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Stability analysis of decentralized RHC for decoupled systems

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***Abstract*— A detailed study on the stability and design of decentralized Receding Horizon Control (RHC) schemes for decoupled systems is presented. An optimal control problem is formulated for a set of decoupled dynamical systems where cost function and constraints couple the dynamical behavior of the systems. The coupling is described through a connected graph where each system is a node and, cost and constraints of the optimization problem associated to each node are only function of its state and the states of its neighbors. The complexity of the problem is addressed by breaking a centralized RHC controller into distinct RHC controllers of smaller sizes. Each RHC controller is associated to a different node and computes the local control inputs based only on the states of the node and of its neighbors. Stability of the decentralized scheme is analyzed and its properties are compared with alternative decentralized RHC approaches being proposed in the literature.**

I. INTRODUCTION

Research on decentralized control dates back to the pioneering work of [1] and since then, the interest has grown significantly due to various results that attempt to reduce the complexity of the problem [2], [3]. Decentralized control techniques today can be found in a broad spectrum of applications ranging from robotics and formation flight to civil engineering.

Approaches to decentralized control design differ from each other in the assumptions they make on: (*i*) the kind of interaction between different systems or different components of the same system (dynamics, constraints, objective), (*ii*) the model of the system (linear, nonlinear, constrained, continuous-time, discrete-time), (*iii*) the model of information exchange between the systems, and (*iv*) the control design technique used. Dynamically coupled systems have been the most studied.

In this paper, we focus on *decoupled systems*. In a descriptive way, the problem of decentralized control for decoupled systems can be formulated as follows. A dynamical system is composed of (or can be decomposed into) distinct dynamical subsystems that can be independently actuated. The subsystems are dynamically decoupled but have common objectives and constraints which make them interact between each other. Typically the *interaction* is local, i.e. the objective and the constraints of a subsystem are function of only a subset of other subsystems' states. The interaction will be represented by an "interaction graph", where the nodes represent the subsystems and an arc between two nodes denotes a coupling term in the objectives and/or in the constraints associated to the nodes. Also, typically it is assumed that the *exchange of information* has a special structure, i.e., it is assumed that each subsystem can sense and/or exchange information with only a subset of other subsystems. Often the *interaction graph* and the *information exchange graph* coincide. A decentralized control scheme consists of distinct controllers, one for each subsystem, where the inputs to each subsystem are computed only based on local information, i.e. on the states of the subsystem and its neighbors.

Our interest in decentralized control for dynamically decoupled systems arises from the abundance of networks of independently actuated systems and the necessity of avoiding centralized design when this becomes computationally prohibitive. Networks of vehicles in formation, production units in a power plant, network of cameras at an airport, mechanical actuators for deforming surface are just a few examples. Other application examples and current approaches for decentralized control design can be found in [3]–[6].

In this paper we make use of Receding Horizon Control (RHC) schemes. The main idea of RHC is to use the *model* of the plant to *predict* the future evolution of the system [7]. Based on this prediction, at each time step t a certain performance index is optimized under operating constraints with respect to a sequence of future input moves. The first of such optimal moves is the *control* action applied to the plant at time t. At time $t+1$, a new optimization is solved over a shifted prediction horizon.

The complexity of large scale control problems is usually approached by using decentralization. The work in this paper investigates decentralized RHC for decoupled systems [8]. A centralized RHC controller is broken into distinct RHC controllers of smaller sizes. Each RHC controller is associated to a different node and computes the local control inputs based only on the states of the node and of its neighbors.

We first present a systematic and rigorous mathematical framework which takes explicitly into account constraints and uses the model of the neighbors to predict their behavior. Along with the benefits of a decentralized design, inherent issues in ensuring stability and feasibility of the system have to be faced. Local RHC designs might lead to instability of the entire system due to the mismatch of predictions that neighboring subsystems make about each other. Moreover, a critical issue in decentralized RHC schemes is that the inputs computed locally are, in general, not guaranteed to be globally feasible for the overall team. As in classical RHC design, one can enforce stability and feasibility in different ways such as modifying cost and constraints. In decentralized RHC schemes the communications structure and exchange of information between local controllers is an additional degree of freedom which can be used for such goal. To this end, a detailed study of stability properties of the proposed decentralized scheme is presented. We also use our framework

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to analyze stability properties of alternative decentralized RHC approaches being proposed in the literature [8]–[12]. In particular, the importance of information exchange between neighbors is highlighted and its role in stabilizing the entire system is investigated.

The framework presented in this paper has been applied in simulation to a number of large scale control problems with success. References to formation flight application examples using a hovering ducted-fan unmanned air vehicle can be found in [8], [13], [14]. The proposed scheme has also been applied to a paper machine control problem as described in [13], [15].

