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Localized Adaptive Bounds for Online Approximation Based Control

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Abstract— **This article develops new methods for adaptively bounding approximation accuracy with methods that involve localized forgetting. The existing results use global forgetting. The importance of local versus global forgetting is motivated in the text. Such bounds have utility for self-organizing approximators that could adjust the number of basis elements N by adding additional approximation resources in the regions where the approximation error bound is large.**

Keywords: *Adaptive control, nonlinear systems, adaptive bounds.*

I. Introduction

Most nonlinear adaptive control methods are proposed to address model uncertainties that are assumed to be the multiplication of known nonlinearities and uncertain parameters [4], [10]. Since the first stability results appeared, adaptive robust nonlinear control has been extensively developed to retain closed-loop stability properties in the presence not only of large parametric uncertainty, but also modeling errors such as additive disturbances and unmodeled dynamics [4]. On-line approximation methods [1], [2], [5], [8], [9], [11], [12], [14], [15], [16] are designed to achieve stability and accurate reference input tracking for systems with partially unknown nonlinearities, by implementing approximations to the unknown nonlinear dynamics during the operation of the system.

Nonlinear close-loop systems which incorporate on-line approximators can be analyzed using Lyapunov stability methods. Both the feedback control law and the approximator parameter estimation equations are derived such that the time derivative of a Lyapunov function has some desirable properties (e.g., negative definiteness). The theory for approximation based nonlinear control is provided in [2], [5], [8], [9], [12], [14]. The design and analysis of adaptive systems have been extensively addressed in [2], [12], [14], including controller structure selection, automatic adjustment of the control law, and complete proofs of stability. Its application based on the feedback linearization method is developed in e.g., [8], [9]. On-line approximation based control by backstepping methods is considered in e.g., [5].

Since on-line approximation based control can never achieve an exact modeling of unknown nonlinearities, *inherent approximation errors* could arise even if optimal approximator parameters were selected. Usually, a restrictive assumption is made that a magnitude bound on the inherent approximation error is known. Articles [5], [12], [13] relax the assumption of a known bound on the inherent approximation errors. With a partially known bound, these articles discuss estimation of the bounding parameters and the design of adaptive robust controllers to guarantee global uniform ultimate boundedness.

However, the global features of the leakage modification for parameter updates in [5], [12], [13] causes each parameter estimate to drift toward certain design parameters as the operating point x leaves a local region for which the parameter is applicable. Thus, both the approximated function and the bounding function will lose local accuracy and any knowledge learned from past experience will not be retained for future use. This issue of global forgetting was addressed in [17], by deriving localized leakage based adaptation algorithms for both the approximator parameters and bounding parameters. The analysis of [17] focused on the scalar single-input-single-output system: $\dot{x} = f(x) + g(x)u$ with $g(x) = 1$ and $x \in \Re^1$.

In this paper, we use the backstepping extension proposed in [3] and develop an adaptive robust control scheme for higher order (i.e., $x \in \Re^n$, $n > 1$) single-input-single-output systems by incorporating on-line approximation of the unknown bounding functions on approximation errors. The existing localized adaptation algorithms [17] for function approximator parameters and bounding parameters are extended to higher order systems with $g_i(x) \neq 1, \forall i = 1, \cdots, n$. Filtering techniques [3] are applied to calculate time derivatives of intermediate state commands for the backstepping approach. The stability and robustness results yield a smaller m.s.s. bound on the tracking error than those in the literature; in addition, the bounding function and function approximation information are retained as a function of the operating point even as the operating point moves around the operating envelope.

II. Problem Formulation

Consider the following class of nth-order single-input-single-output nonlinear systems

$$\dot{x}_i(t) = f_i(x) + g_i(x)x_{i+1}(t), \ 1 \leq i \leq n-1, \quad (1)$$
$$\dot{x}_n(t) = f_n(x) + g_n(x)u(t) \quad (2)$$

where $x = [x_1, \cdots, x_n]^\top$ is the state vector and u is the control signal. It is assumed that the system is strictly feedback passive (see p.46 in [7]). The functions $f_i(x), \ g_i(x), \ i = 1, \cdots, n$ represent nonlinear effects that are unknown at the design stage. Each of these functions is assumed to be continuous on a known compact set $\mathcal{D}$. To ensure controllability, it is necessary to assume that each g_i is bounded away from zero and of known sign. Therefore, without loss of generality, we will invoke the following assumption:

Assumption 1: It is assumed that $g_i(x)$ has lower bound such that $g_i(x) \geq g_{l_i}(x) \geq g_l > 0$, $\forall x \in D$, where $g_{l_i}(x)$ is a known function and g_l is a known constant.

A. Reference Trajectory

There is a desired trajectory $x_d(t)$ with derivative $\dot{x}_d(t)$, both of which are available and lie in the region $\mathcal{D}$ for all $t > 0$. The region $\mathcal{D}$ is a compact domain of operation that is specified at the design stage. The region $\mathcal{D}$ contains all trajectories $x_c = [x_d, x_{2c}, \cdots, x_{nc}]^\top$ for which the system is expected to operate. In fact, we will assume existence of a constant $\gamma > 0$ such that

$$\gamma \leq \min_{x \in \{\Re - \mathcal{D}\}} (\|x_c(t) - x\|), \tag{3}$$

for any $t \geq 0$. This condition states that the desired trajectory is at least a distance γ from the boundary of $\mathcal{D}$. The region $\mathcal{D}$ also defines the largest region over which approximations to f and g will be developed. Our goal is to design the control signal u to steer $x_1(t)$ to track the reference input $x_d(t)$ and to achieve boundedness for the states x_i for $i = 2, \ldots, n$. Note that existing approaches in the literature (e.g., [6], [7]) would require knowledge of the first n derivatives of $x_d(t)$. The approach herein only requires knowledge of $x_d(t)$ and its first derivative [3].

B. Approximator Definition

For $x \in \mathcal{D}$, we define approximations to the unknown functions $f_i(x)$ and $g_i(x)$ as $\hat{f}_i(x) = \theta_{f_i}^\top \Phi_{f_i}(x)$ and $\hat{g}_i(x) = \theta_{g_i}^\top \Phi_{g_i}(x)$ for $i = 1, \ldots, n$, where the parameter vectors θ_{f_i} and θ_{g_i} will be adapted on-line. For $x \notin \mathcal{D}$, $\hat{f}_i(x) = 0$ and $\hat{g}_i(x) = g_l$. The vector $\Phi_{f_i}(x)$ is a user specified regressor vector containing the basis functions for the approximation. Denote the support of the k-th basis function of $\Phi_{f_i}(x)$ vector by $S_{f_i,k} = \{x \in \mathcal{D} \,|\, \Phi_{f_i,k}(x) \neq 0\}$. Let $\bar{S}_{f_i,k}$ denote the closure of $S_{f_i,k}$. Note that each $\bar{S}_{f_i,k}$ is a compact set. For each i, the $\Phi_{f_i}(x)$ vector is defined as a set of positive, locally supported[1] functions $\Phi_{f_i,k}(x)$ for $k = 1, \cdots, N$ such that each set $S_{f_i,k}$ is connected with $\mathcal{D} = \bigcup_{k=1}^{N} S_{f_i,k}$ where N is a finite integer. This ensures that for any $x \in \mathcal{D}$, there exists at least one k such that $\Phi_{f_i,k}(x) \neq 0$. Therefore, $\{S_{f_i,k}\}_{k=1}^{N}$ forms a finite cover for $\mathcal{D}$. Similarly, we define the support of the k-th basis function of $\Phi_{g_i}(x)$ as $S_{g_i,k}$ with closure $\bar{S}_{g_i,k}$. The sets $\bar{S}_{g_i,k}, k = 1 \cdots, N$ also form a finite cover of region $\mathcal{D}$.

