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A Note on Convergence in Maximal Solution Problems for Infinite Markov Jump Linear Systems

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***Abstract*— In the context of infinite Markov jump linear systems (IMJLS), stochastic stability (a sort of L_2-stability) is a structural concept intimately related to a certain bounded linear operator ($\mathcal{D}$). Infinite (or finite) here, has to do with the state space of the Markov chain being finite or infinite countable. In the path to solving the maximal solution problem in the infinite countable case, a certain sequence of bounded linear operators (which converges trivially to $\mathcal{D}$ in the finite case) arises and convergence in the norm topology (uniform operator topology) becomes a relevant point. In this paper, we provide a condition that insures that this convergence also holds in the infinite countable case. This condition is automaticaly satisfied when we reduce the problem to the finite case. The issue of whether this is a restrictive condition or not, is brought to light using arguments that stems from the probabilistic nature of the Markov chain. This, in conjunction with a class of counterexamples, unveil further differences between the finite and the infinite countable case. We also establish a (weaker) condition for the spectrum of the limit of the above sequence of operators being in the closed left half-plane of the complex numbers.**

I. INTRODUCTION

We address to a class of dynamical systems, where parameters vary according to a Markov chain that takes values in a countably infinite state space (see [11]), described by the following stochastic differential equation:

$$(A,B,\Lambda): \begin{cases} \dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t), & t \geq 0 \\ x(0) = x_0, \quad \theta(0) = \theta_0. \end{cases} \tag{1}$$

In the above equation, $x(t) \in \mathbb{C}^n$ denotes the state vector, $u(t) \in \mathbb{C}^m$ the control input and $\{\theta(t),\ s \leq t \leq T\}$ a standard conservative Markov chain with infinitesimal matrix Λ and a countably infinite state space $\mathcal{S} = \{1, 2, ...\}$. We consider (x_0, θ_0) an initial joint random variable. Randomness is introduced in the parameters by means of some correspondence $i \mapsto \eta_i$, for $\theta(t) = i$, η_i standing for the system matrices A_i or B_i, which are all norm bounded uniformly on i. In the specialized control literature, the so-called Infinite Markov Jump Linear Systems (IMJLSs) are those according to the above stochastic equation.

Markov Jump Linear Systems (MJLS) model physical systems that have their structures subject to abrupt changes. Without any intention of being exhaustive, we mention [8], [10], [14], [18], [20] and the references therein as a sample of works dealing with stability, optimal control, filtering, H_∞-control and Riccati differential equations. Results on maximal solution, which is the subject matter of this paper, can be found in [14] and [15]. We mention also [3], [16] and [19] as works dealing with applications. Potential applications include safety-critical and high-integrity systems (e.g., aircraft, chemical plants, nuclear power station, robotic manipulator systems and large scale flexible structures for space stations such as antenna, solar arrays, etc.). A common feature in these papers is that they deal with the case where the state space of the Markov chain is finite, i.e., $\mathcal{S} = \{1, ..., N\}$. First results for infinite horizon continuous-time control problems, with MJLSs within the framework of a countably infinite state space for the Markov chain, were obtained in [11], [12] and [13]. These problems are tackled with the appropriate concept of stochastic stability (SS), to which a certain bounded linear operator ($\mathcal{D}$) in an infinite dimensional Banach space ($\mathcal{H}_1^n$) is intimately related. More explicitly, a MJLS is stochastically stable iff the spectrum of $\mathcal{D}$ lies in the open left-half plane of the complex numbers. This parallel the standard deterministic case where stability means having the eigenvalues of the system's matrix in this same subset.

Another problem, where stochastic stability plays an important role, and therefore the operator $\mathcal{D}$, is concerned with the existence of maximal solution to a certain infinite countable set of coupled algebraic Riccati equation associated to IMJLSs, which is denoted in [2] by *BPARE* (an abreviation for Banach space perturbed algebraic Riccati equation). This sort of problem has been treated in [7] for the finite dimensional case by using stability in the usual sense plus an inconvenient contraction assumption (assumption 2.1 of this reference) originally introduced in [21] (essentially, this assumption imposes the rate of transition between modes not to be too large). In the path to solving the maximal solution problem in the infinite countable case and free from the contraction assumption (see [2]), a certain sequence of bounded linear operators (which converges trivially to $\mathcal{D}$ in the finite case) arises and convergence in the norm topology (uniform operator topology) becomes a relevant point. Another point of interest is whether the maximal solution is a strong solution to the BPARE.

In this paper, motivated by the fact that, unlike the finite dimensional case, spectral continuity may not occur everywhere in the space of all bounded linear operators (see, e.g., [4] and [17, pp 56]), we exhibit a weaker sufficient condition than that of spectral continuity to having a strong (maximal) solution. Spectral continuity is therefore shown not to be a

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necessary condition to this purpose. In the sequel, we provide a condition, given by (13)/(14), that insures convergence in the norm topology of the sequence of operators mentioned above. This condition is automatically satisfied when we reduce our problem to the finite case. The issue whether this is a restrictive condition or not (we shall conclude that it is not) is brought to light using arguments that stem from the probabilistic nature of the Markov chain and so avoiding a direct convergence analysis. This, in conjunction with a class of counterexamples, unveil further differences between the finite and the infinite countable case. Except for the discussion about condition (13)/(14), the arguments and proofs here encompasses a wider class of operators sequences than the one afore mentioned.

