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Simultaneous boundary control of a Rao-Nakra sandwich beam

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Abstract— **We consider the problem of boundary control of a system of three coupled partial differential equations that describe a three layer (Rao-Nakra type) sandwich beam with damping proportional to shear included in the core layer. In the case where one control is applied to each equation, we obtain exact controllablity modulo a finite dimensional quotient in a time determined by the three wave speeds. We show that in a longer time, under some mild conditions on the parameters, we can recover a similar exact controllability result using only two, or possibly even one appropriately chosen control function.**

I. INTRODUCTION

The classical Rao-Nakra [3] sandwich beam model consists of two outer "face plate" layers (which are assumed to be relatively stiff) which "sandwich" a much more compliant "core layer". The Rao-Nakra model is derived using Euler-Bernoulli beam assumptions for the face plate layers, Timoshenko beam assumptions for the core layer and a "no-slip" assumption for the displacements along the interface. If the bending stiffness and longitudinal inertia of the core layer is small compared to those of the outer layers the following set of equations are obtained [4]:

$$m\ddot{w} - \alpha D_x^2\ddot{w} + KD_x^4 w - D_x N h_2(G_2\varphi + \tilde{G}_2\dot{\varphi}) = 0 \quad (1)$$

$$\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\ddot{\mathbf{v}}_{\mathcal{O}} - \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}D_x^2\mathbf{v}_{\mathcal{O}} + \mathbf{B}^{\mathbf{T}}(G_2\varphi + \tilde{G}_2\dot{\varphi}) = 0 \quad (2)$$

on $(0, L) \times (0, \infty)$ where $\varphi = h_2^{-1}\mathbf{B}v_{\mathcal{O}} + Nw_x$. In addition we consider the following controlled boundary conditions valid for $t > 0$:

$$\begin{aligned} w(0,t) = D_x^2 w(0,t) = D_x\mathbf{v}_{\mathcal{O}}(0,t) = w(L,t) = 0,\\ D_x^2 w(L,t) = M(t),\ D_x\mathbf{v}_{\mathcal{O}}(L,t) = \mathbf{g}_{\mathcal{O}}(t). \end{aligned} \quad (3)$$

In the above, w denotes the transverse displacement of the beam, φ denotes the shear angle of the core layer, $\mathbf{v}_{\mathcal{O}} = (v_1, v_3)^T$ is the vector of longitudinal displacement along the neutral axis of the outer layers. ($i = 1, 3$ is for the outer layers, $i = 2$ is for the core layer.) The density of the ith layer is denoted ρ_i, the thickness h_i, the Young's modulus E_i, the shear modulus of the core layer is G_2. We let $m = \sum h_i\rho_i$ denote the mass density per length, $\alpha = \rho_1 h_1^3/12 + \rho_3 h_3^3/12$ is a moment of inertia parameter, $K = E_1 h_1^3/12 + E_3 h_3^3/12$ is the bending stiffness. In addition,

$$\mathbf{p}_{\mathcal{O}} = \text{diag}\,(\rho_1, \rho_3),\ \mathbf{h}_{\mathcal{O}} = \text{diag}\,(h_1, h_3),$$

$$\mathbf{E}_{\mathcal{O}} = \text{diag}\,(E_1, E_3),\ \mathbf{B} = (-1, 1),\ N = \frac{h_1 + h_2 + h_3}{h_2}.$$

In the above and throughout this paper the $\mathcal{O}$ subscript refers to quantities associated with the outer (odd-indexed) layers. The boundary control functions acting at the right end of the beam are $M(t) = -\frac{\tilde{M}(t)}{K}$, where $\tilde{M}(t)$ is the applied moment, and $\mathbf{g}_{\mathcal{O}}(t) = (g_1(t), g_3(t))^T = (\frac{\hat{g}_1}{E_1 h_1}, \frac{\hat{g}_3}{E_3 h_3})$, where $(\hat{g}_1(t), \hat{g}_3(t))$ are the longitudinal forces. For precise definition of the forces see [4]. For background, history and application of sandwich beam and plate theories (including the Rao-Nakra theory), see e.g., Sun and Lu [11].