In this paper, we are not concerned with the selection of particular basis vectors Φ_{f_i} or Φ_{g_i}. Any basis vectors which satisfy the above assumptions are qualified candidates for the regressor vectors. Splines, radial basis functions, certain wavelets, etc. satisfy these assumptions.

We define a set of parameters $\theta_{f_i}^*$ that are optimal in the sense:

$$\theta_{f_i}^* = \arg\min_{\theta} \left(\max_{x \in \mathcal{D}} \left| f_i(x) - \theta^\top \Phi_{f_i}(x) \right| \right)$$

[1] 'Locally supported' means that $\rho(S_{f_i,k}) < \mu \ll \rho(\mathcal{D})$, where for set A, $\rho(A) = \max_{x,y \in A} (\|x - y\|)$.

Note that these optimal parameters are unknown. They are not used in the implemented control law, but are useful for the analysis that follows. Since $\mathcal{D}$ is compact and each f_i is continuous, the vector $\theta_{f_i}^*$ exists and is well-defined. Define the parameter estimation error vector

$$\tilde{\theta}_{f_i} \quad = \quad \theta_{f_i} - \theta_{f_i}^*.$$

Let

$$\delta_{f_i}(x) = f_i(x) - (\theta_{f_i}^*)^\top \Phi_{f_i}(x)$$

represent the *inherent* or *residual approximation error*. Note that by the definition of $\theta_{f_i}^*$ above, the maximum value of $\delta_{f_i}(x)$ on $\mathcal{D}$ is bounded. This maximum value can be affected by the choice of the dimension and type of corresponding basis vector $\Phi_{f_i}(x)$, but for a given choice of basis vector it cannot be decreased by the choice of the parameter vector θ_{f_i}. The upper bound on the magnitude of the residual approximation error only depends on the designer's choice of approximator. The quantities $\theta_{g_i}^*$, $\tilde{\theta}_{g_i}$ and $\delta_{g_i}(x)$ are defined similarly.

With the above definitions, system equations (1)-(2) can be expressed as

$$\begin{aligned}
\dot{x}_i(t) \quad &= \quad (\theta_{f_i}^*)^\top \Phi_{f_i}(x) + \delta_{f_i}(x) \\
&\quad\quad + \left((\theta_{g_i}^*)^\top \Phi_{g_i}(x) + \delta_{g_i}(x) \right) x_{i+1}, \ 1 \leq i < n, \\
\dot{x}_n(t) \quad &= \quad (\theta_{f_n}^*)^\top \Phi_{f_n}(x) + \delta_{f_n}(x) \\
&\quad\quad + \left((\theta_{g_n}^*)^\top \Phi_{f_n}(x) + \delta_{g_n}(x) \right) u.
\end{aligned}$$

C. Bound Approximation

By the definition of the δ_{f_i} and δ_{g_i}, the magnitude of these inherent approximation error functions are bounded on $\mathcal{D}$; however, the bound is not known. Our control approach will utilize an estimate of these upper bound functions. Therefore, we assume a form for the bounding functions with multiplicative parameters that will be estimated. To save computational effort, we reuse the same basis elements; however, the approach easily extends to the case of different basis elements.

By the above discussion, there exists a positive constant vector $\Psi_{fi}^*, \ i = 1, \cdots, n$, referred as the *optimal bounding parameter*, such that

$$|\delta_{f_i}| \leq (\Psi_{f_i}^*)^\top \Phi_{f_i}, \quad \forall x \in \mathcal{D}.$$

The vector $\Psi_{f_i}^*$ is not unique since any $\bar{\Psi}_{f_i}^* > \Psi_{f_i}^*$ satisfies this assumption. To avoid confusion, the *optimal bounding parameter* is defined to be the vector with the smallest 1-norm such that the assumption is satisfied. A vector $\Psi_{g_i}^*$ yielding a bound on $|\delta_{g_i}|$ is defined similarly. Note that the optimal bounding parameter vectors $\Psi_{f_i}^*$ and $\Psi_{g_i}^*$ are unknown. They are used only for analytical purpose. The control law will use estimates Ψ_{f_i} and Ψ_{g_i} of the optimal bounding parameter vectors. Therefore, the approximated bounding functions are $\Psi_{f_i}^\top \Phi_{f_i}$ for $|\delta_{f_i}|$ and $\Psi_{g_i}^\top \Phi_{g_i}$ for $|\delta_{g_i}|$, where the vectors Ψ_{f_i} and Ψ_{g_i} will be estimated on-line. For the following analysis, we define bounding parameter estimation errors as

$$\tilde{\Psi}_{f_i} = \Psi_{f_i} - \Psi_{f_i}^M \text{ and } \tilde{\Psi}_{g_i} = \Psi_{g_i} - \Psi_{g_i}^M.$$

where each element of $\Psi_{f_i}^M$ is defined as $\Psi_{f_i,k}^M = \max\{\Psi_{f_i,k}^*, \Psi_{f_i,k}^0\},\ k = 1, \cdots, N$ with the vector $\Psi_{f_i}^0 = [\Psi_{f_i,1}^0, \cdots, \Psi_{f_i,N}^0]^\top$ selected in the design stage. With these estimated upper bounds, we will select proper terms in the control signal or the intermediate state commands to properly handle the inherent approximation errors.