II. NOTATIONS AND PRELIMINARIES

As usual, $\mathbb{C}^n$ stands for the complex n-space. We denote by $\mathcal{S}$ the countably infinite set $\mathcal{S} = \mathbb{N} = \{1, 2, ...\}$. In the case of control problems involving linear systems with Markov jump parameters, $\mathcal{S}$ corresponds to the state space of the Markov chain. We use the superscript * for conjugate transpose of a matrix. We call $\mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ the normed linear space of all n by m complex matrices and, for simplicity, write $\mathbb{M}(\mathbb{C}^n)$ whenever $n = m$. The notation $L \geq 0$ is adopted if a self-adjoint matrix is nonnegative and we write $\mathbb{M}(\mathbb{C}^n)^+ = \{L \in \mathbb{M}(\mathbb{C}^n); L = L^* \geq 0\}$. Furthermore, I_n stands for the identity operator in $\mathbb{M}(\mathbb{C}^n)$.

We denote by $\|\cdot\|$ the norm in $\mathbb{C}^n$ or the spectral induced norm in $\mathbb{M}(\mathbb{C}^n)$. We set $\mathcal{H}_1^{m,n}$ (respectively $\mathcal{H}_\infty^{m,n}$) the linear space of all infinite sequences of complex matrices $H = (H_1, H_2, ...)$, $H_i \in \mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ such that $\sum_{i=1}^\infty \|H_i\| < \infty$ (respectively $\sup\{\|H_i\|, \ i \in \mathcal{S}\} < \infty$) and write $\mathcal{H}_1^n$ and $\mathcal{H}_\infty^n$ whenever $n = m$. For $H \in \mathcal{H}_1^{m,n}$ (respectively $H \in \mathcal{H}_\infty^{m,n}$) we define $\|H\|_1 = \sum_{i=1}^\infty \|H_i\|$ (respectively $\|H\|_\infty = \sup\{\|H_i\|, \ i \in \mathcal{S}\}$) the norm in the Banach space $(\mathcal{H}_1^{m,n}, \|\cdot\|_1)$ (respectively $(\mathcal{H}_\infty^{m,n}, \|\cdot\|_\infty)$).

We define the nonnegative sets $\mathcal{H}_1^{n+} = \{H \in \mathcal{H}_1^n, H_i \in \mathbb{M}(\mathbb{C}^n)^+, i \in \mathcal{S}\}$ and $\mathcal{H}_\infty^{n+} = \{H \in \mathcal{H}_\infty^n, H_i \in \mathbb{M}(\mathbb{C}^n)^+, i \in \mathcal{S}\}$, the strictly positive set $\breve{\mathcal{H}}_\infty^{n+} = \{H \in \mathcal{H}_\infty^{n+}, H_i > \alpha_H I$ for some $\alpha_H > 0$, $i \in \mathcal{S}\}$ and the sets $\mathcal{H}_1^{n*} = \{H \in \mathcal{H}_1^n, H_i^* = H_i, i \in \mathcal{S}\}$ and $\mathcal{H}_\infty^{n*} = \{H \in \mathcal{H}_\infty^n, H_i^* = H_i, i \in \mathcal{S}\}$. For $H = (H_1, H_2, ...)$ and $L = (L_1, L_2, ...)$ in $\mathcal{H}_1^{n*}$ or $\mathcal{H}_\infty^{n*}$, we say that $H \leqslant L$ if $H_i \leqslant L_i$ for each i in $\mathcal{S}$ and, for L and H in $\mathcal{H}_\cdot^{n+}$, we have that $H \leqslant L \Rightarrow \|H\|_1 \leqslant \|L\|_1$ and $\|H\|_\infty \leqslant \|L\|_\infty$. For $C = (C_1, C_2 ...) \in \mathcal{H}_\infty^n$, we denote $C^* = (C_1^*, C_2^* ...) \in \mathcal{H}_\infty^n$ and $C^{-1} = (C_1^{-1}, C_2^{-1} ...) \in \mathcal{H}_\infty^n$ whenever C_i^{-1}, $i \in \mathcal{S}$, are invertible.

We represent by $(l_1, \|\cdot\|_1)$ and $(l_\infty, \|\cdot\|_\infty)$ the spaces made up of all infinite sequences of complex numbers $x = (x_1, x_2, ...)$ such that $\|x\|_1 = \sum_{i=1}^\infty |x_i| < \infty$ and $\|x\|_\infty = \sup\{|x_i|, \ i = 1, 2, ...\} < \infty$, respectively. It is easy to verify that $(\mathcal{H}_\infty^{m,n}, \|\cdot\|_\infty)$ and $(l_\infty, \|\cdot\|_\infty)$ are uniformly homeomorphic. Since $(l_\infty, \|\cdot\|_\infty)$ is a Banach space, $(\mathcal{H}_\infty^{m,n}, \|\cdot\|_\infty)$ is also a Banach space. The same stands for $(\mathcal{H}_1^{m,n}, \|\cdot\|_1)$ and $(l_1, \|\cdot\|_1)$.