The problem of controlling an initial finite energy state to another in time T with controls $\mathbf{g}_{\mathcal{O}}$, $M(t)$ belonging to $L^2(0,T)$ has been considered in [6]. Under the condition that the wave speeds $\sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}$ are distinct, if $T > \tau$ where

$$\tau = 2L\left[\min\left(\sqrt{\frac{K}{\alpha}}, \sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}\right)\right]^{-1}, \quad (4)$$

the system (1)–(3) is exactly controllable modulo a finite dimensional quotient. Put another way, the uncontrollable subspace is at most finite dimensional. If the damping $\tilde{G}_2$ is sufficiently small, this finite dimensional quotient reduces to the one-dimensional space determined by the "zero energy" uncontrollable state $w = 0, (v_1, v_3) = (1, 1)$ (See Theorem 4, Corollary 2). In this paper, we extend this result in several ways.

First, we find that this same result is true even for the case of identical wave speeds. Secondly, we give sufficient conditions for the same result to hold with a reduced number of boundary controls. This type of problem has been referred to as a *simultaneous boundary control problem* in e.g., [10], [12], since one or two controls are to be designed to do the work of three control inputs.

In particular, we consider two physically motivated choices of (simultaneous) boundary controls. In the case that the top and bottom of the beam at the endpoint $x = L$ are subject to surface tractions $\hat{g}_1(t)$, and $\hat{g}_3(t)$, the controls in (3) take the form (see [4]),

$$\mathbf{g}_{\mathcal{O}} = (\frac{\hat{g}_1(t)}{E_1 h_1}, \frac{\hat{g}_3(t)}{E_3 h_3})^T, M(t) = (\frac{h_1}{2K}\hat{g}_1(t) - \frac{h_3}{2K}\hat{g}_3(t)). \quad (5)$$

In the case that the surface tractions are applied in equal and opposite amounts, the controls take the form

$$\mathbf{g}_{\mathcal{O}} = (\frac{u(t)}{h_1 E_1}, -\frac{u(t)}{h_3 E_3})^T,\ M(t) = \frac{h_1 + h_3}{2K}u(t). \quad (6)$$

The spectrum associated with the PDE (1)-(2) consists of three branches of eigenvalues lying in a vertical strip

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of the complex plane with the real parts of each branch asymptotically approaching a limit a_i, $i = 0, 1, 3$ (corresponding to each of the wave speeds in (4)). Under some very mild conditions on the parameters, the three numbers a_i are distinct. In this case, we are able to prove that a single control $u(t)$ of the form (6) can be used to obtain exact controllability modulo a finite dimensional quotient (but in a longer control time).

In the case the wave speeds $\sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}$ are not distinct, it turns out that two of the numbers a_i are zero, (one is negative) and hence more than one control is necessary to obtain the same type of exact controllability result. Nevertheless, we are able to show that controls of the form (5) are sufficient to obtain the same type of controllability result.

We remark that in all of the cases, there is a one dimensional uncontrollable subspace corresponding to constant translational motion. In Corollary 2 we give some sufficient conditions under which the uncontrollable subspace is exactly this space of translational motions.

The paper is organized as follows. In Section II we describe the semigroup formulation of (1), (3). In Section III we describe spectral properties of the system (1),(2) and the Riesz basis property for the associated eigenfunctions; (see Theorem 2). In Section IV we analyze the moment problem and describe controllability results with three controls. In Section V we describe our simultaneous controllability results.

II. Semigroup formulation

Let $(u, v) = \int_0^L u \cdot \bar{v}\, dx$, where u may be either scalar or vector valued. Define quadratic forms a and c by

$$c(w, \mathbf{v}_{\mathcal{O}}) = (mw, w) + \alpha(w_x, w_x) + (\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\mathbf{v}_{\mathcal{O}}, \mathbf{v}_{\mathcal{O}})$$

$$a(w, \mathbf{v}_{\mathcal{O}}) = K(w_{xx}, w_{xx}) + (\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}\mathbf{v}_{\mathcal{O}x}, \mathbf{v}_{\mathcal{O}x}) + (G_2 h_2 \varphi, \varphi).$$

The energy of the beam is given by

$$\mathcal{E}(t) = \frac{R}{2}(c(\dot{w}, \dot{\mathbf{v}}_{\mathcal{O}}) + a(w, \mathbf{v}_{\mathcal{O}}))$$

where R is the width of the beam. Let $U = (u, \mathbf{u})^T := (w, \mathbf{v}_{\mathcal{O}})^T$, $V = (v, \mathbf{v})^T := (\dot{w}, \dot{\mathbf{v}}_{\mathcal{O}})^T$, $Y = (U, V)$. Also define $J : H^2(0, L) \cap H_0^1(0, L) \to L^2(0, L)$ by $J\theta = m\theta - \alpha D_x^2\theta$. The first order form of (1) with M and $\mathbf{g}_{\mathcal{O}}$ set to zero is