III. Adaptive Backstepping-based Design

Define the tracking error vector as $\tilde{x} = [\tilde{x}_1, \cdots, \tilde{x}_n]^\top$ where

$$\tilde{x}_1 = x_1 - x_d \quad (4)$$

$$\tilde{x}_i = x_i - x_{ic} \text{ for } i = 2, \ldots, n \quad (5)$$

where the x_{ic} are defined below. The pseudocontrol signals α_i of the backstepping procedure [6], [7] are defined as

$$\alpha_1 = \frac{u_{a1}}{\hat{g}_1 + \beta_{g_1}} \quad (6)$$

$$\alpha_i = \frac{u_{ai}}{\hat{g}_i + \beta_{g_i}} \quad (7)$$

where

$$u_{a1} = -k_1\tilde{x}_1 + \dot{x}_d - \hat{f}_1 - \beta_{f_1} + u_{s_1}$$

and

$$u_{ai} = -k_i\tilde{x}_i + \dot{x}_{ic} - \hat{f}_i - \beta_{f_i} - (\hat{g}_{i-1} + \beta_{g_{i-1}})\bar{x}_{i-1} + u_{s_i}$$

for $i = 2, \ldots, (n-1)$. The control gains, $k_i,\ i = 1, \cdots, n-1$ are designer specified positive constants that will determine the decay rate for disturbances and initial condition errors. The $u_{s_i}(t)$ terms are defined as

$$u_{s_i}(t) = -r_i(t)sign(\bar{x}_i) \quad (8)$$

where $\bar{x}_i$, which represent compensated tracking errors, will be defined below in eqn. (10). The $u_{s_i}(t)$ terms are defined to return state x to the approximation region $\mathcal{D}$ and keep it there (i.e., to ensure that $\mathcal{D}$ is an invariant set). Here the gain $r_i(t)$ is given by

$$r_i(t) = \begin{cases} 0, & \text{when } x \in \mathcal{D} \\ \bar{b}_{f_i} + \bar{b}_{g_i}|x_{i+1}|, & \text{when } x \notin \mathcal{D} \end{cases} \quad (9)$$

where $\bar{b}_{f_i}, \bar{b}_{g_i}$ are known upper bounds on $|f_i(x)|$ and $|g_i(x)|$, respectively. Note that if constants $\bar{b}_{f_i}$ and $\bar{b}_{g_i}$ are not known, then they could be estimated using the methods suggested in [13], [12]. We do not present such an adaptive bounding approach herein for $x \notin \mathcal{D}$ as it is not the main topic of this article.

The compensated tracking error signals $\bar{x}_i$ for $i = 1, \cdots, n$ are defined as [3]

$$\bar{x}_i = \tilde{x}_i - \xi_i, \text{ for } i = 1, \cdots, n \quad (10)$$

where the ξ_i are defined below.

The signal x_{ic} required for eqn. (5) and its derivative $\dot{x}_{ic}$ required for eqn. (6-7) are defined by the following the following procedure [3].

1) For $i = 2, \cdots, n$,

   a) Define

$$x_{ic}^0 = \alpha_{i-1} - \xi_i.$$

   The signals x_{ic} and $\dot{x}_{ic}$ are defined as

$$\dot{x}_{ic} = -K_i(x_{ic} - x_{ic}^0) \quad (11)$$

   with $K_i > k_i$ being a designer specified constant. Since the filter of (11) is being used as a means to compute x_{ic} and $\dot{x}_{ic}$ without differentiation, the designer would typically select $K_i \gg k_i$ so that x_{ic} accurately tracks x_{ic}^0 over the bandwidth of x_{ic}^0. Since (11) is a stable linear filter, x_{ic} and $\dot{x}_{ic}$ will be bounded if the input x_{ic}^0 is bounded.

   b) Define

$$\dot{\xi}_{i-1} = -k_{i-1}\xi_{i-1} + (\hat{g}_{i-1} + \beta_{g_{i-1}})(x_{ic} - x_{ic}^0).$$

   This is a stable low pass filter. Its input is the product of $(\hat{g}_{i-1} + \beta_{g_{i-1}})$ which we will prove to be bounded and $(x_{ic} - x_{ic}^0)$ which is small. For $x_{ic}, x_{ic}^0 \in \mathcal{D}$ we always have that $|x_{ic} - x_{ic}^0| < 2\rho(\mathcal{D})$ where

$$\rho(\mathcal{D}) = \max_{x_1, x_2 \in \mathcal{D}} \|x_1 - x_2\|$$

   is the diameter of set $\mathcal{D}$. For any x, each ξ_i is bounded by $\bar{b}_\xi$, i.e., $|\xi_i| \leq \bar{b}_\xi$, where

$$\bar{b}_\xi = \frac{2\rho(\mathcal{D})}{\underline{k}} \max_i \left[\sup_{\forall t}\left(|\hat{g}_{i-1} + \beta_{g_{i-1}}|\right)\right] \quad (12)$$

   $\underline{k} = \min_i k_i$.

2) Define

$$u = u_{ad} + u_{s_n} \quad (13)$$

   where u_{ad} and u_{s_n} are defined as

$$u_{ad} = \frac{u_{a_n}}{\hat{g}_n + \beta_{g_n}},$$

$$u_{a_n} = -k_n\tilde{x}_n + \dot{x}_{nc} - \hat{f}_n - \beta_{f_n} - (\hat{g}_{n-1} + \beta_{g_{n-1}})\bar{x}_{n-1}$$

   and

$$u_{s_n} = -r_n(t)sign(\bar{x}_n) \quad (14)$$

$$r_n(t) = \begin{cases} 0, & \text{when } x \in \mathcal{D} \\ \frac{\bar{b}_{f_n} + \bar{b}_{g_n}|u_{ad}|}{g_l}, & \text{when } x \notin \mathcal{D}. \end{cases} \quad (15)$$

   where $\bar{b}_{f_n}, \bar{b}_{g_n}$ are defined as known upper bounds on $|f_n(x)|$ and $|g_n(x)|$ for $x \notin \mathcal{D}$, respectively. Similarly, we do not present a discussion herein for the case when $\bar{b}_{f_n}$ and $\bar{b}_{g_n}$ are unknown. For completeness, the signal $\xi_n = 0$ and k_n is a designer specified positive constant.

A. Tracking Error Dynamics

This subsection uses the control approach defined above to derive the dynamics of the tracking error. This analysis can be divided into three cases.

1) For $i = 1$:

$$\begin{aligned} \dot{\tilde{x}}_1 &= \hat{f}_1 + (\hat{g}_1 + \beta_{g_1})(\alpha_1 - \xi_2) - \dot{x}_d - \tilde{\theta}_{f_1}^\top \Phi_{f_1} \\ &\quad -\beta_{g_1}x_{2c} + (\hat{g}_1 + \beta_{g_1})(x_{2c} - x_{2c}^0) \\ &\quad +(g_1x_2 - \hat{g}_1x_{2c}) + \delta_{f_1} \\ &= -k_1\tilde{x}_1 - \tilde{\theta}_{f_1}^\top \Phi_{f_1} - \beta_{f_1} + u_{s_1} + \delta_{f_1} \\ &\quad -\beta_{g_1}x_{2c} + (\hat{g}_1 + \beta_{g_1})(x_{2c} - x_{2c}^0) \\ &\quad +(g_1x_2 - \hat{g}_1x_{2c}) - (\hat{g}_1 + \beta_{g_1})\xi_2. \end{aligned} \quad (16)$$