II. PROBLEM FORMULATION

A concise description of the decentralized RHC scheme proposed in [8] follows. Consider a set of N_v linear decoupled dynamical systems, the i-th system being described by the discrete-time time-invariant state equation

$$x^i_{k+1} = f^i(x^i_k, u^i_k), \tag{1}$$

where $x^i_k \in \mathbb{R}^{n^i}$, $u^i_k \in \mathbb{R}^{m^i}$, $f^i : \mathbb{R}^{n^i} \times \mathbb{R}^{m^i} \to \mathbb{R}^{n^i}$ are state, input and state update function of the i-th system, respectively. Let $\mathcal{X}^i \subseteq \mathbb{R}^{n^i}$ and $\mathcal{U}^i \subseteq \mathbb{R}^{m^i}$ be given polytopes and denote the set of feasible states and inputs of the i-th system, respectively.

We will refer to the set of N_v constrained systems as the *overall system*. Let $\tilde{x}_k \in \mathbb{R}^{N_v n^i}$ and $\tilde{u}_k \in \mathbb{R}^{N_v m^i}$ be the vectors which collect the states and inputs of the overall system at time k, i.e. $\tilde{x}_k = [x^1_k, \ldots, x^{N_v}_k]$, $\tilde{u}_k = [u^1_k, \ldots, u^{N_v}_k]$, with

$$\tilde{x}_{k+1} = f(\tilde{x}_k, \tilde{u}_k). \tag{2}$$

We denote by (x^i_e, u^i_e) the equilibrium pair of the i-th system and $(\tilde{x}_e, \tilde{u}_e)$ the corresponding equilibrium for the overall system.

So far the systems belonging to the overall system are completely decoupled. We consider an optimal control problem for the overall system where cost function and constraints couple the dynamic behavior of individual systems. We use a graph topology to represent the coupling in the following way. We associate the i-th system to the i-th node of the graph, and if an edge (i, j) connecting the i-th and j-th node is present, then the cost and the constraints of the optimal control problem will have a component which is a function of both x^i and x^j. This leads to an undirected interconnection graph $\mathcal{G} = \{\mathcal{V}, \mathcal{A}\}$, where $\mathcal{V}$ is the set of nodes $\mathcal{V} = \{1, \ldots, N_v\}$ and $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$ the sets of arcs (i, j) with $i \in \mathcal{V},\ j \in \mathcal{V}$.

Once the graph structure has been fixed, the optimization problem is formulated as follows. Denote with $\tilde{x}^i$ the states of all neighboring systems of the i-th system, i.e. $\tilde{x}^i = \{x^j \in \mathbb{R}^{n^j} | (j, i) \in \mathcal{A}\}$, $\tilde{x}^i \in \mathbb{R}^{\tilde{n}^i}$ with $\tilde{n}^i = \sum_{j|(j,i)\in\mathcal{A}} n^j$. Analogously, $\tilde{u}^i \in \mathbb{R}^{\tilde{m}^i}$ denotes the inputs to all the neighboring systems of the i-th system. Let

$$g^{i,j}(x^i, x^j) \le 0 \tag{3}$$

define the interconnection constraints between the i-th and the j-th systems, with $g^i : \mathbb{R}^{n^i} \times \mathbb{R}^{n^j} \to \mathbb{R}^{n^{i,j}_c}$.

Consider the following overall cost

$$\begin{aligned} l(\tilde{x}, \tilde{u}) &= \sum_{i=1}^{N_v} l^i(x^i, u^i, \tilde{x}^i, \tilde{u}^i) \qquad (4)\\ &= \sum_{(i,j)\in\mathcal{A}} l^{i,j}(x^i, u^i, x^j, u^j) + \sum_{\substack{(q,r)\in\mathcal{A},\\ (i,q)\in\mathcal{A},(i,r)\in\mathcal{A}}} l^{q,r}(x^q, u^q, x^r, u^r), \end{aligned}$$

where $l^i : \mathbb{R}^{n^i} \times \mathbb{R}^{m^i} \times \mathbb{R}^{\tilde{n}^i} \times \mathbb{R}^{\tilde{m}^i} \to \mathbb{R}$ is the cost associated to the i-th system and is a function only of its states and the states of its neighbor nodes. $l^{i,j} : \mathbb{R}^{n^i} \times \mathbb{R}^{m^i} \times \mathbb{R}^{n^j} \times \mathbb{R}^{m^j} \to \mathbb{R}$ represents a cost function involving two adjacent nodes. Assume that l is a positive convex function and that $l^i(x^i_e, u^i_e, \tilde{x}^i_e, \tilde{u}^i_e) = 0$.