$$\frac{dY}{dt} = \mathcal{A}Y := \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix},$$

where $A_1 U =$

$$\begin{pmatrix} J^{-1}(-KD_x^4 u + D_x N h_2 G_2[h_2^{-1}(\mathbf{B}\mathbf{u} + h_2 N D_x u)] \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}[\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}} D_x^2 \mathbf{u} - \mathbf{B}^T G_2[h_2^{-1}(\mathbf{B}\mathbf{u} + h_2 N D_x u)]] \end{pmatrix}.$$

and

$$A_2 V = \begin{pmatrix} J^{-1}(D_x N h_2 \tilde{G}_2[h_2^{-1}(\mathbf{B}\mathbf{v} + h_2 N D_x v)] \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}[-\mathbf{B}^T \tilde{G}_2[h_2^{-1}(\mathbf{B}\mathbf{v} + h_2 N D_x v)]] \end{pmatrix}.$$

The energy inner product is defined by

$$< Y, \hat{Y} >_e = a(U; \hat{U}) + c(V; \hat{V}), \qquad (\hat{Y} = (\hat{U}, \hat{V})),$$

where $a(\cdot\,;\cdot)$ and $c(\cdot\,;\cdot)$ are the bilinear forms that coincide with the previously defined quadratic forms $a(\cdot)$, $c(\cdot)$ on the diagonal. Let

$$\begin{aligned} X_1 &= \{u, \mathbf{u}\} \in H^2(0, L) \cap H_0^1(0, L) \times (H^1(0, L))^2 \\ X_0 &= \{u, \mathbf{u}\} \in H_0^1(0, L) \times (L^2(0, L))^2. \end{aligned}$$

It can be shown [4] that the equations of motion are well-posed on the energy space $(U, \dot{U}) \in C([0, T]; X_1 \times X_0)$. It is not hard to prove the same for semigroup solutions. The domain of this semigroup is $\mathcal{D}(\mathcal{A}) = X_2 \times X_1$, where $X_2 = \{(u, \mathbf{u}) \in X_1 : u \in H^3(0, L),\ \mathbf{u} \in (H^2(0, L))^2 +$ BC's $\}$ where "+BC's" means $D_x^2 u$ and $D_x \mathbf{u}$ vanish at each end.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$ be as above. Then $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to X_1 \times X_0$ is the generator of a C_0 dissipative semigroup on $X_1 \times X_0$.*

One may formulate the equations of motion (1)–(2) as follows:

$$\frac{d}{dt}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{B}\{\hat{M}, \hat{\mathbf{g}}_{\mathcal{O}}\} \end{pmatrix}; \tag{7}$$

where

$$\mathcal{B}\{\hat{M}, \hat{\mathbf{g}}_{\mathcal{O}}\} = \begin{pmatrix} J^{-1}\hat{M}(t)\delta_L'(x) \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}\hat{\mathbf{g}}_{\mathcal{O}}\delta_L(x) \end{pmatrix}. \tag{8}$$

For the inputs defined in (7)-(8), it has been shown in [6] that (7) is well posed on $X_1 \times X_0$. As a consequence, we have:

Corollary 1. *If the initial data $(\{U_0, V_0\}^T)$ belongs to $X_1 \times X_0$ and $u(t) \in L^2(0, T)$ then there exists a unique solution $\{U, V\}^T \in C([0, T]; X_1 \times X_0)$ to (7)-(8). Furthermore for some $C > 0$ we have*

$$||\{U, V\}(t)||_{X_1 \times X_0} \leq C(||\{U_0, V_0\}||_{X_1 \times X_0} + ||u||_{L^2(0,T)})$$

for all $t \in [0, T]$.