2) For $1 < i < n$:

$$\begin{aligned}\dot{\tilde{x}}_i &= \hat{f}_i + (\hat{g}_i + \beta_{g_i})(\alpha_i - \xi_{i+1}) - \dot{x}_{ic} - \tilde{\theta}_{f_i}^\top \Phi_{f_i} \\ &\quad -\beta_{g_i} x_{i+1,c} + (\hat{g}_i + \beta_{g_i})(x_{i+1,c} - x_{i+1,c}^0) \\ &\quad +(g_i x_{i+1} - \hat{g}_i x_{i+1,c}) + \delta_{f_i} \\ &= -k_i \tilde{x}_i - (\hat{g}_{i-1} + \beta_{g_{i-1}})\bar{x}_{i-1} - \tilde{\theta}_{f_i}^\top \Phi_{f_i} \\ &\quad -\beta_{f_i} + u_{s_i} + \delta_{f_i} - \beta_{g_i} x_{i+1,c} \\ &\quad +(g_i x_{i+1} - \hat{g}_i x_{i+1,c}) - (\hat{g}_i + \beta_{g_i})\xi_{i+1} \\ &\quad +(\hat{g}_i + \beta_{g_i})(x_{i+1,c} - x_{i+1,c}^0). \end{aligned} \tag{17}$$

3) For $i = n$:

$$\begin{aligned}\dot{\tilde{x}}_n &= f_n + g_n(u_{ad} + u_{s_n}) - \dot{x}_{nc} \\ &= \hat{f}_n + (\hat{g}_n + \beta_{g_n})u_{ad} - \dot{x}_{nc} - \tilde{\theta}_{f_n}^\top \Phi_{f_n} \\ &\quad -\beta_{g_n} u_{ad} + (g_n - \hat{g}_n)u_{ad} + \delta_{f_n} + g_n u_{s_n} \\ &= -k_n \tilde{x}_n - (\hat{g}_{n-1} + \beta_{g_{n-1}})\bar{x}_{n-1} - \beta_{f_n} \\ &\quad -\tilde{\theta}_{f_n}^\top \Phi_{f_n} - \beta_{g_n} u_{ad} + (g_n - \hat{g}_n)u_{ad} \\ &\quad +\delta_{f_n} + g_n u_{s_n}. \end{aligned} \tag{18}$$

The equations of this section will be used in the following subsection to derive the dynamics of the compensated tracking errors defined in (10).

B. Compensated Tracking Error Dynamics

From Step 1b of the procedure described in Section III, the variables $\xi_i,\ i = 1, \cdots, n-1$ are produced by filtering the unachieved portion of $x_{i+1,c}^0$. The variables $\bar{x}_i$ are referred as *compensated tracking errors*. These variables are obtained by removing the filtered unachieved portion of $x_{i+1,c}^0$ from the tracking error, as specified in eqn. (10). The dynamics of the compensated tracking errors are derived according to the three different cases in Section III-A.

1) For $i = 1$

$$\begin{aligned}\dot{\bar{x}}_1 &= -k_1 \bar{x}_1 - \tilde{\theta}_{f_1}^\top \Phi_{f_1} - \beta_{f_1} - \beta_{g_1} x_{2c} \\ &\quad +(g_1 x_2 - \hat{g}_1 x_{2c}) - (\hat{g}_1 + \beta_{g_1})\xi_2 + \delta_{f_1} + u_{s_1} \\ &= -k_1 \bar{x}_1 - \tilde{\theta}_{f_1}^\top \Phi_{f_1} - \beta_{f_1} - \beta_{g_1} x_2 \\ &\quad +(g_1 - \hat{g}_1)x_2 + (\hat{g}_1 + \beta_{g_1})\bar{x}_2 + \delta_{f_1} + u_{s_1} \\ &= -k_1 \bar{x}_1 - \tilde{\theta}_{f_1}^\top \Phi_{f_1} - \tilde{\theta}_{g_1}^\top \Phi_{g_1} x_2 - \beta_{f_1} - \beta_{g_1} x_2 \\ &\quad +(\hat{g}_1 + \beta_{g_1})\bar{x}_2 + \delta_{f_1} + \delta_{g_1} x_2 + u_{s_1}. \end{aligned} \tag{19}$$

2) Similarly, for $1 < i < n$:

$$\begin{aligned}\dot{\bar{x}}_i &= -k_i \bar{x}_i - (\hat{g}_{i-1} + \beta_{g_{i-1}})\bar{x}_{i-1} + u_{s_i} \\ &\quad -\tilde{\theta}_{f_i}^\top \Phi_{f_i} - \tilde{\theta}_{g_i}^\top \Phi_{g_i} x_{i+1} - \beta_{f_i} - \beta_{g_i} x_{i+1} \\ &\quad +(\hat{g}_i + \beta_{g_i})\bar{x}_{i+1} + \delta_{f_i} + \delta_{g_i} x_{i+1}. \end{aligned} \tag{20}$$

3) For $i = n$, because $\bar{x}_n = \tilde{x}_n$:

$$\begin{aligned}\dot{\bar{x}}_n &= -k_n \bar{x}_n - (\hat{g}_{n-1} + \beta_{g_{n-1}})\bar{x}_{n-1} - \beta_{f_n} \\ &\quad -\tilde{\theta}_{f_n}^\top \Phi_{f_n} - \beta_{g_n} u_{ad} - \tilde{\theta}_{g_n}^\top \Phi_{g_n} u_{ad} \\ &\quad +\delta_{g_n} u_{ad} + \delta_{f_n} + g_n u_{s_n}. \end{aligned} \tag{21}$$

Given eqns. (19) - (21), we are now ready to analyze the stability of the specified control law.

IV. Stability and Parameter Adaptation

We consider the following Lyapunov function candidate

$$V = \sum_{i=1}^{n} V_i(\bar{x}_i, \tilde{\theta}_{f_i}, \tilde{\theta}_{g_i}, \tilde{\Psi}_{f_i}, \tilde{\Psi}_{g_i}) \tag{22}$$

where

$$\begin{aligned}V_i &= \frac{1}{2}\Big(\bar{x}_i^2 + \tilde{\theta}_{f_i}^\top \Gamma_{f_i}^{-1}\tilde{\theta}_{f_i} + \tilde{\theta}_{g_i}^\top \Gamma_{g_i}^{-1}\tilde{\theta}_{g_i} \\ &\qquad + \tilde{\Psi}_{f_i}^\top \Gamma_{\Psi f_i}^{-1}\tilde{\Psi}_{f_i} + \tilde{\Psi}_{g_i}^\top \Gamma_{\Psi g_i}^{-1}\tilde{\Psi}_{g_i}\Big).\end{aligned}$$

with $\Gamma_{f_i},\ \Gamma_{g_i},\ \Gamma_{\Psi f_i},\ \Gamma_{\Psi g_i}, i = 1, \cdots, n$ being defined as positive definite matrices representing the learning rates. The time derivative of the V is $\dot{V} = \sum_{i=1}^{n} \dot{V}_i$, and $\dot{V}_i$ along solutions of eqns. (19 - 21) are:

