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A New Approach to Detectability of Discrete-Time Infinite Markov Jump Linear Systems

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Abstract— **This paper deals with detectability for the class of discrete-time Markov jump linear systems (MJLS) with the underlying Markov chain having countably infinite state space. The formulation here relates the convergence of the output with that of the state variables. Our approach introduces invariant subspaces for the autonomous system and exhibits the role that they play. This allows us to show that detectability can be written equivalently in term of two conditions: stability of the autonomous system in a certain invariant space and convergence of general state trajectories to this invariant space under convergence of input and output variables. This, in turn, provides the tools to show that detectability here generalizes uniform observability ideas as well as previous detectability notions for MJLS with finite state Markov chain, and allows us to solve the jump-linear-quadratic control problem. In addition, it is shown for the MJLS with finite Markov state that the second condition is redundant and that detectability retrieves previously well-known concepts in their respective scenarios.**

Index Terms— **detectability, stochastic systems, Markov jump systems, infinite Markov state space, optimal control**

I. INTRODUCTION

Structural concepts such as observability and detectability have a solid ground in system theory, as the imposing literature for linear and linear-Gaussian systems conveys (see, e.g., [15]). For instance, in control problems, detectability firmly associates the solution for the optimal problems with stability of the corresponding controlled system, whereas, for filtering, it makes the system observations meaningful for state estimates by connecting convergence of the output with convergence of the state. Although the theory involving these concepts is quite developed and a number of results are available in the context of linear deterministic systems, there is still a great deal of research activity in this area (see, e.g., [13], [17] and references therein).

Among the most important properties of detectability for the linear deterministic scenario, we mention that: (i) detectability can be expressed in terms of the parameters of the autonomous version of the system, e.g., by requiring that nonobserved modes of the autonomous system are stable; (ii) Detectability generalizes observability.

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Another important but less acknowledged property is that (iii) detectability is a necessary and sufficient condition to guarantee convergence of the state from convergence of the output (under regular nonsingular linear state feedback controls).

Property (iii) ensures that the optimal control solution is stabilizing and makes output observations meaningful in filtering problems. Due to its generic formulation, these properties constitute a paradigm for more general contexts. The challenge then is how to devise a detectability concept for a certain class of systems that allows one to employ the structure of the system to retrieve properties (i)–(iii).

In this spirit, some of the authors have recently developed a notion of detectability (called weak detectability) that generalizes previous detectability ideas for MJLS with finite Markov chain state, retrieves the properties (i)–(iii), and allows an associate observability matrix, in an extension to the well-known deterministic concepts, see [1] and [2]. In this process all but one[1] of the linear deterministic concepts are retrieved.

However, as far as the authors are aware, these ideas have no parallel in more complex scenarios such as the MJLS with countably infinite state space of the Markov chain. This is a rather general class of systems that includes the classes of finite MJLS and linear deterministic systems, as well as deterministic time varying systems. For this class of systems, up to this date there is no detectability concept that retrieves properties (i)–(iii) above. For instance, the stochastic notion in [7] can be expressed in terms of the autonomous system data, thus satisfying (i), but (ii) does not hold and only the sufficiency part of (iii) holds; in [4] we derive a detectability notion in the perspective of (iii) for which (ii) holds, but it does not satisfy (i).

These shortfalls come from the analytical complexity inherent to the infinite many Markov state case. In particular, the main difficulty arises from the fact demonstrated in this paper that converging input and output *do not* ensure convergence of state trajectory to the observed space; see Example 1 in connection. In the simpler case of finitely many Markov states, the above convergence relation holds, and apart from ensuring stability within the observed space, with detectability it guarantees convergence of the state trajectory to the origin. This is the mechanism that fails here, and in this regard we can conclude that any detectability concept with the perspective of (i) (stable nonobserved modes) by

[1] The observability idea that after a number of observations that equals the system dimension, the initial state value can be precisely retrieved. This is inherently a nonstochastic idea.

itself cannot provide the property in (iii) and thus, it cannot ensure that the optimal control is stabilizing.

With the aim of studying detectability for MJLS with countably infinite state space of the Markov chain and to retrieve (i)–(iii), we introduce a novel point of view toward detectability by considering the paradigmatic property in (iii) as a general, direct, and intuitive notion of detectability, which relates the convergence of the input and output with that of the state variables. Then we introduce certain invariant subspaces for the autonomous system, which play a key role to relate detectability with stability and convergence of the state trajectory; this allows us to show that this detectability sense generalizes uniform observability ideas as well as previous detectability notions for MJLS with finite state Markov chain, and to solve the jump-linear-quadratic control problem. To show some subtleties of the approach, and to clarify the role of some tools, we also analyze the MJLS with finite state Markov chain and present illustrative examples.