In the preliminary study [8], the complexity associated to a centralized optimal control design for such class of large scale systems is tackled by formulating N_v decentralized finite time optimal control problems, each one associated to a different node as detailed next. Each node has information about its current states and its neighbors' current states. Based on such information, each node computes its optimal inputs and its neighbors' optimal inputs. The input to the neighbors will only be used to predict their trajectories and then discarded, while the first component of the optimal input to the node will be implemented where it was computed. Let the following finite time optimal control problem $\mathcal{P}_i$ with optimal value function $J^{i*}_N(x^i_t, \tilde{x}^i_t)$ be associated to the i-th system at time t

$$\begin{aligned} \min_{\tilde{U}^i_t} \quad & \sum_{k=0}^{N-1} l^i_t(x^i_{k,t}, u^i_{k,t}, \tilde{x}^i_{k,t}, \tilde{u}^i_{k,t}) + l^i_N(x^i_{N,t}, \tilde{x}^i_{N,t}) \\ \text{subj. to} \quad & x^i_{k+1,t} = f^i(x^i_{k,t}, u^i_{k,t}), \qquad (5a)\\ & x^i_{k,t} \in \mathcal{X}^i, \quad u^i_{k,t} \in \mathcal{U}^i, \qquad (5b)\\ & \quad k = 1, \ldots, N-1 \\ & x^j_{k+1,t} = f^j(x^j_{k,t}, u^j_{k,t}), \quad (i,j) \in \mathcal{A}, \qquad (5c)\\ & x^j_{k,t} \in \mathcal{X}^i, \quad u^j_{k,t} \in \mathcal{U}^j, \quad (i,j) \in \mathcal{A}, \qquad (5d)\\ & \quad k = 1, \ldots, N-1 \\ & g^{i,j}(x^i_{k,t}, u^i_{k,t}, x^j_{k,t}, u^j_{k,t}) \le 0, \qquad (5e)\\ & \quad (i,j) \in \mathcal{A}, k = 1, \ldots, N-1 \\ & g^{q,r}(x^q_{k,t}, u^q_{k,t}, x^r_{k,t}, u^r_{k,t}) \le 0, \qquad (5f)\\ & \quad (q,r) \in \mathcal{A},\ (q,i) \in \mathcal{A},\ (r,i) \in \mathcal{A}, \\ & \quad k = 1, \ldots, N-1 \\ & x^i_{N,t} \in \mathcal{X}^i_f, \quad x^j_{N,t} \in \mathcal{X}^j_f, \quad (i,j) \in \mathcal{A} \qquad (5g)\\ & x^i_{0,t} = x^i_t, \quad \tilde{x}^i_{0,t} = \tilde{x}^i_t, \qquad (5h) \end{aligned}$$

where $\tilde{U}^i_t \triangleq [u^i_{0,t}, \tilde{u}^i_{0,t}, \ldots, u^i_{N-1,t}, \tilde{u}^i_{N-1,t}] \in \mathbb{R}^s$, $s \triangleq (\tilde{m}^i + m^i)N$ denotes the optimization vector, $x^i_{k,t}$ denotes the state vector of the i-th node predicted at time $t+k$ obtained by starting from the state x^i_t and applying to system (1) the input sequence $u^i_{0,t}, \ldots, u^i_{k-1,t}$. The tilded vectors denote the prediction vectors associated to the neighboring systems. Denote by $\tilde{U}^{i*}_t = [u^{*i}_{0,t}, \tilde{u}^{*i}_{0,t}, \ldots, u^{*i}_{N-1,t}, \tilde{u}^{*i}_{N-1,t}]$ an optimizer of problem $\mathcal{P}_i$.

Note that problem $\mathcal{P}_i$ involves only the state and input variables of the i-th node and its neighbors at time t. We will define the following decentralized RHC scheme. At time t

1) Each node i solves problem $\mathcal{P}_i$ based on measurements of its state x_t^i and the states of all its neighbors $\tilde{x}_t^i$.
2) Each node i implements the first sample of $\tilde{U}_t^{i*}$

$$u_t^i = u_{0,t}^{*i}. \quad (6)$$

3) Each node repeats steps 1 to 3 at time $t+1$, based on the new state information x_{t+1}^i, $\tilde{x}_{t+1}^i$.

The solution of the i-th subproblem will yield a control policy for the i-th node of the form $u_t^i = c^i(x_t^i, \tilde{x}_t^i)$, where $c^i : \mathbb{R}^{n^i} \times \mathbb{R}^{\tilde{n}^i} \to \mathbb{R}^{m^i}$ is a time-invariant feedback control law implicitly defined by the optimization problem $\mathcal{P}_i$.

Even if we assume N to be infinite, the decentralized RHC approach described so far does not guarantee that solutions computed locally are globally feasible and stable. The reason is simple: at the i-th node the prediction of the neighboring state x^j is done independently from the prediction of problem $\mathcal{P}_j$. Therefore, the trajectory of x^j predicted by problem $\mathcal{P}_i$ and the one predicted by problem $\mathcal{P}_j$, based on the same initial conditions, are different (since in general, $\mathcal{P}_i$ and $\mathcal{P}_j$ will be different). This will imply that constraint fulfillment will be ensured by the optimizer u_t^{*i} for problem $\mathcal{P}_i$ but not for the centralized problem involving the states of all nodes.