For any complex Banach space Y, we write $Blt\,(Y)$ for the Banach space of all bounded linear transformations of Y into Y with the norm topology (generated by the uniform induced norm and denoted by $\|\cdot\|_B$) and, for $L \in Blt\,(Y)$, we refer to $\sigma\,(L)$ as the spectrum of L. $Blt\,(\mathcal{H}_1^n)$ here is a Banach algebra with identity (the identity operator in $Blt\,(\mathcal{H}_1^n)$). In addition, we define the product of an element $A \in \mathcal{H}_\eta^{m,n}$ by another element $B \in \mathcal{H}_\nu^{q,m}$ by

$$AB = (A_1 B_1, A_2 B_2, ...), \tag{2}$$

where η and ν stands either for ∞ or 1. AB then belongs either to $\mathcal{H}_\infty^{q,n}$ or $\mathcal{H}_1^{q,n}$, as we shall see below (it is worth noticing that both $\mathcal{H}_\infty^n$ and $\mathcal{H}_1^n$ equipped with (2) are Banach algebras with identity $(I_n, I_n, ...)$). Finally, we denote by $E[\cdot]$ the expectation operator.

We conclude this section with an auxiliary result.

Lemma 1: For every $A \in \mathcal{H}_\infty^{m,n}$, $B \in \mathcal{H}_1^{q,m}$, $C \in \mathcal{H}_1^{m,n}$ and $D \in \mathcal{H}_\infty^{q,m}$,

(i) AB and CD belong to $\mathcal{H}_1^{q,n}$ where $\|AB\|_1 \leqslant \|A\|_\infty \|B\|_1$ and $\|CD\|_1 \leqslant \|C\|_1 \|D\|_\infty$, and

(ii) AD belongs to $\mathcal{H}_\infty^{q,n}$ where $\|AD\|_\infty \leqslant \|A\|_\infty \|D\|_\infty$.

Proof: Each entry of AB, CD and AD is finite and, from (2),

(i) $\|AB\|_1 = \sum_{i=1}^\infty \|A_i B_i\| \leqslant \sum_{i=1}^\infty \|A_i\| \|B_i\| \leqslant \|A\|_\infty \|B\|_1$. Similarly for $\|CD\|_1$.

(ii) $\|AD\|_\infty = \sup_{i \in \mathcal{S}} \|A_i D_i\| \leqslant \sup_{i \in \mathcal{S}} \|A_i\| \sup_{i \in \mathcal{S}} \|D_i\| = \|A\|_\infty \|D\|_\infty$ ■

III. STABILITY ASPECTS AND PROBLEM STATEMENT

Let the operator $\Gamma = (\Gamma_1, \Gamma_2, ...) \in Blt(\mathcal{H}_1^n)$ be as it appears in control problems involving MJLSs with infinite countable state space $\mathcal{S}$ (see, e.g., [11] and references therein):

$$\Gamma_i(W) = \sum_{j=1, j \neq i}^\infty \lambda_{ji} W_j, \ i \in \mathcal{S} \tag{3}$$

$W = (W_1, W_2, ...) \in \mathcal{H}_1^n$. Or else, viewing W as an infinite column of matrices, $\Gamma W = ((\Lambda - \ diag(\lambda_{ii})) \otimes I_n)^* W$, where $\Lambda = [\lambda_{ij}]_{i,j \in \mathcal{S}}$ is the infinitesimal matrix of a standard conservative Markov chain $\{\theta\}$ with values in $\mathcal{S}$, $\lambda_{ij} \geq 0$, $i \neq j$, $0 < -\lambda_{ii} = \sum_{j=1, j \neq i}^\infty \lambda_{ij} \leqslant cte < \infty$.

For $A \in \mathcal{H}_\infty^n$, $B \in \mathcal{H}_\infty^{m,n}$ and $K \in \mathcal{H}_\infty^{n,m}$, define the operator $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2, ...) \in Blt(\mathcal{H}_1^n)$ such that, for every $W = (W_1, W_2, ...) \in \mathcal{H}_1^n$ and each $i \in \mathcal{S}$,

$$\begin{aligned} \mathcal{D}_i(W) &= (A_i + \tfrac{1}{2}\lambda_{ii} I_n - B_i K_i) W_i \\ &+ W_i (A_i + \tfrac{1}{2}\lambda_{ii} I_n - B_i K_i)^* + \Gamma_i(W). \end{aligned} \tag{4}$$

Note that Γ_i is responsible for the interconnection among the individual components $\mathcal{D}_i$. In view of (2) and (4), $\mathcal{D}$ also writes

$$\begin{aligned} \mathcal{D}(W) &= (A + \tfrac{1}{2}\lambda I - BK) W \\ &+ W (A + \tfrac{1}{2}\lambda I - BK)^* + \Gamma(W), \end{aligned} \tag{5}$$

where $\lambda := (\lambda_{11}, \lambda_{22}, ...)$ and $I := (I_n, I_n, ...)$. Clearly, $A + \frac{1}{2}\lambda I - BK \in \mathcal{H}_\infty^n$. To see that $\mathcal{D} \in Blt\,(\mathcal{H}_1^n)$ refer to [11]). In order to emphasize the structural aspect, which associate the equation to the relevant parameters, we denote by (A, B, Γ) the differential equation

$$\dot{W}(t) = \mathcal{D}(W(t)), \quad t \geq 0. \tag{6}$$

We define now the following L_2-type of stability, where we preserve the nomenclature of the MJLS scenario:

Definition 2 (Stochastic Stability for (6)): We say that (A, B, Γ) is stochastically stabilizable (SS) if there exists (a stabilizing) $K \in \mathcal{H}_{\infty}^{n,m}$ such that, for any $W(0) \in \mathcal{H}_1^n$,

$$\int_0^{\infty} \|W(t)\|_1 \, dt < \infty, \tag{7}$$

where $W(t) \in \mathcal{H}_1^n$ is given by (6).