An outline of the content of the paper is as follows. In section II we provide the bare essential of notations, state the model, and discuss the general ideas of the paper. Section III provides some preliminaries. Necessary and sufficient conditions for detectability are treated in section IV, and some sufficient conditions are presented in section V. The finite MJLS is analyzed in section VI, and the control problem is studied in section VII. Some illustrative examples are exhibited in section VIII. Finally, section IX presents some conclusions.

II. Problem formulation and general ideas

Let $\mathbb{R}^n$ represent the usual linear space of all n-dimensional vectors and $\mathscr{R}^{r,n}$ (respectively, $\mathscr{R}^n$) the normed linear space formed by all $r\times n$ real matrices (respectively, $n\times n$). For $V\in\mathscr{R}^{n,r}$, V' denotes the transpose of V. $\sigma^+(V)$ and $\sigma^-(V)$ stand, respectively, for the largest and smallest singular value of V and $\|V\|=\sigma^+(V)$. For $V,W\in\mathscr{R}^n$, $V>W$ $(V\geq W)$ indicates that $V-W$ is positive definite (semidefinite).

Let $\mathscr{H}_\infty^{r,n}$ denote the linear space formed by sequences of matrices $H=\{H_i\in\mathscr{R}^{r,n}; i\in\mathscr{Z}\}$ such that $\sup_{i\in\mathscr{Z}}\|H_i\|<\infty$; also, $\mathscr{H}_\infty^n\equiv\mathscr{H}_\infty^{n,n}$ and $\|H\|_\infty=\sup_{i\in\mathscr{Z}}\|H_i\|$. For $H,V\in\mathscr{H}_\infty^n$, $H\geq V$ indicates that $H_i\geq V_i$ for each $i\in\mathscr{Z}$; similarly, for $H\in\mathscr{H}_\infty^{r,n}$ and $V\in\mathscr{H}_\infty^{n,s}$, the "product" HV indicates the element of $\mathscr{H}_\infty^{r,s}$ formed by the sequence $\{H_iV_i, i\in\mathscr{Z}\}$, and equivalent understanding should apply to any basic mathematical operation involving elements of $\mathscr{H}_\infty^{r,n}$. In what follows, capital letters denote elements of $\mathscr{H}_\infty^{r,n}$, and capital letters with an index denote elements of $\mathscr{R}^{r,n}$.

The system we deal with is the discrete-time MJLS with infinite countably Markov chain, defined in a fixed stochastic basis $(\Omega,\mathfrak{F},(\mathfrak{F}_k),\mathscr{P})$ by

$$\Psi:\begin{cases}x(k+1)=A_{\theta(k)}x(k)+B_{\theta(k)}u(k), & k\geq 0,\\ y(k)=C_{\theta(k)}x(k)+D_{\theta(k)}u(k), & x(0)=x,\ \theta(0)=\theta,\end{cases}\tag{1}$$

where y is the output process and u is the input, an $(\mathfrak{F}_k)$-adapted process. The mode θ is the state of an underlying discrete-time Markov chain $\Theta=\{\theta(k);k\geq 0\}$ taking values in $\mathscr{Z}=\{1,2,\dots\}$ and having a stationary transition probability matrix $\mathbb{P}=[p_{ij}]$, $i,j\in\mathscr{Z}$. The state of the system is the compound variable $(x(k),\theta(k))$. The matrices A_i belong to the sequence of matrices $A\in\mathscr{H}_\infty^n$, and similarly for $B\in\mathscr{H}_\infty^{n,r}$, $C\in\mathscr{H}_\infty^{q,n}$, and $D\in\mathscr{H}_\infty^{q,r}$. In addition, without loss of generality, we also assume that $C'D=0$.

In the paper we deal with detectability for systems described by (1). The departure point is the concept of detectability that follows from property (iii) of section I. We emphasize that the specific notion of convergence is not relevant; the essence of the concept is the relation among convergence of state, input and output.

Definition 1 (detectability): The system Ψ is detectable if the state converges provided that the output and the input converge.

With the detectability concept above at hands, the issues pursued here are primarily summarized as follows: (I) Relate the concept with the autonomous version of the system, aiming at mimicking item (i) mentioned in the introduction; (II) Show that it retrieves property (ii) mentioned in the introduction; (III) Investigate the extent to which the above concept is related to the weak detectability concept in [1] and [2] for MJLS, and the usual concept for deterministic linear systems.

We consider a cost functional that is an ℓ_2-measurement of the output (the expected accumulated energy in the output),

$$\mathscr{Y}_u(x,\theta)=E_{x,\theta}\left\{\sum_{k=0}^{\infty}|y(k)|^2\right\},\tag{2}$$

defined for an admissible control u whenever $x(0)=x$ and $\theta(0)=\theta$. We also denote for the autonomous system obtained from Ψ with $u\equiv 0$,

$$\mathscr{Y}_0(x,\theta)=\mathscr{Y}_{u\equiv 0}(x,\theta).\tag{3}$$

In agreement with (2), we adopt the corresponding ℓ_2-convergence notion for each Ψ-processes, namely, we say that the output y converges whenever $\mathscr{Y}_u(\cdot,\cdot)<\infty$; similar notion holds for u and x.