Stability and feasibility of decentralized RHC schemes are currently active research areas. In the following section the stability of the decentralized RHC scheme (5)-(6) is analyzed in detail.

III. STABILITY ANALYSIS

Without loss of generality, we assume the origin to be an equilibrium for the overall system. In this section, we rely on the general problem formulation introduced in Section II and focus on systems *with* input and state constraints, *no* coupling constraints and terminal point constraint to the origin $\mathcal{X}_f^i = \mathbf{0}$.

In order to illustrate the fundamental issues regarding stability in a simple way, we first consider two systems ($N_v = 2$). The general formulation for an arbitrary number of nodes is treated later in Section III-B. We consider two decentralized RHC problems $\mathcal{P}_1$ and $\mathcal{P}_2$ according to (5). We will make the following assumption on the structure of individual cost functions l^1 and l^2:

Assumption 1: The cost term l^i in (4) associated to the i-th system can be written as follows

$$\begin{aligned} &l^1(x^1, u^1, x^2, u^2) = l^1(x^1, u^1, x^2, u^2) \\ &= \|Qx^1\|_p + \|Qx^2\|_p + \|Q(x^1 - x^2)\|_p + \|Ru^1\|_p + \|Ru^2\|_p. \end{aligned} \quad (7)$$

Remark 1: The cost function structure in Assumption 1 can be used to describe several practical applications including formation flight, paper machine control and monitoring network of cameras [13].

Consider the RHC problem $\mathcal{P}_1$ and $\mathcal{P}_2$ and assume that they are feasible at time $t = 0$. In classical RHC schemes, stability and feasibility is proven by using the value function as a Lyapunov function. In the decentralized framework proposed in this paper, we will investigate three different approaches to analyzing and ensuring stability of the overall system:

1) Use of individual cost functions as Lyapunov functions for each node (Section III-A).
2) Use of the sum of individual cost functions as Lyapunov function for the entire system (Section III-C).
3) Exchange of optimal solutions between neighbors (Section III-D).

A. Individual value functions as Lyapunov functions

In this section, we give sufficient conditions for the stability of each individual node, which lead to stability of the entire system. The following notation will be used to describe state and input signals. For a particular variable, the first superscript refers to the index of the corresponding system, the second superscript refers to the location where it is computed. For instance the input $u^{i,j}$ represents the input to the i-th system calculated by solving problem $\mathcal{P}_j$. Similarly, the state variable $x^{i,j}$ stands for the states of system i predicted by solving $\mathcal{P}_j$. The lower indices conform to the standard time notation of RHC schemes. For example, variable $x_{k,t}$ denotes the k-step ahead prediction of the states made at time instant t.

In order to simplify notation, we define

$$\ell^1(x_t^1, U_t^{1,1}, x_t^2, U_t^{2,1}) = \sum_{k=1}^{N} l^1(x_{k,t}^{1,1}, u_{k,t}^{1,1}, x_{k,t}^{2,1}, u_{k,t}^{2,1}), \quad (8)$$

where x_t^1 and x_t^2 are the initial states of systems 1 and 2 at time t, and $U_t^{1,1} = \{u_{0,t}^{1,1}, \ldots, u_{N-1,t}^{1,1}\}$, $U_t^{2,1} = \{u_{0,t}^{2,1}, \ldots, u_{N-1,t}^{2,1}\}$ are the control sequences for node 1 and 2 calculated by node 1. Let $[U_0^{1,1*}, U_0^{2,1*}]$ be an optimizer of problem $\mathcal{P}_1$ for $t = 0$:

$$U_0^{1,1*} = \{u_{0,0}^{1,1}, \ldots, u_{N-1,0}^{1,1}\}, \quad U_0^{2,1*} = \{u_{0,0}^{2,1}, \ldots, u_{N-1,0}^{2,1}\}, \quad (9)$$

and $\mathbf{x}_0^{1,1} = \{x_{0,0}^{1,1}, \ldots, x_{N,0}^{1,1}\}$, $\mathbf{x}_0^{2,1} = \{x_{0,0}^{2,1}, \ldots, x_{N,0}^{2,1}\}$, be the corresponding optimal state trajectories of node 1 and 2 predicted at node 1 by $\mathcal{P}_1$.