1) For $i = 1$,

$$\begin{aligned}\dot{V}_1 &= -k_1 \bar{x}_1^2 + \bar{x}_1(\hat{g}_1 + \beta_{g_1})\bar{x}_2 + \bar{x}_1 u_{s_1} + \Delta_1 \\ &\quad +\tilde{\theta}_{f_1}^\top \Gamma_{f_1}^{-1}\left(\dot{\theta}_{f_1} - \Gamma_{f_1}\Phi_{f_1}\bar{x}_1\right) \\ &\quad +\tilde{\theta}_{g_1}^\top \Gamma_{g_1}^{-1}\left(\dot{\theta}_{g_1} - \Gamma_{g_1}\Phi_{g_1}\bar{x}_1 x_2\right). \end{aligned} \tag{23}$$

2) For $i = 2, \cdots, (n-1)$,

$$\begin{aligned}\dot{V}_i &= -k_i \bar{x}_i^2 - \bar{x}_{i-1}(\hat{g}_{i-1} + \beta_{g_{i-1}})\bar{x}_i \\ &\quad +\bar{x}_i(\hat{g}_i + \beta_{g_i})\bar{x}_{i+1} + \bar{x}_i u_{s_i} + \Delta_i \\ &\quad +\tilde{\theta}_{f_i}^\top \Gamma_{f_i}^{-1}\left(\dot{\theta}_{f_i} - \Gamma_{f_i}\Phi_{f_i}\bar{x}_i\right) \\ &\quad +\tilde{\theta}_{g_i}^\top \Gamma_{g_i}^{-1}\left(\dot{\theta}_{g_i} - \Gamma_{g_i}\Phi_{g_i}\bar{x}_i x_{i+1}\right). \end{aligned} \tag{24}$$

3) For i=n,

$$\begin{aligned}\dot{V}_n &= -k_n \bar{x}_n^2 - \bar{x}_n(\hat{g}_{n-1} + \beta_{g_{n-1}})\bar{x}_{n-1} \\ &\quad +\bar{x}_n g_n u_{s_n} + \Delta_n \\ &\quad +\tilde{\theta}_{f_n}^\top \Gamma_{f_n}^{-1}\left(\dot{\theta}_{f_n} - \Gamma_{f_n}\Phi_{f_n}\bar{x}_n\right) \\ &\quad +\tilde{\theta}_{g_n}^\top \Gamma_{g_n}^{-1}\left(\dot{\theta}_{g_n} - \Gamma_{g_n}\Phi_{g_n}\bar{x}_n u_{ad}\right). \end{aligned} \tag{25}$$

In the above, for $i < n$:

$$\begin{aligned}\Delta_i &= \bar{x}_i\left(-\beta_{f_i} - \beta_{g_i} x_{i+1} + \delta_{f_i} + \delta_{g_i} x_{i+1}\right) \\ &\quad +\tilde{\Psi}_{f_i}^\top \Gamma_{\Psi f_i}^{-1}\dot{\Psi}_{f_i} + \tilde{\Psi}_{g_i}^\top \Gamma_{\Psi g_i}^{-1}\dot{\Psi}_{g_i}, \end{aligned} \tag{26}$$

and for $i = n$:

$$\begin{aligned}\Delta_n &= \bar{x}_n\left(-\beta_{f_n} - \beta_{g_n} u_{ad} + \delta_{f_n} + \delta_{g_n} u_{ad}\right) \\ &\quad +\tilde{\Psi}_{f_n}^\top \Gamma_{\Psi f_n}^{-1}\dot{\Psi}_{f_n} + \tilde{\Psi}_{g_n}^\top \Gamma_{\Psi g_n}^{-1}\dot{\Psi}_{g_n}. \end{aligned} \tag{27}$$

We choose the localized adaptive laws of θ_{f_i} and $\theta_{g_i},\ i = 1, \cdots, n$ for $\|\bar{x}\| > \sqrt{\frac{\bar{\rho}+\mu}{c}}$ as

$$\dot{\theta}_{f_i} = \Gamma_{f_i}\bar{x}_i\Phi_{f_i} \tag{28}$$

$$\dot{\theta}_{g_i} = \begin{cases} Proj\{\Gamma_{g_i}\ \bar{x}_i\ x_{i+1}\ \Phi_{g_i}\}, & \text{if } i < n \\ Proj\{\Gamma_{g_n}\ \bar{x}_n\ u_{ad}\ \Phi_{g_n}\}, & \text{if } i = n \end{cases} \tag{29}$$

where a projection modification $Proj\{\cdot\}$ is used to ensure that $\hat{g}_i, i = 1, \cdots, n$ are bounded away from zero. When $\|\bar{x}\| \le \sqrt{\frac{\bar{\rho}+\mu}{c}}$, $\dot{\theta}_{f_i} = 0$ and $\dot{\theta}_{g_i} = 0$. The design parameters $\bar{\rho},\ \mu,\ c$ are defined in the discussion related to Theorem

1. Substituting (28) and (29) in (23) - (25), we obtain the derivative of V defined in eqn. (22) as

$$\dot{V} = -\sum_{i=1}^{n} k_i \bar{x}_i^2 + \sum_{i=1}^{n-1} (\bar{x}_i u_{s_i} + \Delta_i) + (g_n \bar{x}_n u_{s_n} + \Delta_n) . \quad (30)$$

Next, we will only consider the case when $x \in \mathcal{D}$ and perfect approximation is not possible. We are interested in developing bounds on the approximation error and using those bounds in the control law to achieve robustness to the approximation error. This goal is attained by defining smooth functions β_{f_i} and β_{g_i}, $i = 1, \dots, n$ as

$$\beta_{f_i} = \Psi_{f_i}^\top \Phi_{f_i}\ tanh\left(\frac{\bar{x}_i}{\epsilon}\right) = \Psi_{f_i}^\top \cdot \Omega_{f_i} \quad (31)$$

$$\beta_{g_i} = \Psi_{g_i}^\top \cdot \Omega_{g_i} \quad (32)$$

and

$$\Omega_{g_i} = \begin{cases} \Phi_{g_i}\ tanh\left(\frac{\bar{x}_i x_{i+1}}{\epsilon}\right), & \text{if } i < n \\ \Phi_{g_n}\ tanh\left(\frac{\bar{x}_n u_{an}}{\epsilon}\right), & \text{if } i = n \end{cases} \quad (33)$$

where $\epsilon > 0$ is a small design constant.

Lemma 1 of [12] provides the inequality

$$0 \leq |m| - m \cdot tanh\,(m/\epsilon) \leq \eta\epsilon$$

for any $\epsilon > 0$ and for any $m \in R$, where η is a constant that satisfies $\eta = e^{-(\eta+1)}$ (i.e. $\eta = 0.2785$). This Lemma is used below.