Our approach starts from a novel point of view, which hinges on the following steps. We first locate an invariant linear subspace for the autonomous system, in the sense that the trajectories remain almost surely confined to it. Then we indicate the role that the invariant space plays in the convergence of an arbitrary state trajectory, showing that the existence of an invariant space for which the autonomous system is stable, together with the convergence to this set of an arbitrary trajectory, is equivalent to convergence to the origin of such a trajectory (see section IV and Theorem 1).

In order to make the above result suitable to deal with (I), we seek the largest of such an invariant space. It turns out to be the linear subspace $\mathscr{F} = \{(x,\theta) : \mathscr{Y}_0(x,\theta) < \infty\}$, and in Theorem 2 we state that detectability according to Definition 1 is equivalent to requiring that

(A1) the autonomous system is stable in $\mathscr{F}$,

(A2) the state x converges to $\mathscr{F}$ provided that both y and u converge.

Notice that condition (A1) accounts for the autonomous version of system Ψ only, and it is consistent with the notion of detectability for finite dimensional linear deterministic systems. Together with condition (A2) for system Ψ (not only the autonomous version), they build the essentials to complete the mechanism yielding (iii). Due to (A2), a complete counterpart for property (i) is not viable in the present setup, and any attempt to enlarge $\mathscr{F}$ is worthless, as we show in Lemma 5.

Section V addresses (II), where we show that detectability according to Definition 1 generalizes uniform observability as in [1], [3], [5], [12], which, by its turn, generalizes previous observability concepts for MJLS, like those in [11]. We also show that an earlier ℓ_2-detectability concept in [7] is stricter than detectability. Moreover, we introduce a notion of uniform observability in the invariant space $\mathscr{F}$ that serves as a sufficient condition for (A2).

Regarding (III), in section VI we show that $\mathscr{F}^\perp$ is uniformly observable in the finite Markov chain case, which renders condition (A2) always true. Thus, we have that detectability is equivalent to (A1) in the finite case, allowing us to show that the weak detectability in [2] and the usual detectability concept in the deterministic linear case are necessary and sufficient conditions (in their particular contexts) for detectability according to Definition 1. The fact that (A2) holds true for the case in which the Markov chain is finite explains why no such condition appears in those simpler scenarios. By contrast, (A2) may fail in the infinite Markov chain case, as illustrated in Example 1.

Another important feature of the setting and results here is that, unlike previous ones, the focus is not constrained (i.e., is not *ad hoc*) to the optimal jump-linear-quadratic (JLQ) control and/or controls in linear feedback form, where detectability appears as a dual notion to stabilizability. It covers any $(\mathfrak{F}_k)$-adapted converging control that induces a finite cost $\mathscr{Y}_u$ for each initial state, assuring that it is stabilizing, and clearly encompassing the optimal solution. In particular for the JLQ control, we show that the solution to the associated infinite coupled algebraic Riccati equation is unique (see section VII).

III. Preliminaries

In this section we consider the autonomous version of (1), which will be essential to relate detectability with stability and convergence of the state trajectory (see (A1) and (A2) in section 2). All proofs in this concise version are omitted. We consider the autonomous version of system (1):

$$\Psi_0 : \begin{cases} x_0(k+1) = A_{\theta(k)} x_0(k), & k \geq 0, \\ y_0(k) = C_{\theta(k)} x_0(k), & x_0(0) = x, \theta(0) = \theta. \end{cases}$$

Sometimes we refer to the autonomous system by the pair $(A,\mathbb{P})$ or by the triplet $(A,C,\mathbb{P})$. In addition, in what follows, for each $i \in \mathscr{X}$, let $\mathscr{S}_i \subset \mathbb{R}^n$ stand for a vector subspace and let $\mathscr{S} = \{\mathscr{S}_i, i \in \mathscr{X}\}$.

Definition 2 (Ψ_0-invariant space): Consider the autonomous system Ψ_0. We say that $\mathscr{S}$ is an invariant space if $x_0(k) \in \mathscr{S}_{\theta(k)}$ implies that $x_0(t) \in \mathscr{S}_{\theta(t)}$ almost surely (a.s.) for each $t \geq k$.

Definition 3 (projections onto $\mathscr{S}^\perp$): For each $i \in \mathscr{X}$, let $P_i \in \mathscr{R}^n$ denote the orthogonal projection onto $\mathscr{S}_i^\perp$. Clearly, $P = \{P_i, i \in \mathscr{X}\} \in \mathscr{H}_\infty^n$.

Definition 4 (Ψ_0-convergence): We say that $x(\cdot)$ converges (in the ℓ_2 sense) to the Ψ_0-invariant space $\mathscr{S}$ if

$$\sum_{k=0}^{\infty} E_{x,\theta}\{|P_{\theta(k)} x(k)|^2\} < \infty.$$

We say that $x(\cdot)$ converges if it converges to the trivial Ψ_0-invariant space $\mathscr{S} = 0$.