Invoking Lemmas 4.3 and 6.6 of [11], we may link stability with the spectrum of $\mathcal{D}$ as below.

Lemma 3: The following assertions are equivalent.

(a) (A, B, Γ) is stochastically stabilizable (SS) with stabilizing K.

(b) $\sup\{\ Re\lambda : \lambda \in \sigma(\mathcal{D})\} < 0$.

By its turn, the concept of stochastic stability for MJLSs is as follows (see [11] and [12]).

Definition 4 (Stochastic Stability for (1)): We say that (A, B, Λ) is stochastically stabilizable (SS) if there exists (a stabilizing) $K \in \mathcal{H}_{\infty}^{n,m}$ such that, for any joint initial distribution ϑ_0,

$$\int_0^{\infty} E[\|x(t)\|^2] dt < \infty,$$

where $x(t)$ is given by (1) with $u(t) = -K_{\theta(t)}x(t)$, or else, by

$$(A, B, \Lambda) : \begin{cases} \dot{x}(t) = (A_{\theta(t)} - B_{\theta(t)}K_{\theta(t)})x(t), & t \geq 0 \\ x(0) = x_0, \quad \theta(0) = \theta_0. \end{cases} \tag{8}$$

The relevance of the operator $\mathcal{D}$ stems from the fact that the above definitions and assertion (b) of Lemma 3 are equivalent. An argument for this is that equation (6) describes the behavior of a version of the state correlation matrix running in the MJLSs. We shall omit references to the BPARE stated in the introduction and deal with a broader set of elements (S) in $\mathcal{H}_{\infty}^n$ to which solutions to equations of the sort of the BPARE belong. Hence, let us state the following definitions.

Definition 5: S is a stabilizing element to (A, B, Γ) if $S \in \mathcal{H}_{\infty}^n$ and $K = R^{-1}B^*S$ stabilizes (A, B, Γ) (clearly $K \in \mathcal{H}_{\infty}^{n,m}$).

Definition 6: S is a strong element to (A, B, Γ) if $S \in \mathcal{H}_{\infty}^n$ and

$$\sup\{Re\lambda : \lambda \in \sigma(\mathcal{D})\} \leq 0,$$

with $\mathcal{D}$ equipped with $K = R^{-1}B^*S$.

Now set an arbitrary sequence $(S^j)_{j\in\mathbb{N}}$ of stabilizing elements to (A, B, Γ) and an element $\bar{S} \in \mathcal{H}_{\infty}^n$ such that

$$\bar{S}_i := \lim_{j\to\infty} S_i^j, \quad \forall i \in \mathcal{S}. \tag{9}$$

Also define the sequence of (stable) operators $(\mathcal{D}^j)_{j\in\mathbb{N}}$ and $\mathcal{D}$ in $Blt(\mathcal{H}_1^n)$ with

$$\mathcal{D}^j \textit{ as in } (5) \textit{ equipped with } K = R^{-1}B^*S^j. \tag{10}$$

and

$$\mathcal{D} \textit{ as in } (5) \textit{ equipped with } K = R^{-1}B^*\bar{S}. \tag{11}$$

In the maximal solution problem $(S^j)_{j\in\mathbb{N}}$ is a particular sequence of stabilizing elements to (A, B, Γ) such that the element $\bar{S}$ given by (9) turns out to be the maximal solution to the BPARE. Our aim is finding conditions, other than that of spectral continuity, that insures $\bar{S}$ being a strong element to (A, B, Γ) (remind that, unlike the finite dimensional case, spectral continuity may not occur everywhere in the space of all bounded linear operators (see, e.g., [4] and [17, pp 56])). A basic need is having $(\mathcal{D}^j)_{j\in\mathbb{N}} \to \mathcal{D}$ in the norm topology. Hence, we also look for a condition that insures this convergence. We use arguments that stem from the probabilistic nature of the Markov chain to evaluate the restrictions imposed by this condition.

IV. RESULTS

A. Sufficient conditions for $\bar{S}$ to be a strong element to (A, B, Γ)

Theorem 8 below gives us sufficient conditions for $\bar{S}$ being a strong element to (A, B, Γ), or else, for $\mathcal{D}$ as in (11) being such that $\sup\{Re\lambda : \lambda \in \sigma(\mathcal{D})\} \leq 0$.

We start with the following definitions. Let $(\alpha_n)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ of non-empty compact sets.