With this definition, starting from (26) and (27), we can reduce the expression for Δ_i:

$$\begin{aligned} \Delta_i \leq\ & (\Psi_{f_i}^M)^\top \Phi_{f_i} |\bar{x}_i| + (\Psi_{g_i}^M)^\top \Phi_{g_i} |\bar{x}_i x_{i+1}| \\ & -(\Psi_{f_i}^M + \tilde{\Psi}_{f_i})^\top \Phi_{f_i} \bar{x}_i\ tanh\left(\frac{\bar{x}_i}{\epsilon}\right) \\ & -(\Psi_{g_i}^M + \tilde{\Psi}_{g_i})^\top \Phi_{g_i} \bar{x}_i x_{i+1}\ tanh\left(\frac{\bar{x}_i x_{i+1}}{\epsilon}\right) \\ & +\tilde{\Psi}_{f_i}^\top \Gamma_{\Psi f_i}^{-1} \dot{\Psi}_{f_i} + \tilde{\Psi}_{g_i}^\top \Gamma_{\Psi g_i}^{-1} \dot{\Psi}_{g_i} \\ \leq\ & \eta\epsilon\left((\Psi_{f_i}^M)^\top \Phi_{f_i} + (\Psi_{g_i}^M)^\top \Phi_{g_i}\right) \\ & +\tilde{\Psi}_{f_i}^\top \Gamma_{\Psi f_i}^{-1} \left(\dot{\Psi}_{f_i} - \Gamma_{\Psi f_i} \Omega_{f_i} \bar{x}_i\right) \\ & +\tilde{\Psi}_{g_i}^\top \Gamma_{\Psi g_i}^{-1} \left(\dot{\Psi}_{g_i} - \Gamma_{\Psi g_i} \Omega_{g_i} \bar{x}_i x_{i+1}\right) \end{aligned} \quad (34)$$

and similarly

$$\begin{aligned} \Delta_n \leq\ & \eta\epsilon(\Psi_{f_n}^M)^\top \Phi_{f_n} + \frac{|u_{ad}|}{|u_{an}|}\eta\epsilon(\Psi_{g_n}^M)^\top \Phi_{g_n} \\ & +\tilde{\Psi}_{f_n}^\top \Gamma_{\Psi f_n}^{-1} \left(\dot{\Psi}_{f_n} - \Gamma_{\Psi f_n} \Omega_{f_n} \bar{x}_n\right) \\ & +\tilde{\Psi}_{g_n}^\top \Gamma_{\Psi g_n}^{-1} \left(\dot{\Psi}_{g_n} - \Gamma_{\Psi g_n} \Omega_{g_n} \bar{x}_n u_{ad}\right) \end{aligned} \quad (35)$$

where the extension of Lemma 1 in [12], which provides the inequality[2]

$$|\bar{x}_n u_{ad}| - \bar{x}_n u_{ad}\ tanh\,(\bar{x}_n u_{an}/\epsilon) \leq \left|\frac{\bar{x}_n u_{ad}}{\bar{x}_n u_{an}}\right| \eta\epsilon,$$

is used.

[2]Note that u_{an} and u_{ad} have the same sign since the denominator of the control equation is ensured to be bounded away from zero such that $\theta_{g_n}^\top \Phi_{g_n} + \beta_{g_n} > g_l > 0$.

Based on the inequalities (34–35), for $\|\bar{x}\| > \sqrt{\frac{\bar{\rho}+\mu}{c}}$ the localized adaptive laws of Ψ_{f_i} and Ψ_{g_i} with σ-modification are selected as

$$\begin{aligned} \dot{\Psi}_{f_i} &= \Gamma_{\Psi f_i}\Big[\bar{x}_i tanh\left(\frac{\bar{x}_i}{\epsilon}\right) \\ &\quad - \sigma_{\Psi f i}\ diag(\Psi_{f_i} - \Psi_{f_i}^0)\Big]\Phi_{f_i}, \quad (36) \\ \dot{\Psi}_{g_i} &= Proj\{\tau_{\Psi g_i}\} \quad (37) \end{aligned}$$

where

$$\tau_{\Psi g_i} = \begin{cases} \Gamma_{\Psi g_i}\Big(\bar{x}_i x_{i+1}\ tanh\left(\frac{\bar{x}_i x_{i+1}}{\epsilon}\right) \\ \quad - \sigma_{\Psi g i}\ diag(\Psi_{g_i} - \Psi_{g_i}^0)\Big)\Phi_{g_i}, \text{ if } i < n \\ \Gamma_{\Psi g_n}\Big(\bar{x}_n u_{ad}\ tanh\left(\frac{\bar{x}_n u_{an}}{\epsilon}\right) \\ \quad - \sigma_{\Psi g n}\ diag(\Psi_{g_n} - \Psi_{g_n}^0)\Big)\Phi_{g_n}, \text{ if } i = n \end{cases}$$

where $diag(v)$ is the square diagonal matrix with diagonal components equal to the vector v; $\sigma_{\Psi f_i}, \sigma_{\Psi g_i} > 0$; and $\Psi_{f_i}^0$ and $\Psi_{g_i}^0$ are design parameters (vectors). When $\|\bar{x}\| \leq \sqrt{\frac{\bar{\rho}+\mu}{c}}$, $\dot{\Psi}_{f_i} = 0$ and $\dot{\Psi}_{g_i} = 0$.

Note that all u_{s_i} terms in (30) are zero for $x \in \mathcal{D}$. If we substitute (34–35) and (36–37) into (30), we attain

$$\begin{aligned} \dot{V} \leq\ & -\sum_{i=1}^{n} k_i \bar{x}_i^2 + \eta\epsilon \sum_{i=1}^{n} (\Psi_{f_i}^M)^\top \Phi_{f_i} \\ & +\eta\epsilon\left(\sum_{i=1}^{n-1} (\Psi_{g_i}^M)^\top \Phi_{g_i} + \frac{|u_{ad}|}{|u_{an}|}(\Psi_{g_n}^M)^\top \Phi_{g_n}\right) \\ & -\sum_{i=1}^{n}\Big(\sigma_{\Psi f i} \tilde{\Psi}_{f_i}^\top R_{f_i} (\Psi_{f_i} - \Psi_{f_i}^0) \\ & \quad +\sigma_{\Psi g i} \tilde{\Psi}_{g_i}^\top R_{g_i} (\Psi_{g_i} - \Psi_{g_i}^0)\Big) \end{aligned}$$

where $R_{f_i} = diag(\Phi_{f_i})$ and $R_{g_i} = diag(\Phi_{g_i})$. After applying the equation

$$\begin{aligned} \tilde{a}^\top R(a - a^0) =\ & \frac{1}{2}\tilde{a}^\top R \tilde{a} + \frac{1}{2}(a - a^0)^\top R (a - a^0) \\ & -\frac{1}{2}(a^* - a^0)^\top R (a^* - a^0) \end{aligned}$$

to the two terms in the last summation, with the vector a replaced by Ψ_{f_i} and Ψ_{g_i}, $i = 1, \cdots, n$, respectively, we have