Definition 5 (ℓ_2-stability): Consider the autonomous system Ψ_0. We say that $(A,\mathbb{P})$ is ℓ_2-stable in the invariant space $\mathscr{S}$ if $x_0(\cdot)$ converges for each initial condition $\theta \in \mathscr{X}$ and $x \in \mathscr{S}_\theta$. We say that $(A,\mathbb{P})$ is ℓ_2-stable if it is ℓ_2-stable in $\mathscr{S}$ with $\mathscr{S}_i = \mathbb{R}^n$, $i \in \mathscr{X}$

Notice that $x(\cdot)$ converges if and only if $\sum_{k=0}^{\infty} E\{|x(k)|^2\} < \infty$, since $P = I$ whenever $\mathscr{S}$ is trivial. Also, ℓ_2-stability of $(A,\mathbb{P})$ is equivalent to convergence of $x_0(\cdot)$ for each initial condition $\theta \in \mathscr{X}$ and $x \in \mathbb{R}^n$.

We will need the following property related with the concept of ℓ_2-stability in $\mathscr{S}$ and the projections P.

Lemma 1: Assume that $(A,\mathbb{P})$ is ℓ_2-stable in $\mathscr{S}$. Then, $(A - AP,\mathbb{P})$ is ℓ_2-stable.

Let $\mathscr{H}_1^n$ denote the linear space formed by sequences of matrices $H = \{H_i = H_i' \geq 0; i \in \mathscr{X}\}$ such that $\sum_{i\in\mathscr{X}} tr\{H_i\} < \infty$. Let $\mathscr{H}_F^n \subset \mathscr{H}_1^n$ denote the closed cone formed by sequences of symmetric positive semidefinite matrices $H = \{H_i = H_i' \geq 0; i \in \mathscr{X}\}$. For $H, V \in \mathscr{H}_F^n$ we define the inner product $\langle H, V\rangle = \sum_{i\in\mathscr{X}} tr\{H_i' V_i\}$ and the Frobenius norm

$$\|H\|_F = \langle H, I\rangle. \qquad (4)$$

Recall from the definition of the Ψ_0-invariant subspace $\mathscr{S}$ that $\mathscr{S}_i = \{x : P_i x = 0\}$. In connection, we define the spaces $\bar{\mathscr{S}} = \{H \in \mathscr{H}_F^n : PHP' = 0\} \subset \mathscr{H}_F^n$ and $\bar{\mathscr{S}}^\perp = \{H \in \mathscr{H}_F^n : H - PHP' = 0\}$. PHP' is the orthogonal projection of H onto $\bar{\mathscr{S}}^\perp$; indeed, P inherits from P_i the property that $P^2 = P$, and it is easy to check that $\langle PHP', H - PHP'\rangle = \langle H, PHP' - P^2HP^{2\prime}\rangle = 0$.

Definition 6 (convergence in $\mathscr{H}_F^n$): We refer to convergence of sequences in $\mathscr{H}_F^n$ in the ℓ_1 sense: we say that a sequence $H(\cdot) \in \mathscr{H}_F^n$ converges to the space $\bar{\mathscr{S}}$ whenever

$\sum_{k=0}^{\infty} \|PH(k)P'\|_F < \infty$; we say that $H(\cdot)$ converges if it converges to the trivial space $\bar{\mathscr{S}} = 0$.

We define $X(\cdot) \in \mathscr{H}_F^n$ and $U(\cdot) \in \mathscr{H}_F^r$ as

$$X_i(k) = E\{x(k)x(k)'1_{\{\theta(k)=i\}}\} \qquad (5)$$
$$U_i(k) = E\{u(k)u(k)'1_{\{\theta(k)=i\}}\} \; \forall i \in \mathscr{X}, k \geq 0,$$

where $1_{\{\cdot\}}$ is the Dirac indicator function. We write $X_0(\cdot)$ when we refer to the autonomous system. We define $\mathscr{Y}_u^{t,T}$ similarly to the functional $\mathscr{Y}$ in (2) as follows:

$$\mathscr{Y}_u^{t,T}(x,\theta) = E_{x,\theta}\left\{\sum_{k=t}^{t+T-1} |y(k)|^2\right\} = \sum_{k=t}^{t+T-1}\Big(\langle X(k), C'C\rangle + \langle U(k), D'D\rangle\Big) \qquad (6)$$

whenever $x(0) = x, \theta(0) = \theta$; for simplicity we write $\mathscr{Y}_u^{t=0,T}(x,\theta) = \mathscr{Y}_u^T(x,\theta)$ and also $\mathscr{Y}_{u\equiv 0}^T(x,\theta) = \mathscr{Y}_0^T(x,\theta)$.