Definition 7: (see, e.g., [4] and [17, pp 56])

(i) $\limsup_n \alpha_n = \{\lambda \in \mathbb{C} : \lambda = \lim_{k\to\infty} \lambda_{n_k}$ where $\lambda_{n_k} \in \alpha_{n_k}$ and $n_{k+1} > n_k\}$,

(ii) $\liminf_n \alpha_n = \{\lambda \in \mathbb{C} : \lambda = \lim_{n\to\infty} \lambda_n$ where $\lambda_n \in \alpha_n\}$ and

(iii) $(\alpha_n)_{j\in\mathbb{N}}$ is called convergent iff $\limsup_n \alpha_n = \liminf_n \alpha_n$, in which case this common value is called $\lim_{n\to\infty} \alpha_n$.

Theorem 8: Suppose $\mathcal{D}^j \to \mathcal{D}$ in the norm topology (uniform operator topology), with $(\mathcal{D}^j)_{j\in\mathbb{N}}$ and $\mathcal{D}$ given by (10) and (11), respectively. Then, the equivalent conditions (a1) and (a2) as well as the weaker condition (b) are sufficient for $\bar{S}$ being a strong element to (A, B, Γ):

(a1) the spectrum $\sigma(\cdot)$ (a set-valued function defined in $Blt(\mathcal{H}_1^n)$) is lower semicontinuous at $\mathcal{D}$, i.e. (see [17, pp 56]),

$$\sigma(\mathcal{D}) \subset \liminf_j \sigma(\mathcal{A}^j)$$

$\forall\ (\mathcal{A}^j)_{j\in\mathbb{N}}$ such the $\mathcal{A}^j \to \mathcal{D}$ in the norm topology.

(a2) the spectrum is continuous at $\mathcal{D}$, i.e.,

$$\sigma(\mathcal{D}) = \lim_{j\to\infty} \sigma(\mathcal{A}^j)$$

whenever $\mathcal{A}^j \to \mathcal{D}$ in the norm topology.

(b)

$$\sigma(\mathcal{D}) \subset \limsup_j \sigma(\mathcal{A}^j)$$

whenever $\mathcal{A}^j \to \mathcal{D}$ in the norm topology.

Proof: Upper semicontinuity of the spectrum holds in an infinite dimensional Banach algebra with identity, which is the case of $Blt(\mathcal{H}_1^n)$. Since by assumption the spectrum is also lower semicontinuous, then it is continuous and $(a1) \Longrightarrow (a2)$ follows. The equivalence of $(a1)$ and $(a2)$ is therefore straightforward. Now, (a) implies (b), so it suffices to prove that condition (b) implies $\bar{S}$ to be a strong element to (A, B, Γ). We proceed as follows. Since K^j stabilize (A, B, Γ), then from Lemma 3 $(a) \Longrightarrow (b)$,

$$\sup\{Re\lambda : \lambda \in \sigma(\mathcal{D}^j)\} < 0,\ j \in \mathbb{N}.$$

Hence, if $c \in \mathbb{C}$ with $Re(c) \geq 0$, then $c \notin \sigma(\mathcal{D}^j)$, $j \in \mathbb{N}$, or else,

$$\begin{aligned} &c \notin \{\gamma \in \mathbb{C} : \exists j \in \mathbb{N} \textit{ such that } \gamma \in \sigma(\mathcal{D}^j)\} \\ &=: \cup_{j=1}^{\infty} \sigma(\mathcal{D}^j) \supset \cap_{j=1}^{\infty} \cup_{k=j}^{\infty} \sigma(\mathcal{D}^k). \end{aligned}$$

But the closure of $\cap_{j=1}^{\infty} \cup_{k=j}^{\infty} \sigma(\mathcal{D}^k)$ equals $\limsup_j \sigma(\mathcal{D}^j)$ and so $\sup\{Re\lambda : \lambda \in \limsup_j \sigma(\mathcal{D}^j)\} \leq 0$. Since, in particular, (b) assumes that $\sigma(\mathcal{D}) \subset \limsup_j \sigma(\mathcal{D}^j)$, then $\sup\{Re\lambda : \lambda \in \sigma(\mathcal{D})\} \leq 0$. ■

Since condition (b) does not imply (a) and (a) implies (b), it follows that continuity (or lower semicontinuity) is not necessary for $\bar{S}$ to be a strong element to (A, B, Γ). Condition (b) is a weaker sufficient condition than that of continuity for $\bar{S}$ to be a strong element to (A, B, Γ).

It is noteworthy that we can disregard considerations about topology in $Blt(\mathcal{H}_1^n)$ (in particular the convergence $\mathcal{D}^j \to \mathcal{D}$), and state (more loosely) the following version of the above theorem: each of the conditions

(a'1) $\sigma(\mathcal{D}) \subset \liminf_j \sigma(\mathcal{D}^j)$,

(a'2) $\sigma(\mathcal{D}) = \lim_{j\to\infty} \sigma(\mathcal{D}^j)$ and

(b') $\sigma(\mathcal{D}) \subset \limsup_j \sigma(\mathcal{D}^j)$

are sufficient for $\bar{S}$ to be a strong (respectively stabilizing) element to (A, B, Γ), where lim sup and lim inf are given by Definition 7 (respectively, $\limsup \sigma_n = \cap_{n\geq 1} \cup_{k\geq n} \sigma_k$ and $\liminf \sigma_n = \cup_{n\geq 1} \cap_{k\geq n} \sigma_k$). Note that conditions (a'1) and (a'2) are no more equivalent. More explicitly, (a'2)$\Longrightarrow$(a'1)$\Longrightarrow$(b').