Analogously, let $[U_0^{1,2*}, U_0^{2,2*}]$ be an optimizer of problem $\mathcal{P}_2$ for $t = 0$:

$$U_0^{1,2*} = \{u_{0,0}^{1,2}, \ldots, u_{N-1,0}^{1,2}\}, \quad U_0^{2,2*} = \{u_{0,0}^{2,2}, \ldots, u_{N-1,0}^{2,2}\}, \quad (10)$$

and $\mathbf{x}_0^{1,2} = \{x_{0,0}^{1,2}, \ldots, x_{N,0}^{1,2}\}$, $\mathbf{x}_0^{2,2} = \{x_{0,0}^{2,2}, \ldots, x_{N,0}^{2,2}\}$, be the corresponding optimal state trajectories of node 1 and 2 predicted at node 2 by $\mathcal{P}_2$. By hypothesis, neighboring systems either measure or exchange state information, so the initial states for both problems are the same at each time step, i.e. $x_{0,0}^{1,1} = x_{0,0}^{1,2}$ and $x_{0,0}^{2,1} = x_{0,0}^{2,2}$.

We denote the set of states of node i at time k feasible for problem $\mathcal{P}_i$ by

$$\begin{aligned} &\mathcal{X}_k^i = \left\{x^i \mid \exists u^i \in \mathcal{U}^i \text{ such that } f^i(x^i, u^i) \in \mathcal{X}_{k+1}^i\right\} \cap \mathcal{X}^i, \\ &\quad \text{with } \mathcal{X}_N^i = \mathcal{X}_f^i. \end{aligned} \quad (11)$$

Since we are neglecting coupling constraints, the set of feasible states for the decentralized RHC scheme (5)-(6) applied to the overall system is the cross product of the feasible set of states associated to each node:

$$\mathcal{X}_k = \bigtimes_{i=1}^{N_v} \mathcal{X}_k^i, \quad (12)$$

where the symbol $\times$ denotes the standard Cartesian product of sets.

Denote with $c(\tilde{x}_k) = [u_{0,k}^{1,1*}(\tilde{x}_k), u_{0,k}^{2,2*}(\tilde{x}_k)]$ the control law obtained by applying the decentralized RHC policy (5)-(6) with cost function (8), when the current state is $\tilde{x}_k = [x_k^1, x_k^2]$. Consider the overall system model (2) consisting of two nodes ($N_v = 2$), and denote with

$$\tilde{x}_{k+1} = f\left(\tilde{x}_k, c(\tilde{x}_k)\right), \tag{13}$$

the closed-loop dynamics of the entire system. In the following theorem we state sufficient conditions for the asymptotic stability of the closed-loop system.

Theorem 1: Assume that

(A0) $Q = Q' \succ 0, R = R' \succ 0$ if $p = 2$ and Q, R are full column rank matrices if $p = 1, \infty$.

(A1) The state and input constraint sets $\mathcal{X}^1, \mathcal{X}^2$ and $\mathcal{U}^1, \mathcal{U}^2$ contain the origin in their interior.

(A2) The following inequality is satisfied for all $x_0^i \in \mathcal{X}_0^i$, $x_0^j \in \mathcal{X}_0^j$ with $i = 1$, $j = 2$ and $i = 2$, $j = 1$:

$$\varepsilon \leq \|Qx_0^i\|_p + \|Qx_0^j\|_p + \|Q(x_0^i - x_0^j)\|_p + \|Ru_{0,0}^{i,i}\|_p + \|Ru_{0,0}^{j,i}\|_p, \tag{14}$$

where

$$\varepsilon = \sum_{k=1}^{N-1} \left(2\|Q(x_{k,0}^{j,j} - x_{k,0}^{j,i})\|_p + \|R(u_{k,0}^{j,j} - u_{k,0}^{j,i})\|_p\right). \tag{15}$$

Then, the origin of the closed loop system (13) is asymptotically stable with domain of attraction $\mathcal{X}_0^1 \times \mathcal{X}_0^2$.

Proof: Consider first problem $\mathcal{P}_1$ and its optimal solution $U_0^{1,1*}$ and $U_0^{2,1*}$ at initial time 0. The shifted sequences $U_1^{1,1} = \{u_{1,0}^{1,1}, \ldots, u_{N-1,0}^{1,1}, \mathbf{0}\}$ and $U_1^{2,1} = \{u_{1,0}^{2,1}, \ldots, u_{N-1,0}^{2,1}, \mathbf{0}\}$ of problem $\mathcal{P}_1$, are not necessarily feasible at the next time step $t = 1$ since the state of system 2 at time 1 is $x_{1,0}^{2,2}$ and not $x_{1,0}^{2,1}$, even assuming no model uncertainty. However, one can construct a feasible shifted sequence by using the optimizer of problem $\mathcal{P}_2$

$$U_1^{2,2} = \{u_{1,0}^{2,2}, \ldots, u_{N-1,0}^{2,2}, \mathbf{0}\}. \tag{16}$$

This is possible, since the dynamics of both subsystems are decoupled. Furthermore, we have assumed no coupling constraints, which implies that $U_1^{1,1}$ and $U_1^{2,2}$ will be feasible at time $t = 1$ for problem $\mathcal{P}_1$.