III. Spectral analysis of $\mathcal{A}$

In this section we describe the spectrum of the operator $\mathcal{A}$. In the case of distinct wave speeds, the calculations can be found in [6]. The case of identical wave speeds requires a separate analysis which, for reasons of brevity, we omit. The spectrum of $\mathcal{A}$ consists entirely of the eigenvalues

$$\sigma(\mathcal{A}) = \bigcup_{k=0}^{\infty} S_k,$$

where S_0 consists of the double eigenvalue at 0 and the roots of the following:

$$\lambda^2 + \lambda R\tilde{G}_2/h_2 + RG_2/h_2 = 0, \quad R = \mathbf{B}\mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}\mathbf{B}^T > 0, \tag{9}$$

and for $k \in \mathbb{N}$,

$$S_k = \{\lambda_{k,0}^+, \lambda_{k,0}^-, \lambda_{k,1}^+, \lambda_{k,1}^-, \lambda_{k,3}^+, \lambda_{k,3}^-\},$$

where $\lambda^+_{k,j}, \lambda^-_{k,j}$ are complex conjugates. In the case of *distinct wave speeds* we have

$$\lambda^+_{k,0} = -\frac{N^2\tilde{G}_2 h_2}{2\alpha} + i\sigma_k\sqrt{\frac{K}{\alpha}} + \mathcal{O}(k^{-1}) \quad (10)$$

$$\lambda^+_{k,j} = -\frac{\tilde{G}_2}{2h_2 h_j \rho_j} + i\sigma_k\sqrt{\frac{E_j}{\rho_j}} + \mathcal{O}(k^{-1}),\ j=1,3, \quad (11)$$

where $\sigma_k = \frac{k\pi}{L}$. In the case of *identical wave speeds*, they are as follows:

$$\lambda^+_{k,j} \quad = \quad r_j + i\mu\sigma_k + \mathcal{O}(k^{-1}), \quad (12)$$

where

$$\{r_0, r_1, r_3\} = \{0, 0, \frac{-\tilde{G}_2}{2}(\frac{N^2 h_2}{\alpha} + \frac{1}{h_2}(\frac{1}{h_1\rho_1} + \frac{1}{h_3\rho_3}))\}. \quad (13)$$

For $\mathbf{k=0}$, there is a null vector of the form

$$\mathbf{u} = \vec{1}_{\mathcal{O}}, \qquad u = 0, \qquad V = 0. \quad (14)$$

and an associated generalized null vector of the form

$$U = 0, \qquad v = 0, \qquad \mathbf{v} = \vec{1}_{\mathcal{O}}. \quad (15)$$

Also, for eigenvalues satisfying (9), each λ is associated with an eigenvector of the form

$$U = (0, h_3\rho_3, -h_1\rho_1)^T, V = \lambda U. \quad (16)$$

In the case that $\tilde{G}_2^2 R/h_2 = 4G_2$, $\lambda = -\tilde{G}_2 R/(2h_2)$ is a double root and a corresponding generalized eigenvector can be found from (9).

For $\mathbf{k = 1, 2, 3}$..., eigenvectors and generalized eigenvectors corresponding to $\lambda = \lambda^\pm_{k,j}$ are of the form

$$Y = (U, V), U = \frac{1}{\lambda}\left(\begin{array}{c} u \\ \mathbf{u} \end{array}\right), V = \lambda U \quad (17)$$

$$\left(\begin{array}{c} u \\ \mathbf{u} \end{array}\right) = \left(\begin{array}{c} u_{k,j} \\ \mathbf{u}_{k,j} \end{array}\right) = \mathbf{D}_k(\vec{d_j} + \mathcal{O}(k^{-1})), \quad (18)$$

$$\mathbf{D}_k = \text{diag }((1/\sigma_k)\sin\sigma_k x, \cos\sigma_k x, \cos\sigma_k x), \quad (19)$$

In the case of distinct wave speeds, $\sqrt{\frac{E_1}{\rho_1}} \neq \sqrt{\frac{E_3}{\rho_3}}$,

$$\vec{d_0} = \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right), \vec{d_1} = \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right), \vec{d_3} = \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right), \quad (20)$$

while in the case of identical wave speeds we have

$$\{\vec{d_0}, \vec{d_1}, \vec{d_3}\} = \quad (21)$$
$$\left\{\left(\begin{array}{c} 1 \\ Nh_2/2 \\ -Nh_2/2 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right), \left(\begin{array}{c} -Nh_2 \\ \alpha/h_2h_1\rho_1 \\ -\alpha/h_2h_3\rho_3 \end{array}\right)\right\}$$

In either case, the eigenvectors are block orthogonal with respect to blocks of eigenvectors corresponding to the eigenvalues in S_k, $k \in \mathbb{N}$.

Theorem 2. *The eigenvectors associated with (1)-(2), (3) form a Riesz basis for the finite energy space* $X_0 \times X_1$.