Remark 9: The sets $\limsup_n \alpha_n$ and $\liminf_n \alpha_n$ defined above slightly differ from $\limsup_n \alpha_n = \cap_{n\geq 1} \cup_{k\geq n} \alpha_k$ and $\liminf_n \alpha_n = \cup_{n\geq 1} \cap_{k\geq n} \alpha_k$, respectively. In fact, the former sets are the closure of the latter. This difference is perceptible in both finite (when only the point spectrum is present) and infinite dimensional cases. This is the reason for obtaining in theorem 8 "a strong element to (A, B, Γ)" instead of "a stabilizing element to (A, B, Γ)".

B. Convergence of $(\mathcal{D}^j)_{j\in\mathbb{N}}$ in the norm topology

Theorem 8 requires $(\mathcal{D}^j)_{j\in\mathbb{N}} \to \mathcal{D}$ in the operator's norm topology, with $\mathcal{D}^j$ and $\mathcal{D}$ given by (10) and (11), respectively. Lemma 10 below reduces this convergence to that of $S^j \to \bar{S}$ in the $\mathcal{H}_\infty^n$ norm, which in turn is insured by (13) of Lemma 11. The discussion whether (13) is a restrictive condition or not is brought up in the final part of this section by viewing $\left(S^j\right)$ as a statistic of the Markov chain that embeds the operator Γ (note that S^j may be viewed as a function of Λ, for given parameters A and B).

Lemma 10: $(S^j)_{j\in\mathbb{N}} \to \bar{S}$ in the $\mathcal{H}_\infty^n$ norm, or equivalently, $(S_i^j)_{j\in\mathbb{N}} \to \bar{S}_i$ in the $\mathbb{M}(\mathbb{C}^n)$ norm uniformly with respect to i, implies $(\mathcal{D}^j)_{j\in\mathbb{N}} \to \mathcal{D}$ in the norm topology.

Proof: (except for the equivalence part).

$$\begin{aligned} &\left\|(\mathcal{D}^j - \mathcal{D})(W)\right\|_1 = \left\|\mathcal{D}^j(W) - \mathcal{D}(W)\right\|_1 \\ &= \left\|(\hat{F}^j - \hat{F})W - W(\hat{F}^j - \hat{F})^*\right\|_1 \leq 2\left\|\hat{F}^j - \hat{F}\right\|_\infty \|W\|_1 \\ &\leq 2\left\|BR^{-1}B^*\right\|_\infty \left\|S^j - \bar{S}\right\|_\infty \|W\|_1, \end{aligned}$$

so that

$$\begin{aligned} &\left\|(\mathcal{D}^j - \mathcal{D})\right\|_B = \sup\nolimits_{\|W\|_1=1} \left\|(\mathcal{D}^j - \mathcal{D})(W)\right\|_1 \\ &\leq 2\left\|BR^{-1}B^*\right\|_\infty \left\|(S^j - \bar{S})\right\|_\infty \to 0 \textit{ as } j \to \infty. \end{aligned}$$

■

Define

$$U^j := S^j - \bar{S}. \tag{12}$$

Lemma 11: The mild assumption

$$\begin{gathered} \frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} \not\longrightarrow 0 \text{ as } j \to \infty \text{ for some } i \in \mathbb{N} \\ (\text{ or equivalently, } \limsup_{j\to\infty} \frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} > 0 \\ \text{for some } i \in \mathbb{N}) \end{gathered} \tag{13}$$

and

$$\begin{gathered} \left(\left\|U^j\right\|_\infty\right) \text{ does not exhibits, simultaneously,} \\ \text{infinitely many zeros and nonzero numbers.} \end{gathered} \tag{14}$$

implies

$$\lim_{j\to\infty} \left\|U^j\right\|_\infty = 0. \tag{15}$$

Proof: Suppose $\left\|U^j\right\|_\infty \neq 0\ \forall j$ except for a finite number of j. Then, for arbitrary i, $\lim_{j\to\infty} \left\|U_i^j\right\| = 0$, or else, $\lim_{j\to\infty} \frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} \left\|U^j\right\|_\infty = 0$. This shows that having $\frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} \not\longrightarrow 0$ as $j \to \infty$ for some i suffices to obtain $\lim_{j\to\infty} \left\|U^j\right\|_\infty = 0$. The equivalence is straightforward: if the limit exist, then $\lim_{j\to\infty} \frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} = \limsup_{j\to\infty} \frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty}$. If there is no limit, then there must be $\varepsilon > 0$ such that $\frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} \geq \varepsilon$ holds for infinitely many j, which means that $\limsup \cdot$ must be greater than zero. If $\left\|U^j\right\|_\infty = 0\ \forall j$ except for a finite number of j, (15) trivially holds. ■

Remark 12: In the finite dimensional case, it is a simple task to obtain (15) without assumption (13), since

$$\lim_{j\to\infty}\|U^j\|_{\mathcal{H}^{nN}} := \lim_{j\to\infty}\max\{\left\|U_i^j\right\|, i=1,..,N\}$$
$$\le \lim_{j\to\infty}\sum_{i=0}^{N}\left\|U_i^j\right\| = \sum_{i=0}^{N}\lim_{j\to\infty}\left\|U_i^j\right\| = 0,$$

also reminding that norms are equivalent in finite dimensional spaces.