Using the notation above we can write $E_{x,\theta}\{|x(k)|^2\} = \|X(k)\|_F$ and this provides a connection between convergence in the ℓ_1 sense of $X(\cdot) \in \mathscr{H}_F^n$ with the ℓ_2 convergence of $x(\cdot)$. A further connection is presented in the next lemma.

Lemma 2: $x(\cdot)$ converges to $\mathscr{S}$ if and only if $X(\cdot)$ converges to $\bar{\mathscr{S}}$.

Now, let us define for $V \in \mathscr{H}_\infty^{n,r}$ the linear operator $\mathscr{L}_V : \mathscr{H}_F^r \to \mathscr{H}_F^n$

$$\mathscr{L}_{Vi}(H) = \sum_{j\in\mathscr{X}} p_{ji} V_j H_j V_j'. \qquad (7)$$

It is shown in [7] that the limit in (7) is well defined. We denote $\mathscr{L}^0(H) = H$, and for $k \geq 1$, we can define $\mathscr{L}^k(H)$ recursively by $\mathscr{L}^k(H) = \mathscr{L}(\mathscr{L}^{k-1}(H))$. Also, $r_\sigma(\mathscr{L})$ denotes the spectral radius of $\mathscr{L}$. Operator $\mathscr{L}$ is related to system Ψ as follows; the result is adapted from [7].

Proposition 1: The following assertions hold:

(i) $X_0(k+1) = \mathscr{L}_A(X_0(k))$, $k \geq 0$;

(ii) $(A,\mathbb{P})$ is ℓ_2-stable if and only if $r_\sigma(\mathscr{L}_A) < 1$.

We finish the section with the following facts that we believe are worth mentioning. $\bar{\mathscr{S}}$ inherits from $\mathscr{S}$ the property that it is a Ψ_0-invariant subspace, that is, $PX_0(k)P' = 0$, $k \geq 0$, implies that $PX_0(t)P' = 0$, $t \geq k$. The notion of convergence in $\mathscr{H}_F^n$ is usual, in the sense that a sequence $H(\cdot) \in \mathscr{H}_F^n$ converges to the space $\bar{\mathscr{S}}$ if and only if $\sum_{k=0}^{\infty} \inf_{V\in\bar{\mathscr{S}}} \|H(k) - V\|_F < \infty$. Actually, the proof follows immediately from the fact that for each $H(k)$ there exists $V \in \bar{\mathscr{S}}$ for which $\|H(k) - V\|_F = \|PH(k)P'\|_F$ (indeed, $V = H(k) - PH(k)P'$).

IV. A NECESSARY AND SUFFICIENT CONDITION FOR DETECTABILITY

We show in section IV-A that a general state trajectory $x(\cdot)$ converges if and only if there exists an invariant space $\mathscr{S}$ for which: (i) $(A,\mathbb{P})$ is ℓ_2-stable in $\mathscr{S}$ and (ii) $x(\cdot)$ converges to $\mathscr{S}$. In section IV-B we introduce the Ψ_0-invariant space $\mathscr{F}$ and we show that $\mathscr{F}$ is adequate to formulate the equivalence between detectability and conditions (A1) and (A2).

A. Conditions for state convergence

Theorem 1: Consider system Ψ and assume that the input converges. The state $x(\cdot)$ converges if and only if there exists an invariant space $\mathscr{S}$ such that the following conditions hold:

(i) $(A,\mathbb{P})$ is ℓ_2-stable in $\mathscr{S}$;

(ii) $x(\cdot)$ converges to $\mathscr{S}$.

Proof. (Necessity.) Since $x(\cdot)$ converges to the origin, $\mathscr{S} = 0$ trivially satisfies (i) and (ii). (Sufficiency.) omitted. ■

B. The main result

The first result of this section follows in a straightforward manner from Theorem 1 and the definition of detectability.

Lemma 3: System Ψ is detectable if and only if there exists an invariant space $\mathscr{S}$ such that:

(i) $(A,\mathbb{P})$ is ℓ_2-stable in $\mathscr{S}$;

(ii) $x(\cdot)$ converges to $\mathscr{S}$ provided that $y(\cdot)$ and $u(\cdot)$ converge.

Notice that, for $\mathscr{S}$ trivial, item (i) holds trivially and item (ii) reduces to the definition of detectability. The larger the invariant space $\mathscr{S}$ is, the more significant the result will be. Along this line, we introduce the set $\mathscr{F} = \{\mathscr{F}_i, i \in \mathscr{X}\}$ as

$$\mathscr{F}_i = \{x \in \mathbb{R}^n : \mathscr{Y}_0(x,i) < \infty\} \quad \forall i \in \mathscr{X} \qquad (8)$$

and we show that $\mathscr{F}$ is the largest of such Ψ_0-invariant space.

Lemma 4: $\mathscr{F}$ is a Ψ_0-invariant space.

Next we show that $\mathscr{F}$ is the largest Ψ_0-invariant space that possibly meets the condition (i) in Lemma 3.