$$\dot{V} \leq -c\,\|\bar{x}\|^2 + d_0 + \rho \quad (38)$$

where c, d_0 and ρ are all positive constants given by

$$c = \min_{i=1,\dots,n} \{k_i\} \quad (39)$$

$$\begin{aligned} d_0 =\ & \eta\epsilon \sum_{i=1}^{n} (\Psi_{f_i}^M)^\top \Phi_{f_i} + \eta\epsilon\Bigg(\sum_{i=1}^{n-1} (\Psi_{g_i}^M)^\top \Phi_{g_i} \\ & + \frac{|u_{ad}|}{|u_{an}|}(\Psi_{g_n}^M)^\top \Phi_{g_n}\Bigg) \end{aligned} \quad (40)$$

$$\begin{aligned} \rho =\ & \frac{1}{2}\sum_{i=1}^{n}\left[\sigma_{\Psi f i}(\Psi_{f_i}^M - \Psi_{f_i}^0)^\top R_{f_i} (\Psi_{f_i}^M - \Psi_{f_i}^0)\right. \\ & \left. +\sigma_{\Psi g i}(\Psi_{g_i}^M - \Psi_{g_i}^0)^\top R_{g_i} (\Psi_{g_i}^M - \Psi_{g_i}^0)\right] . \end{aligned} \quad (41)$$

Let $\bar{\rho} > (d_0 + \rho)$ be a strict upper bound on $(d_0 + \rho)$. Select a small design constant $\mu > 0$. For $\|\bar{x}\| > \sqrt{\frac{\bar{\rho}+\mu}{c}}$, $\dot{V} < -\mu < 0$. When $\|\bar{x}\| \leq \sqrt{\frac{\bar{\rho}+\mu}{c}}$, a deadzone is included in the adaptive laws. Without such a deadzone, stability is not guaranteed and the parameters could drift.

Note that $m\ tanh(m/\epsilon) \geq 0$; therefore, without leakage terms $\dot{\Psi}_{f_i}$ and $\dot{\Psi}_{g_i}$ would always be non-negative .

The stability properties are summarized in the following theorem:

Theorem 1: Assuming the upper bound $\bar{\rho} > d_0 + \rho > 0$ is known, for the system described by (1)-(2) with the adaptive feedback control law of eqns. (13-15) and the parameter adaptation laws of eqns. (28–29) and (36–37), we have the following stability properties:

1) For $x(0) \notin \mathcal{D}$, $x(t)$ for $t > 0$ converges to region $\mathcal{D}$ in finite time.
2) When $x \in \mathcal{D}$ and $\delta_{f_i} = \delta_{g_i} = 0$:
   a) $\bar{x}_i \in \mathcal{L}_2$;
   b) $\bar{x}_i \to 0$ as $t \to \infty$;
   c) $\bar{x}_i, \theta_{f_i}, \theta_{g_i} \in \mathcal{L}_\infty$.
3) When $x \in \mathcal{D}$ and $\delta_{f_i} \neq 0$ or $\delta_{g_i} \neq 0$:
   a) $\bar{x}_i$, $\tilde{x}_i$, $\tilde{\theta}_{f_i}$, $\tilde{\theta}_{g_i}$, $\tilde{\Psi}_{f_i}$, $\tilde{\Psi}_{g_i} \in \mathcal{L}_\infty$;
   b) x_i, θ_{f_i}, θ_{g_i}, Ψ_{f_i}, $\Psi_{g_i} \in \mathcal{L}_\infty$;
   c) $\dot{\bar{x}}_i$, $\dot{\theta}_{f_i}$, $\dot{\theta}_{g_i}$, $\dot{\Psi}_{f_i}$, $\dot{\Psi}_{g_i} \in \mathcal{L}_\infty$;
   d) $\bar{x}$ is $\bar{\rho}$-small in the m.s.s. [4].
   e) The total time outside the deadzone is finite.
   f) $\|\bar{x}\|$ is ultimately bounded by $\|\bar{x}\| \leq \sqrt{\frac{\bar{\rho}_2+\mu}{c}}$, as $t \to \infty$.

Pf.

1) When $x \notin \mathcal{D}$: We want to show that all initial conditions will return to and stay within region $\mathcal{D}$. Note that the Φ_{f_i}, Φ_{g_i}, $\dot{\theta}_{f_i}$, $\dot{\theta}_{g_i}$, β_{f_i}, β_{g_i} terms are all zero, and $\delta_{f_i} = f_i(x)$ and $\delta_{g_i} = g_i(x) - g_l$ for $i = 1, \cdots, n$. Therefore, For $x \notin \mathcal{D}$, we consider the Lyapunov function as

$$\bar{V} = \frac{1}{2}\sum_{i=1}^{n}\bar{x}_i^2. \tag{42}$$

The derivative of $\bar{V}$ can be easily shown to be similar to (30), where Δ_i terms defined in eqns. (26 - 27) are simplified to

$$\Delta_i = \begin{cases} \bar{x}_i\left(f_i + (g_i - g_l)x_{i+1}\right), & \text{for } 1 \leq i < n \\ \bar{x}_n\left(f_n + (g_n - g_l)u_{ad}\right), & \text{for } i = n. \end{cases}$$

Applying the sliding control of (8) and (14), we obtain the derivative of $\bar{V}$ defined in eqn. (42) as

$$\begin{aligned} \frac{d\bar{V}}{dt} \leq & -\sum_{i=1}^{n}k_i\bar{x}_i^2 + \sum_{i=1}^{n-1}\left(-r_i|\bar{x}_i| + |\Delta_i|\right) \\ & + \left(-g_n r_n|\bar{x}_n| + |\Delta_n|\right). \end{aligned} \tag{43}$$

Since the sliding gains of (9) and (15) yield, for $i < n$

$$r_i|\bar{x}_i| = (\bar{b}_{f_i} + \bar{b}_{g_i}|x_{i+1}|)|\bar{x}_i| \geq |\Delta_i|,$$

and, for $i = n$

$$g_n r_n|\bar{x}_n| = \frac{g_n}{g_l}(\bar{b}_{f_n} + \bar{b}_{g_n}|u_{ad}|)|\bar{x}_n| \geq |\Delta_n|.$$

Then, we attain

$$\frac{d\bar{V}}{dt} \leq -\sum_{i=1}^{n}k_i\bar{x}_i^2 < -\underline{k}\bar{V} \tag{44}$$

$$\bar{V}(t) \leq e^{-\underline{k}t}\bar{V}(0), \text{ for any } t \geq 0. \tag{45}$$

Then, for any t larger than some finite time called T_2, $\bar{V}(t) < \frac{\gamma^2}{8}$ which implies that $\|\bar{x}(t)\| < \frac{\gamma}{2}$. In addition, for $x \notin \mathcal{D}$ $|\xi_i| < \bar{b}_\xi$ where $\bar{b}_\xi = \frac{2\rho(\mathcal{D})g_l}{\underline{k}}$ by methods similar to those used to derive (12). Therefore, we can attain $\|\xi(t)\| < \frac{\gamma}{2}$ by choosing g_l sufficiently small for $x \notin \mathcal{D}$. Therefore, for $t > T_2$,

$$\|\tilde{x}(t)\| \leq \|\bar{x}(t)\| + \|\xi(t)\| < \gamma$$

which implies that x returns to within $\mathcal{D}$ in finite time. Once $x \in \mathcal{D}$, the sliding mode term will not allow x to leave $\mathcal{D}$.