Let us address ourselves to whether (13) is a restrictive assumption. For $\varepsilon \in (0,1]$ (that does not depend on j but may depend on i) and, for each $i \in \mathbb{N}$, define the (unique) $\mathbb{N}$-valued strictly increasing index sequence $(r_{s,i})_{s=1,\dots,M_i}$ such that

$$\frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} \ge \varepsilon \quad or \quad \frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} = \frac{0}{0} \quad iff \quad j = r_{s,i} \; for \; some \; s.$$

This sequence indicates the ε-tracking of the supremum sequence $(\left\|U^j\right\|_\infty)_{j\in\mathbb{N}}$ by $(\left\|U_i^j\right\|)_{j\in\mathbb{N}}$, in that $\left\|U_i^{r_{s,i}}\right\| \ge \varepsilon\left\|U^{r_{s,i}}\right\|_\infty$ for $s=1,\dots,M_i$, and does not exist if $M_i=0$. In analogy to terms used in Markov chain theory, we shall say that i is recurrent if, for some ε, $(\left\|U_i^j\right\|)$ indefinitely provides an ε-tracking of the supremum sequence, or else, if $M_i \to \infty$ for some ε. Clearly, if i is recurrent with respect to ε_0, it is so with respect to $\varepsilon_1 < \varepsilon_0$. The following expressions are therefore equivalent:

(a) i is recurrent

(b) $\forall N \in \mathbb{N}, \exists j \ge N$ such that $\frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} \ge \varepsilon$, for some ε ($j = r_{s,i}$ for some $s \in \mathbb{N}$).

(c) $\frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} \not\longrightarrow 0$ as $j \to \infty$.

We shall say that i is nonrecurrent if, for every $\varepsilon \in (0,1]$, $(\left\|U_i^j\right\|)$ provides an ε-tracking of the supremum sequence for a finite number of times at most, or else, if $M_i(\varepsilon) < \infty\ \forall\varepsilon$. Clearly, "$i$ is nonrecurrent" is the negation of "i is recurrent". The assertions that follow are equivalent:

(a') i is nonrecurrent

(b') $\forall\varepsilon\ \exists N \in \mathbb{N}$ *such that* $\frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} < \varepsilon,\ j \ge N,$

(c') $\frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} \longrightarrow 0$ as $j \to \infty$.

Clearly, if i is nonrecurrent with respect to ε_0, it is so with respect to $\varepsilon_1 > \varepsilon_0$.

It is straightforward from assertion *(c')* that (13) does not hold if and only if *every* i is nonrecurrent.

In the case where S assumes a finite number of entries (i.e., the Markov chain θ assumes a finite number of states), a recurrent state with $\varepsilon = 1$ (and therefore with $\varepsilon \le 1$) must always exist. Indeed, the maximum in this case is always attained by some state, so all of them cannot be nonrecurrent. Consequently, assumption (13) is *always* satisfied in the finite dimensional case. Now, in the infinite dimensional case, we can build sequences (S^j) for which (13) does not hold, as Appendix V-A shows). The matter then is whether this sort of hand-conducted sequences can actually be generated by some Markov chain with given parameters A and B. In fact, this appears to be unnatural, and so the case of complete absence of recurrent states. By its turn, (14) means, for instance, that there are infinitely many $S^j = \bar{S}$, or else, the sequence (S^j) visits its limit point $\bar{S}$ infinitely many times, which is an is an anomalous situation. Hence, (13)/(14) seems in fact to be a mild assumption.

V. APPENDIX

A. Examples with no recurrent states

We consider the space $\mathcal{H}_\infty^1 \equiv l_\infty$ and two situations for the sequence $(U^j)_{j\in\mathbb{N}} \subset \mathcal{H}_\infty^1$:

$$\textbf{I}. \quad U_i^j = \begin{cases} \frac{1}{j+1}, & j < i \\ \frac{1}{j}, & j = i \\ \frac{1}{j^{i+1}}, & j > i \end{cases}$$

$$\textbf{II}. \quad U_i^j = \begin{cases} \frac{1}{j+1}, & j < i \\ \frac{1}{j}, & j = i \\ \frac{1}{j^2}, & j > i \end{cases}$$

We have that $(\left\|U_i^j\right\|)_{j\in\mathbb{N}}$ are non-increasing monotone and converge to zero, as required by [2], (9) and (12).

Our aim is showing that

$$\frac{\left\|U_i^j\right\|}{\left\|U^j\right\|_\infty} \longrightarrow 0 \; as \; j \to \infty \; for \; every \; i, \tag{16}$$

i.e., *every* i is nonrecurrent.