Lemma 5: If $\mathscr{S}$ is such that $(A,\mathbb{P})$ is ℓ_2-stable in $\mathscr{S}$, then $\mathscr{S} \subset \mathscr{F}$.

And the main result of the paper is:

Theorem 2: System Ψ is detectable if and only if the following conditions hold:

(A1) $(A,\mathbb{P})$ is ℓ_2-stable in $\mathscr{F}$;

(A2) $x(\cdot)$ converges to $\mathscr{F}$ provided $y(\cdot)$ and $u(\cdot)$ converge.

Proof: (Sufficiency.) (A1) and (A2) satisfy the conditions for detectability in Lemma 3.

(Necessity.) Since $(A,C,\mathbb{P})$ is detectable, from Lemma 3 we have that there exists $\mathscr{S}$ for which $(A,\mathbb{P})$ is ℓ_2-stable in $\mathscr{S}$ and Lemma 5 provides that $\mathscr{S} \subset \mathscr{F}$. Lemma 3 also yields that $x(\cdot)$ converges to $\mathscr{S}$ provided $y(\cdot)$ and $u(\cdot)$ converges; this fact together with the fact that $\mathscr{S} \subset \mathscr{F}$ lead immediately to (A2).

Now, notice from the concept of detectability that, in particular for the autonomous system Ψ_0, $x_0(\cdot)$ converges whenever the corresponding output $y(\cdot)$ converges or, equivalently, whenever $x(0) \in \mathscr{F}_{\theta(0)}$. This means that $(A,\mathbb{P})$ is ℓ_2-stable in $\mathscr{F}$ and (A1) holds. ■

V. SUFFICIENT CONDITIONS FOR (A1) AND (A2)

In this section of the original paper, we deal with other detectability and observability concepts that appear in the literature of MJLS and we present the role that they play as sufficient conditions (expressed entirely in terms of the autonomous version of the system) for (A1) and (A2), and therefore for the detectability concept here.

VI. Finite MJLS

Recall from the main result of the paper, Theorem 2, that the system is detectable if and only if (A1) and (A2) hold. In this section of the original paper, we show that (A2) is made redundant when the Markov state space is finite, $\mathscr{X} = \{1,\dots,N\}$. This leads to the main result of the section: (A1) is a necessary and sufficient condition for detectability, in parallel with detectability notions for linear deterministic systems and previous concepts for MJLS [2]. That result also generalizes previous results in the literature, which require that the control is in the linear state feedback form.

VII. Detectability and the Jump Linear Quadratic Problem

We are concerned here with the JLQ problem, which consists of obtaining the control $u(\cdot)$ that minimizes the cost functional $\mathscr{Y}_u(x,\theta)$. We assume here with no loss of generality that the control is in linear state feedback form, $u(k) = G_{\theta(k)}x(k)$, $G \in \mathscr{H}_\infty^{r,n}$. Indeed, it is a well-known fact that the optimal control is in this form; see, e.g., [7]. We denote $\mathscr{Y}_G(\cdot) = \mathscr{Y}_u(\cdot)$ to emphasize the dependence on G.

A standard assumption in the JLQ problem, that $\inf_{i\in\mathscr{X}} \sigma^-(D_i'D_i) = \xi > 0$, is in force here. In this situation, the convergence of the input and the output are directly connected and the condition in (A2) (e.g., in Theorem 2) related to the input is not essential; the following lemma formalizes the result.

Lemma 6: If $\inf_{i\in\mathscr{X}} \sigma^-(D_i'D_i) = \xi > 0$ and $\mathscr{Y}_u(x,\theta) < \infty$, then $u(\cdot)$ converges.

Proof: Employing (6) and the assumptions in the lemma, we evaluate $\infty > \mathscr{Y}_u(x,\theta) \geq \sum_{k=0}^{\infty}\langle U(k), D'D\rangle \geq \xi\sum_{k=0}^{\infty}\|U(k)\|_F$. ■

The next result establishes that a linear state feedback control is stabilizing whenever the associated cost is bounded.

Lemma 7: Assume that $(A,C,\mathbb{P})$ is detectable. If $G \in \mathscr{H}_\infty^{r,n}$ is such that $\mathscr{Y}_G(x,\theta) < \infty$ $\forall x \in \mathbb{R}^n, \theta \in \mathscr{X}$, then $(A+BG,\mathbb{P})$ is ℓ_2-stable.