At the next time step ($t = 1$), the current states of the two systems are denoted by $x_{0,1}^{1,1}$ and $x_{0,1}^{2,2}$. Since the neighboring state information is exchanged between nodes, or assumed to be measured, we have $x_{0,1}^{1,2} = x_{0,1}^{1,1}$ and $x_{0,1}^{2,1} = x_{0,1}^{2,2}$ as well. We use the following notation:

$$\begin{aligned} x_0^1 &= x_{0,0}^{1,1} = x_{0,0}^{1,2} & x_1^1 &= x_{0,1}^{1,1} = x_{0,1}^{1,2} \\ x_0^2 &= x_{0,0}^{2,2} = x_{0,0}^{2,1} & x_1^2 &= x_{0,1}^{2,2} = x_{0,1}^{2,1} \\ \tilde{x}_0 &= (x_0^1, x_0^2) & \tilde{x}_1 &= (x_1^1, x_1^2) \end{aligned}$$

We can compute a bound on the value function as follows:

$$J^1(\tilde{x}_1) \leq \ell^1(x_1^1, U_1^{1,1}, x_1^2, U_1^{2,2}) \tag{17a}$$

$$= J^1(\tilde{x}_0) - \|Qx_0^1\|_p - \|Qx_0^2\|_p - \|Q(x_0^1 - x_0^2)\|_p - \|Ru_{0,0}^{1,1}\|_p - \|Ru_{0,0}^{2,1}\|_p \tag{17b}$$

$$- \sum_{k=1}^{N-1} (\|Qx_{k,0}^{2,1}\|_p - \|Qx_{k,0}^{2,2}\|_p) \tag{17c}$$

$$- \sum_{k=1}^{N-1} (\|Ru_{k,0}^{2,1}\|_p - \|Ru_{k,0}^{2,2}\|_p) \tag{17d}$$

$$- \sum_{k=1}^{N-1} (\|Q(x_{k,0}^{1,1} - x_{k,0}^{2,1})\|_p - \|Q(x_{k,0}^{1,1} - x_{k,0}^{2,2})\|_p). \tag{17e}$$

It should be emphasized that in (17a) the cost function ℓ^1 of problem $\mathcal{P}_1$ is evaluated using the feasible shifted input sequence $U_1^{2,2}$ for node 2 and the corresponding state trajectory.

The cost function $J^{1*}(\tilde{x}_0)$ in (17) is associated to the optimal control solution $U_0^{2,1*}$ of $\mathcal{P}_1$. The cost ℓ_1 in (17a) instead is evaluated at the sequence $U_1^{2,2}$ associated to $\mathcal{P}_2$. The mismatch between the two control sequences $U_1^{2,2}$ in (16) and $U_0^{2,1*}$ in (9) generates the terms in (17d). The difference between these control sequences generates also a mismatch between the state trajectories of node 2 predicted at node 1 and predicted at node 2. These are represented by the terms in (17c) and (17e).

Using the homogenity axiom of vector norms and applying $\|\alpha\|_p - \|\beta\|_p \leq \|\alpha - \beta\|_p$ leads to

$$J^{1*}(\tilde{x}_1) \leq J^{1*}(\tilde{x}_0) - (\text{terms in (17b)}) \tag{18a}$$

$$+ \sum_{k=1}^{N-1} \left(2\|Q(x_{k,0}^{2,2} - x_{k,0}^{2,1})\|_p + \|R(u_{k,0}^{2,2} - u_{k,0}^{2,1})\|_p\right). \tag{18b}$$

Notice that the term (18b) arises from the control solution mismatch between $\mathcal{P}_1$ and $\mathcal{P}_2$, and it represents $\varepsilon = \sum_{k=1}^{N-1}(2\|Q(x_{k,0}^{j,j} - x_{k,0}^{j,i})\|_p + \|R(u_{k,0}^{j,j} - u_{k,0}^{j,i})\|_p)$ defined in (15) for $i = 1$, $j = 2$. It follows that if inequality (14) holds, then $J^{1*}(\tilde{x}_1) \leq J^{1*}(\tilde{x}_0)$. This implies that under the assumptions of Theorem 1 $J_N^{1*}(\tilde{x})$ is positive and non-increasing along the closed-loop trajectories, thus can be used as a Lyapunov function for node 1. The same derivation applies to node 2 and its associated cost function.