Now, assertion (16) is equivalent to having item *(b')* of Section IV-B valid for every i. Cases I and II are therefore inspired on a particular situation of this latter condition, which is having the supremum sequence $(\left\|U^j\right\|_\infty)$ ε-tracked by $(\left\|U_i^j\right\|)_{j\in\mathbb{N}}$ with $\varepsilon = 1$ (instead of $0 < \varepsilon \le 1$), one time only for each $i \in \mathbb{N}$. Note that, in the finite dimensional case, even this peculiar situation cannot occur, since a recurrent state must always exist.

Turning back to the proof of (16), define $a_i^j := \left\|U_i^j\right\|$ and $b^j := \left\|U^j\right\|_\infty$, $i,j \in \mathbb{N}$. Thus,

Case I :

$$b^j = \max_i a_i^j$$
$$= \max\{\max_{i>j}\tfrac{1}{j+1}, \tfrac{1}{j}, \max_{i<j}\tfrac{1}{j^{i+1}}\}$$
$$= \max\{\tfrac{1}{j+1}, \tfrac{1}{j}, \tfrac{1}{j^2}\} = \tfrac{1}{j}$$

Case II :

$$b^j = \max_i a_i^j$$
$$= \max\{\max_{i>j}\tfrac{1}{j+1}, \tfrac{1}{j}, \max_{i<j}\tfrac{1}{j^2}\}$$
$$= \max\{\tfrac{1}{j+1}, \tfrac{1}{j}, \tfrac{1}{j^2}\} = \tfrac{1}{j}$$

and so

$$\textbf{Case I} \qquad\qquad \textbf{Case II}$$

$$\frac{a_i^j}{b^j} = \begin{cases} \frac{j}{j+1}, \ j < i \\ 1, \ \ j = i \\ \frac{1}{j^i} \ , \ \ j > i \end{cases} \qquad \frac{a_i^j}{b^j} = \begin{cases} \frac{j}{j+1}, \ j < i \\ 1, \ \ j = i \\ \frac{1}{j} \ , \ \ j > i \end{cases}$$

Hence, for arbitrary i,

$$\textbf{Case I}:$$

$$\left(\frac{a_i^j}{b^j}\right)_{j\in\mathbb{N}}$$

$$= (\tfrac{1}{2}, \tfrac{2}{3}, ..., \overbrace{\frac{i-1}{i}}^{(i-1)^{th}}, \overbrace{1}^{i^{th}}, \overbrace{\frac{1}{(i+1)^i}}^{(i+1)^{th}}, \overbrace{\frac{1}{(i+2)^i}}^{(i+2)^{th}}, ...)$$

$$\textbf{Case II}:$$

$$\left(\frac{a_i^j}{b^j}\right)_{j\in\mathbb{N}}$$

$$= (\tfrac{1}{2}, \tfrac{2}{3}, ..., \overbrace{\frac{i-1}{i}}^{(i-1)^{th}}, \overbrace{1}^{i^{th}}, \overbrace{\frac{1}{i+1}}^{(i+1)^{th}}, \overbrace{\frac{1}{i+2}}^{(i+2)^{th}}, ...)$$

where the $i-1$ first terms in both sequences should be ignored if $i = 1$). So, for $0 < \varepsilon < \frac{1}{2}$, we have that

$$\frac{a_i^j}{b^j} \geq \varepsilon \ for \ j = 1, ..., k(\varepsilon)$$

with k such that

$$\textbf{Case I}: \quad \tfrac{1}{k^i} \geq \varepsilon, \ or \ else, \ k \leq \left(\tfrac{1}{\varepsilon}\right)^{\frac{1}{i}} < \infty$$

$$\textbf{Case II}: \quad \tfrac{1}{k} \geq \varepsilon, \ or \ else, \ k \leq \tfrac{1}{\varepsilon} < \infty$$

(clearly the $k's$ are even smaller for $\varepsilon \geq \frac{1}{2}$). Thus, in both cases, k is finite for every $\varepsilon \in (0, 1]$ and each i, which is the same as saying that every i is nonrecurrent.

In what follows, we prove expression (16) via the concept of rate of convergence, which is a simpler procedure but poorer in interpretation and insight.

In case **I,** the rate of convergence of a_i^j as $j \to \infty$ increases as i increases. This is clear since

$$\frac{a_i^j}{a_{i-1}^j} = \frac{1/j^{i+1}}{1/j^i} = \frac{1}{j} \to 0 \ as \ j \to \infty, \ i = 2, 3, ...$$

Still, $(a_1^j)_{j\in\mathbb{N}} = (1, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}...)$ - which exhibits the lowest convergence rate among all other sequences indexed by $i = 2, 3, ...$, has a convergence rate higher than that of the supremum sequence $(b^j)_{j\in\mathbb{N}}$. In fact $\frac{a_1^j}{b^j} = \frac{1/j^2}{1/j} = \frac{1}{j} \to 0$ as $j \to \infty$. Assertion (16) then follows.

In case **II,** the rate of convergence of every sequence $(a_i^j)_{j\in\mathbb{N}}$ is uniform with respect to i, but still higher than that of the supremum sequence $\{b^j\}_{j\in\mathbb{N}}$. Indeed, $\frac{a_i^j}{b^j} = \frac{1/j^2}{1/j} = \frac{1}{j} \to 0$ as $j \to \infty$. Assertion (16) again follows.

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