Proof: Consider the system Ψ in closed loop form with $u(k) = G_{\theta(k)}x(k)$,

$$\begin{cases} x(k+1) = (A_{\theta(k)} + B_{\theta(k)}G_{\theta(k)})x(k), & k \geq 0,\\ y(k) = (C_{\theta(k)} + D_{\theta(k)}G_{\theta(k)})x(k). \end{cases} \tag{9}$$

For each initial condition $x \in \mathbb{R}^n$ and $\theta \in \mathscr{X}$ we have from the lemma that $\mathscr{Y}_u(x,\theta) = \mathscr{Y}_G(x,\theta) < \infty$, which means that the associated output $y(\cdot)$ converges; moreover, Lemma 6 provides that $u(\cdot)$ converges. In this situation, detectability yields that $x(\cdot)$ converges, and we conclude that $(A+BG,\mathbb{P})$ is ℓ_2-stable. ■

In what follows, we consider the infinite coupled algebraic Riccati equations (ICARE) in the unknown $R \in \mathscr{H}_F^n$ that arises in the JLQ problem (see, e.g., [7]):

$$0 = \left(A_i + B_iG_i\right)' \sum_{j\in\mathscr{X}} p_{ij}R_j\left(A_i + B_iG_i\right) + C_i'C_i + G_i'D_i'D_iG_i, \tag{10}$$

$$G_i = -\left(D_i'D_i + B_i'\sum_{j\in\mathscr{X}} p_{ij}R_jB_i\right)^{-1} B_i'\sum_{j\in\mathscr{X}} p_{ij}R_jA_i, \quad i \in \mathscr{X}. \tag{11}$$

The following results are adapted from [7].

Proposition 2: Assume that $R \in \mathscr{H}_F^n$ satisfies the ICARE (10)–(11). The following assertions hold:

(i) $\mathscr{Y}_G(x,\theta) \leq x'R_\theta x$;

(ii) If $(A+BG,\mathbb{P})$ is ℓ_2-stable, then $R \in \mathscr{H}_\infty^n$ is the unique solution of the ICARE. Moreover, the solution of the JLQ problem is $u(k) = G_{\theta(k)}x(k)$, where G is given by (11).

Theorem 3: Assume that $(A,C,\mathbb{P})$ is detectable according to Definition 1. Then, the ICARE has at most one solution. Moreover, if $R \in \mathscr{H}_F^n$ is the solution of the ICARE, then $(A+BG,\mathbb{P})$ is ℓ_2-stable with the optimal control (11).

Proof: Let $R \in \mathscr{H}_F^n$ be a solution of the ICARE. From Proposition 2 (i) we have that $\mathscr{Y}_G(x,\theta) \leq x'R_\theta x$, for each x,θ, and Lemma 7 provides that $(A+BG,\mathbb{P})$ is ℓ_2-stable. Hence, Proposition 2 (ii) yields that R is the unique solution of the ICARE and the optimal control is given by (11). ■

Remark 1: The results in this section generalize previous result in [7] from the fact that detectability here generalizes the ℓ_2-detectability notion employed there.

VIII. Examples

This section in the original paper contains an example showing that (A2) does not necessarily hold for MJLS with infinite countably Markov chain. Another example shows that the detectability notion according to Definition 1 depends on the collections of matrices B and D, and thus it cannot be related to the autonomous version Ψ_0 only. A third example shows that the detectability concept generalizes the earlier ℓ_2-detectability and uniform observability concepts. Here we present the first example only.

Example 1: This example illustrates that (A2) does not necessarily hold true for MJLS with infinite countably Markov chain. Indeed, we present a system for which the state trajectory does not converge to $\mathscr{F}$ under converging input and output.

Assume that $p_{ii+1} = 1$, $i \in \mathscr{X}$, in such a manner that $\theta(k) = k+i$ a.s. whenever $\theta(0) = i$. Let $n = 1$, $A_i = B_i = 1, D_i = 0, i \in \mathscr{X}$. As regards to $C \in \mathscr{H}_F^1$, we set $C_1 = 0$ and $C_i = (i-1)^{-1/2}$, $i \geq 2$, in order to get that $C_{\theta(k)} = (k+i-1)^{-1/2}$, $k \geq 1$.

It is simple to check for the autonomous system that $\mathscr{Y}_0(x,\theta) = \sum_{k=0}^{\infty} x^2/(k+i-1)$, which converges if and only if $x = 0$, thus leading to $\mathscr{F} = 0$.

Now, for simplicity, we consider fixed initial conditions $x = 1$ and $\theta = 1$. Consider the control given by $u(0) = 0$ and

$u(k) = (k+1)^{-1/2} - k^{-1/2}$, $k \geq 1$. We get that $x(k) = k^{-1/2}$, $k \geq 1$ is the corresponding trajectory. It is a simple matter to check that (see [16, Chap. 2.6])

$$E_{x,\theta}\left\{\sum_{k=0}^{\infty}|u(k)|^2\right\} = \sum_{k=1}^{\infty}\frac{(k^{1/2}-(k+1)^{1/2})^2}{k(k+1)} \leq \sum_{k=1}^{\infty}\frac{1}{k(k+1)}, \tag{12}$$

the last series converges to one and we have that the input converges. As regards to the output, we first evaluate

