $0.5 < x_{s_1} 1 < 1$, $1.5 < x_{s_2} < 2$. For the HMPZ implementation the computation time was 3.6 seconds. All computations were performed in Matlab 6.5 under Windows 2000 SP4 operating system on a P4 3.2 GHz machine with 512 MB of RAM. □

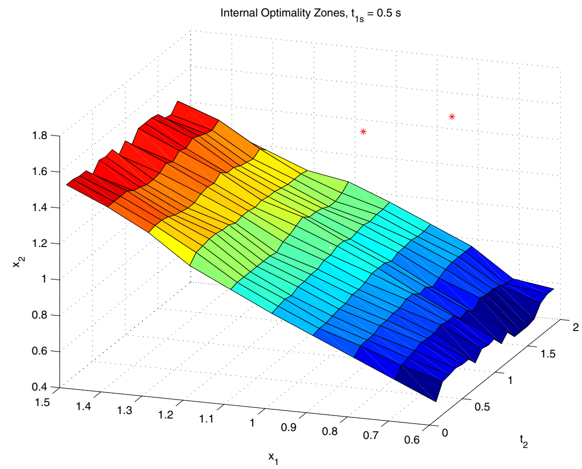

Fig. 4. OZ boundary for x_1, x_2, t_2 varying with $t_1 = 0.5$. Asterisks indicate joint time and state switching pairs at HMPZ iterations 1 and 2.

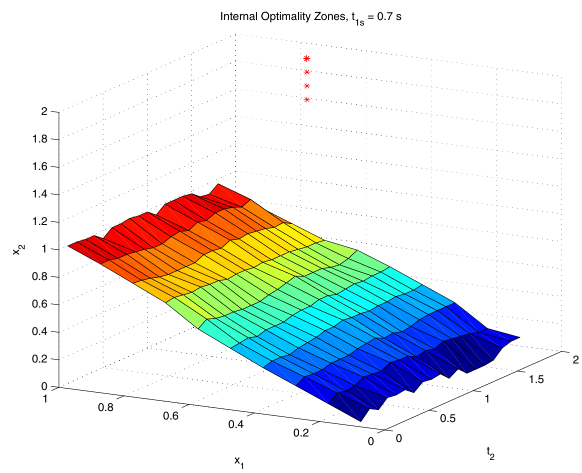

Fig. 5. OZ boundary for x_1, x_2, t_2 varying with $t_1 = 0.7$. Asterisks indicate joint time and state switching pairs at HMPZ iterations 3, 4, 5 and 6.

Example 5.2: To demonstrate the power of the HMPZ algorithm we applied it to solve an HOCP involving the BLQR of Example 5.1 with ten switchings. *It is to be noted that the computation of the Optimality Zones for the two switchings case (Example 5.1) is reused for this ten switchings example without any modification.* The problem data was: $t_0 = 0$, $t_f = 2$, $x_0 = 2.4$, $x_f = 2.6$ and number of switchings was set to 10. The algorithm initially computed (i) ten uniformly distributed switching times between $t_0 = 0$ and $t_f = 2$, (ii) ten randomly distributed switching states between $x_0 = 2.4$ and $x_f = 2.6$, and (iii) the initial switching sequence:

TABLE I

EXECUTION OF ALGORITHM HMPZ

Iter.	Loc. sequence	Cost	(t_{s_1}, x_{s_1})	(t_{s_2}, x_{s_2})
1	$(1,1,1)$	1.2928	$(0.5,1)$	$(1.5,1.5)$
2	$(1,1,2)$	0.56549	$(0.5,0.8)$	$(1.6,1.6)$
3	$(1,2,2)$	0.37839	$(0.7,0.7)$	$(1.7,1.7)$
4	$(1,2,2)$	0.2909	$(0.7,0.7)$	$(1.7,1.8)$
5	$(1,2,2)$	0.22118	$(0.7,0.7)$	$(1.7,1.9)$
6	$(1,2,2)$	0.16674	$(0.7,0.7)$	$(1.7,2)$

TABLE II

EXECUTION OF ALGORITHM HMPZ: TEN SWITCHINGS CASE

Iteration	Location sequence	Cost
1	$(1,1,1,1,2,1,1,2,2,1,1)$	0.75653
2	$(1,1,1,1,2,1,1,2,2,1,1)$	0.70324
3	$(1,1,1,1,2,1,1,2,2,1,1)$	0.68563
4	$(1,1,1,1,2,1,1,2,2,1,1)$	0.63887
5	$(1,1,1,1,2,1,1,2,2,1,1)$	0.61678
6	$(1,1,1,1,2,1,1,2,2,1,1)$	0.60291
7	$(1,1,1,1,2,1,1,2,2,1,1)$	0.58548
8	$(1,1,1,1,2,1,1,2,2,1,1)$	0.54783
9	$(1,1,1,1,2,1,1,2,2,1,1)$	0.49985
10	$(1,1,1,1,2,1,1,2,2,1,1)$	0.47789
11	$(1,1,1,1,2,1,1,2,2,1,1)$	0.43679
12	$(1,1,1,1,2,1,1,2,2,1,1)$	0.39453
13	$(1,1,1,1,2,1,1,2,2,1,1)$	0.35672
14	$(1,1,1,1,2,1,1,2,2,1,1)$	0.33756
15	$(1,1,1,1,2,1,1,2,2,1,1)$	0.31957
16	$(1,1,2,2,1,1,2,1,2,1,1)$	0.21986
17	$(1,1,2,2,1,1,2,1,2,1,1)$	0.21897
18	$(1,1,2,2,1,1,2,1,2,1,1)$	0.21897

$(1,1,1,1,2,1,1,2,2,1,1)$ which corresponds to the initial choice of switching times and states. The initial cost as computed by the algorithm is J = 0.75653 which drops down to J = 0.31957 by the 15th iteration. In the next (i.e. 16th) iteration the algorithm switches to the zone corresponding to the optimal switching sequence: $(1,1,2,2,1,1,2,1,2,1,1)$ giving the optimal cost J = 0.21897 at the 18th iteration. The running time was 45.596 seconds. The iterations of the program execution are shown in Table V. □

VI. ACKNOWLEDGMENT

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