The rest of the proof follows from Lyapunov arguments, close in spirit to the arguments of [16] where it is established that the value function $J^*(\cdot)$ of the receding horizon problem is a Lyapunov function for the closed-loop system. Based on the hypothesis (A0) on the matrices Q and R, inequality (14) is sufficient to ensure that the state of the closed-loop system (13) converges to zero as $k \to \infty$. Stability follows from the fact that $J^1(x)$ and $J^2(x)$ can be lower and upper bounded by functions $\alpha(\|\tilde{x}\|)$ and $\beta(\|\tilde{x}\|)$, where $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are of class K [7]. □

Theorem 1 highlights the relationship between the stability of the decentralized scheme (5)-(6), the prediction mismatch and the initial conditions of the overall system. The term ε in inequality (14) is a function of the error between the trajectories of node 2 predicted by node 1 and the one predicted by node 2 itself. The smaller the error, the larger the set of initial states for which the value function will decrease along the overall system trajectories.

Remark 2: It should be noted that in the two-system scenario presented in this section, problems $\mathcal{P}_1$ and $\mathcal{P}_2$ are identical and multiple optima can arise only from non-strictly convex cost functions. However, in general, non-convex coupling constraints are a source of multiple optimal solutions as well. Furthermore, for larger number of nodes with an arbitrary graph interconnection, neighboring nodes will not be solving the same problem even without coupling constraints. This leads to different optimal solutions and warrants distinguishing between $U^{1,1*}, U^{2,1*}$ in (9) and $U^{1,2*}, U^{2,2*}$ in (10).

Similar ideas can be used if instead of a terminal point constraint, nonzero terminal cost and terminal set constraints $\mathcal{X}_f \neq \mathbf{0}$ are used. In this case, the terminal set has to be control invariant and the terminal cost is chosen as a control Lyapunov function [7], [17], [18].

B. Generalization to arbitrary graph

The development of Section III-A carries over to any number of nodes and general graph structure. Let us denote the decentralized receding horizon control law for the overall system with $c(\tilde{x}_k) = [u_{0,k}^{1,1*}(\tilde{x}_k^1), \ldots, u_{0,k}^{N_v,N_v*}(\tilde{x}_k^{N_v})]$, obtained by applying the decentralized RHC policy (5)-(6) of each subproblem $\mathcal{P}_i$ when the current state is $\tilde{x}_k = [x_k^1, \ldots, x_k^{N_v}]$. Note that since there are no coupling constraints, the feasible states for the overall system is the cross product of the feasible states associated to each node as defined in (11) and (12). Consider the system model (2) and denote by

$$\tilde{x}_{k+1} = f\left(\tilde{x}_k, c(\tilde{x}_k)\right), \tag{19}$$

the closed-loop dynamics of the overall system. Sufficient conditions for asymptotic stability of the closed-loop system are given next.

Theorem 2: Assume

- *(A0)* $Q = Q' \succ 0, R = R' \succ 0$ if $p = 2$ and Q, R are full column rank matrices if $p = 1, \infty$.
- *(A1)* The state and input constraint sets $\mathcal{X}^i$ and $\mathcal{U}^i$ contain the origin for each node in their interior.
- *(A2)* The following inequality is satisfied for each node and all $x_0^i \in \mathcal{X}_0^i$:

$$\sum_{j|(i,j)\in\mathcal{A}} \varepsilon^{i,j} + \epsilon^i \leq J_0^{i*}, \tag{20}$$

where

$$\varepsilon^{i,j} = \sum_{k=1}^{N-1} \left(2\|Q(x_{k,0}^{j,j} - x_{k,0}^{j,i})\|_p + \|R(u_{k,0}^{j,j} - u_{k,0}^{j,i})\|_p\right), \tag{21}$$

$$\epsilon^i = \sum_{k=1}^{N-1} \sum_{\substack{(q,r)\in\mathcal{A},\\ (i,q)\in\mathcal{A},(i,r)\in\mathcal{A}}} \left(\|Q(x_{k,0}^{q,q} - x_{k,0}^{q,i}) - Q(x_{k,0}^{r,i} - x_{k,0}^{r,r})\|_p\right), \tag{22}$$

and

$$J_0^{i*} = \|Qx_0^i\|_p + \|Ru_{0,0}^{i,i}\|_p + \sum_{j|(i,j)\in\mathcal{A}} (\|Qx_0^j\|_p + \|Ru_{0,0}^{j,i}\|_p) + \sum_{j|(i,j)\in\mathcal{A}} \|Q(x_0^i - x_0^j)\|_p + \sum_{\substack{(q,r)\in\mathcal{A},\\ (i,q)\in\mathcal{A},(i,r)\in\mathcal{A}}} \|Q(x_0^q - x_0^r)\|_p. \tag{23}$$

Then, the origin of the closed loop system (19) is asymptotically stable with domain of attraction $\bigtimes_{i=1}^{N_v} \mathcal{X}_0^i$.

Proof: The proof follows along the lines of Theorem 1. The difference is the derivation of stability condition (20) for any particular node within an arbitrary graph interconnection $\mathcal{A}$. This derivation is omitted here for brevity and can be found in [19].