$$E\left\{\sum_{k=0}^{\infty}x(k)'C'_{\theta(k)}C_{\theta(k)}x(k)|x=\theta=1\right\} = \sum_{k=1}^{\infty}\frac{1}{k^{4\times 1/2}}$$
$$= \sum_{k=1}^{\infty}\frac{1}{k^2} \leq \sum_{k=0}^{\infty}\frac{1}{2^k} \leq 2, \tag{13}$$

where, in the last inequality, we employed the evaluation in [16, Chap. 3.1]). Together with (12), they provide that $\mathscr{Y}(1,1) \leq 3$, which means that the output converges. However, we can also write that

$$E_{x,\theta}\left\{\sum_{k=0}^{\infty}|x(k)|^2\right\} = \sum_{k=0}^{\infty}\frac{1}{k} = \infty,$$

and the state does not converge to the trivial $\mathscr{F}$. Thus, (A2) does not hold and the system is not detectable.

IX. Conclusions

This paper deals with detectability for discrete-time Markov jump linear systems with countably infinite Markov state. Beginning with Definition 1, which expresses an idea that at same time is purposeful and captures the abstract notion of detectability, we show that it can be written down in terms of conditions (A1) and (A2). Condition (A1) alone refers to the autonomous systems and its behavior within the invariant space $\mathscr{F}$. It is reminiscent of detectability concepts related with finite dimensional linear systems. Condition (A2) refers to the complete system Ψ and its behavior within set $\mathscr{F}^{\perp}$. It comes as an essential condition, connected to the fact that the observed part of the autonomous system, represented by $\mathscr{F}^{\perp}$, may not be uniformly observable, contrary to the finite dimensional case. Example 1 shows that (A2) may fail in the infinite Markov state case. This clarifies that, unlike the finite dimensional contexts, the detectability notion yielding property (i) (stated in section I) cannot be expressed in terms of the parameters of the autonomous version Ψ_0; thus, (iii) cannot be completely reproduced.

Finally, although the analysis here concludes a circle of ideas toward detectability of MJLS, which has began in [1], [3], we believe that the approach via invariant subspaces proposed here may be useful elsewhere, in contexts such as nonlinear systems or other infinite dimensional systems.

X. References

[1] E. F. Costa and J. B. R. do Val, *On the detectability and observability of discrete-time Markov jump linear systems*, Systems Control Lett., 44 (2001), pp. 135–145.

[2] E. F. Costa and J. B. R. do Val, *On the detectability and observability of continuous-time Markov jump linear systems*, SIAM J. Control Optim., 41 (2002), pp. 1295–1314.

[3] E. F. Costa and J. B. R. do Val, *Weak detectability and the linear quadratic control problem of discrete-time Markov jump linear systems*, Int. J. Control, 16/17 (2002), pp. 1282–1292.

[4] E. F. Costa, J. B. R. do Val, and M. D. Fragoso, *On a detectability concept of discrete-time infinite Markov jump linear systems*, Stochastic Analysis and Applications, to appear.

[5] E. F. Costa, J. B. R. do Val, and M. D. Fragoso, *On a detectability concept of discrete-time infinite Markov jump linear systems*, in Proceedings of the 15th Triennal World Congress IFAC, 2002, pp. 2660–2665.

[6] O. L. V. Costa, *Discrete-time coupled Riccati equations for systems with Markov switching parameters*, J. Math. Anal. Appl., 194 (1995), pp. 197–216.

[7] O. L. V. Costa and M. D. Fragoso, *Discrete-time LQ-optimal control problems for infinite Markov jump parameter systems*, IEEE Trans. Automat. Control, 40 (1995), pp. 2076–2088.

[8] M. D. Fragoso and J. Baczynski, *Optimal control for continuous time problems with infinite Markov jump parameters*, SIAM J. Control Optim., 40 (2001), pp. 270–297.

[9] M. D. Fragoso and J. Baczynski, *Lyapunov coupled equations for continuous-time infinite Markov jump linear systems*, J. Math. Anal. Appl., 274 (2002), pp. 319–335.

[10] M. D. Fragoso and J. Baczynski, *Stochastic versus mean square stability in continuous time linear infinite Markov jump parameter systems*, Stochastic Anal. Appl., 20 (2002), pp. 347–356.

[11] Y. Ji and H. J. Chizeck, *Controllability, stabilizability and continuous time Markovian jump linear quadratic control*, IEEE Trans. Automat. Control, 35 (1990), pp. 777–788.

[12] T. Morozan, *Stability and control for linear systems with jump Markov perturbations*, Stochastic Anal. Appl., 13 (1995), pp. 91–110.

[13] W. S. Gray and J. P. Mesko, *Observability functions for linear and nonlinear systems*, Systems Control Lett., 38 (1999), pp. 99–113.

[14] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1990.

[15] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[16] K. Knopp, *Infinite Sequences and Series*, Dover, New York, 1956.

[17] I. R. Petersen, *Notions of observability for uncertain linear systems with structured uncertainty*, SIAM J. Control Optim., 41 (2002), pp. 345–361.