The reminder of this proof will only be concerned with the case of $x \in \mathcal{D}$, where each sliding control term u_{s_i} is zero. For $x \in \mathcal{D}$, we will continue the analysis of $\dot{V}$ for V defined in (22).

2) When $x \in \mathcal{D}$ and $\delta_{f_i} = \delta_{g_i} = 0$: Note that all u_{s_i} terms in (30) are zero for $x \in \mathcal{D}$. In addition, for the ideal case of perfect approximation, the β_{f_i}, β_{g_i}, δ_{f_i}, and δ_{g_i} terms are identically zero, which yields directly $\Delta_i = 0$. Then, (30) is simplified as

$$\frac{dV}{dt} \leq -\sum_{i=1}^{n}k_i\bar{x}_i^2 \tag{46}$$

which is negative semi-definite. This implies that the variables $\bar{x}_i, \theta_{f_i}, \theta_{g_i}$ are each bounded. Since each term of $\dot{\bar{x}}_i$ is bounded, $\ddot{V}$ can be directly shown to be bounded. Barbalat's lemma implies that each $\bar{x}_i$ approaches zero as t approaches infinity. Finally, integrating both sides of (46) yields

$$V(0) \geq \sum_{i=1}^{n}\int_0^t k_i\bar{x}_i^2(\tau)d\tau,$$

which shows that each $\bar{x}_i$ is in $\mathcal{L}_2$.

3) When $x \in \mathcal{D}$ and $\delta_{f_i} \neq 0$ or $\delta_{g_i} \neq 0$: Starting from the inequality (38), the derivative of V for $\|\bar{x}\| > \sqrt{\frac{\bar{\rho}+\mu}{c}}$ is:

$$\dot{V} \leq -c\,\|\bar{x}\|^2 + d_0 + \rho < -\mu < 0 \tag{47}$$

where c, d_0 and ρ are given as (39–41), respectively. Therefore, if $\|\bar{x}\| > \sqrt{\frac{\bar{\rho}+\mu}{c}}$, then V is decreasing. If $\|\bar{x}\| \leq \sqrt{\frac{\bar{\rho}+\mu}{c}}$ then $\tilde{\theta}_{f_i}, \tilde{\theta}_{g_i}, \tilde{\Psi}_{f_i}$ and $\tilde{\Psi}_{g_i}$ are all constant and $\|\bar{x}\|$ is bounded. Thus, $V(t)$ is bounded by the maximum of $V(0)$ or $\max_{\|\bar{x}\|=\sqrt{\frac{\bar{\rho}+\mu}{c}}}\left(V(\bar{x}, \tilde{\theta}_{f_i}(0), \tilde{\theta}_{g_i}(0), \tilde{\Psi}_{f_i}(0), \tilde{\Psi}_{g_i}(0))\right)$ which shows that $\bar{x}_i$, $\tilde{\theta}_{f_i}$, $\tilde{\theta}_{g_i}$, $\tilde{\Psi}_{f_i}$, $\tilde{\Psi}_{g_i} \in \mathcal{L}_\infty$. The

$\mathcal{L}_\infty$ property of $\tilde{x}_i$ comes from the fact that each ξ_i is bounded. Properties 3b, 3c can be similarly shown. For the proof of 3d, we integrate (38) to obtain

$$c\int_0^t \|\bar{x}\|^2\ d\tau \le V(0) + \int_0^t (d_0+\rho)\ d\tau$$

which implies that $\bar{x}$ is $\bar{\rho}$-small in the mean square sense (m.s.s.).

Next, we will show the Property 3e. Assume x starts at t_0 outside the deadzone, enters the deadzone at t_{2i-1}, and leaves it at t_{2i}, for $i \ge 1$. Then, during the interval $t \in [t_{2i-1}, t_{2i}]$, there is no parameter update; $\|\bar{x}(t_{2i-1})\| = \|\bar{x}(t_{2i})\|$, thus

$$V(t_{2i-1}) = V(t_{2i}),$$

and outside the deadzone according to (47),

$$V(t_{2i+1}) - V(t_{2i}) < -\mu(t_{2i+1} - t_{2i}).$$

Therefore, the total time outside the deadzone is

$$T_d = (t_1 - t_0) + \sum_{i\ge1}(t_{2i+1} - t_{2i}),$$

and

$$\begin{aligned} T_d &< \frac{1}{\mu}\Big(V(t_0) - V(t_1) + \sum_{i\ge1}(V(t_{2i}) - V(t_{2i+1}))\Big) \\ &< \frac{1}{\mu}\Big(V(t_0) - V(t_1) + \sum_{i\ge1}(V(t_{2i-1}) - V(t_{2i+1}))\Big) \\ &< \frac{V(t_0)}{\mu} \end{aligned}$$

which is a finite value. Property 3f comes directly from the Property 3e.

■

The formulation of localized adaptive laws as defined in eqns. (36–37) localizes the effects of leakage terms to the vicinity of the present operating point, thus eliminating the problem with global forgetting. Localized forgetting also decreases the required amount of on-line computation, since all parameters associated with zero elements of basis vectors are left unchanged.

In addition, due to the inclusion in ρ of R_{f_i} and R_{g_i}, which are local functions of the operating point, the m.s.s. bound can be shown to be significantly smaller than the bound derived from the previously existing approaches.

V. Conclusions and Open Issues

We have considered in this paper the robust adaptive control design for a wide class of n-th order uncertain nonlinear systems. A novel robust adaptive backstepping design procedure is proposed by incorporating the locally learned adaptive bounding functions on the *residual approximation errors*. This is an extension of the localized adaptive bounding technique proposed in [17] to higher order systems with $g_i(x) \neq 1$. Furthermore, the complexity of calculating time derivatives of intermediate state commands for the backstepping approach [6], [7] is addressed by the command filtering techniques proposed in [3]. We have proved that the overall adaptive scheme can guarantee the boundedness of both actual tracking errors and compensated tracking errors, by applying the Lyapunov stability analysis.

In addition, we successfully show that the localized adaptation algorithms with deadzone and parameter projection modification is effective to prevent the parameter drift and to guarantee the ultimate boundedness of the compensated tracking errors $\bar{x}$. Since we have shown that the m.s.s. bound on $\bar{x}$ is on the order of the residual function approximation errors, our future extension will focus on the adaptive enhancement of the structure of the approximator to achieve better tracking performance.

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