C. Sum of value functions as Lyapunov function

If we consider the sum of individual cost functions as a Lyapunov function for the entire system, the value function inequality such as the one in (18) will involve significantly more terms than the case presented in the previous sections [19]. In fact, this condition might be less restrictive than the one presented in (20). Even if the individual inequalities for single systems such as (18) presented in the previous section might not hold for every subproblem $\mathcal{P}_i$, the sum of individual value functions could still be used as a Lyapunov function for the entire system. This will be the case if the quantities $\sum_{j|(i,j)\in\mathcal{A}} \varepsilon^{i,j}$ and ϵ^i of nodes with decreasing individual Lyapunov functions $J_N^{i*}(x^i, \tilde{x}^i)$ will be small enough to compensate for those associated with the non-decreasing ones.

D. Exchange of information

Stability conditions derived in the previous sections show that it is the mismatch between the predicted and actual control solutions of neighbors that plays a central role in the stability problem. Therefore we are prompted to investigate how sufficient conditions for stability could be improved by allowing the exchange of optimal solutions between neighbors. By examining the stability condition (14) from this aspect, we can immediately make two general observations:

1) Using bounds on the mismatch between the predicted and actual inputs and states of neighbors, the stability condition (14) could be made less restrictive by reducing the size of positive terms in (18b), which adversely affect the value function variation of (18). In other words, using a coordination scheme based on information exchange, it may be possible to reduce the size of ε to decrease the left side of inequality (14).
2) Also, one can observe that as each node is getting closer to its equilibrium (in our example the origin) the right side of inequality (14) starts to diminish, which leads to more stringent restrictions on the allowable prediction mismatch between neighbors, represented by the left side of the inequality.

These observations suggest that information exchange between neighboring nodes has a beneficial effect in proving stability, *if it leads to reduced prediction mismatch.* As each system converges to its equilibrium, assumptions on the behavior of neighboring systems should get more and more accurate to satisfy the stability condition (14). In fact, as system (13) approaches its equilibrium the right hand side of inequality (14) decreases. In turn, the left hand side of inequality (14) has to diminish as well. This leads to the counter-intuitive conclusion that an increasing information exchange rate between neighbors might be needed when approaching the equilibrium.

These conclusions are in concert with the stability conditions of a distributed RHC scheme proposed in [10], where it is shown that information exchange between neighbors must happen increasingly more frequently as the system equilibrium is approached. However, our simulation examples [8], [13], [20] suggest that the prediction errors between neighbors tend to disappear as each node approaches its equilibrium, and the prediction mismatch converges to zero at a faster rate than the decay in the right hand side.

Exchange of information has been proposed in the approach of [12], [21] as well, where a decentralized RHC scheme is considered for a special graph structure based on a leader-follower architecture. Nodes transmit their optimal solutions each time step to their followers. Each node incorporates the received leader information regarding the current time step in its solution, which is robustified against uncertain predictions about their own followers based on data from the previous time step. Robustification to the followers' RHC problems is based on the expected change in optimal receding horizon solutions from one time step to another. Characterizing the possible one-step changes of optimal solutions is not trivial and might result in a conservative robustification scheme.

E. The effect of horizon length

Increasing the prediction horizon length in decentralized RHC has different consequences than classical results in predictive control suggest. The fundamental difference comes from inaccurate predictions made about neighboring nodes in a decentralized scheme. These prediction errors may increase as the horizon length is enlarged leading to loss in performance [9] and in some cases even instability.

Based on the stability condition given in (14), it is straightforward to see that by increasing the prediction horizon, the number of positive terms in ε increases and the left side of the inequality becomes larger. If the horizon length is too long, the inequality might not hold eventually. Although instability does not necessarily emerge as a consequence of this, simulation examples show that global performance starts to deteriorate after a certain horizon length [8].

IV. ENSURING FEASIBILITY

If coupling constraints are present, ensuring feasibility in a decentralized receding horizon control scheme without introducing excessively conservative assumptions is a challenging problem. Consider for instance the problem of choosing local terminal regions for each decentralized RHC problem (5). Even if we assume terminal point constraints (the most conservative and simplest formulation), infeasibility for the overall system can occur since coupling constraints may be violated due to the mismatch between neighbors' predictions and their closed loop behavior. Indeed, decoupled terminal regions do not enforce feasibility of coupling constraints, which renders them less effective than in a centralized approach. Investigation of different options for reducing the uncertainty about the behavior of neighboring systems in order to ensure feasibility in the presence of coupling constraints seems to be a more appealing approach. Problem (5) needs to be modified to accomplish this objective. For further discussion on this topic we refer to [19], which uses robust constraint fulfillment and proposes other alternative approaches that could be used to tackle the feasibility problem in practice.

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