The proof of Theorem (2) in the case of distinct wave speeds is given in [6]. The general proof uses the same idea, which is to apply Bari's theorem ([13],[1]) to the eigenvectors of the damped and undamped systems. Since the eigenvectors for the undamped case form an orthogonal basis and the damping is sufficiently small, it turns out that the damped eigenvectors are quadratically close to the undamped ones. The ω independence condition required by Bari's theorem is easily proved using the block diagonal structure.

IV. Derivation of the Moment Problem

Let us assume that the initial data given is zero, and determine which states are reachable in time T. We write the terminal state as

$$Y(T) = \sum_{\lambda\in\sigma(\mathcal{A})} c_\lambda Y_\lambda; \qquad c_\lambda =< Y(T), Y^*_\lambda >_e$$

where Y^*_λ is the eigenvector of $\mathcal{A}^*$ with eigenvalue $\bar{\lambda}$. For $k \in \mathbb{N}$,

$$Y^*_\lambda = \left(\begin{array}{c} \frac{V^*_\lambda}{\lambda} \\ V^*_\lambda \end{array}\right) V^*_\lambda = \left(\begin{array}{c} v^*_{k,j} \\ \mathbf{v}^*_{k,j} \end{array}\right)$$

The eigenvectors of $\mathcal{A}^*$ are the same as the eigenvectors of $\mathcal{A}$ except for the negative sign corresponding to any term that multiplies $\tilde{G}_2$. We need the following calculation

$$\begin{aligned} &\left\langle \left(\begin{array}{c} 0 \\ \mathcal{B}\{\hat{M}, \hat{\mathbf{g}}_{\mathcal{O}}\} \end{array}\right), Y^*_\lambda \right\rangle_e = c(\mathcal{B}\{\hat{M}, \hat{\mathbf{g}}_{\mathcal{O}}\}, V^*_\lambda) \\ = \ & c(\left(\begin{array}{c} J^{-1}\hat{M}(t)\delta'_L \\ \mathbf{h}^{-1}_{\mathcal{O}}\mathbf{p}^{-1}_{\mathcal{O}}\hat{\mathbf{g}}_{\mathcal{O}}\delta_L \end{array}\right), \left(\begin{array}{c} v^*_\lambda \\ \mathbf{v}^*_\lambda \end{array}\right)) \\ = \ & < \hat{M}(t)\delta'_L, v^*_\lambda > + < \hat{\mathbf{g}}_{\mathcal{O}}, \delta_L \mathbf{v}^*_\lambda > \\ = \ & \{\hat{M}(t), \hat{\mathbf{g}}_{\mathcal{O}}\} \cdot \{(\vec{d_j} + \mathcal{O}(k^{-1})\} =: h_\lambda, \end{aligned} \quad (22)$$

where $\vec{d_j}$'s are given by (20) and (21). (The "$\cdot$" used in the last line indicates scalar product in $\mathbb{R}^3$.)

The variation of parameters solution can be written

$$\begin{aligned} Y(T) \quad &= \quad \int_0^T e^{\mathcal{A}(T-s)} \left(\begin{array}{c} 0 \\ \mathcal{B}\{\hat{M}, \hat{\mathbf{g}}_{\mathcal{O}}\} \end{array}\right) ds \\ &= \quad \int_0^T e^{\mathcal{A}t} \left(\begin{array}{c} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_{\mathcal{O}}\} \end{array}\right) ds \\ \text{where } \tilde{M}(t) \quad &= \quad \hat{M}(T-t), \quad \tilde{\mathbf{g}}_{\mathcal{O}} = \hat{\mathbf{g}}_{\mathcal{O}}(T-t). \end{aligned}$$

Hence multiplying the above by the eigenvectors Y^*_λ of $\mathcal{A}^*$ gives

$$\begin{aligned} c_\lambda &=< Y(T), Y^*_\lambda >_e \\ &= \int_0^T e^{\lambda t} < \left(\begin{array}{c} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_{\mathcal{O}}\} \end{array}\right), Y^*_\lambda >_e ds \\ &= \int_0^T e^{\lambda t} h_\lambda ds, \end{aligned} \quad (23)$$

where h_λ is given by (22).

A. Distinct wave speeds

Let $\varepsilon_{ij}^k = \mathcal{O}(k^{-1})$, i.e., there exist $C > 0, K \in \mathbb{N}$ such that $|\varepsilon_{ij}^k| \le Ck^{-1}, \forall k \ge K, i,j = 1,2,3$. The moment problem (23) for the distinct wave speed case may be written as

$$c_{k,0}^{\pm} = \int_0^T e^{\lambda_{0,k}^{\pm} t}(\tilde{M}(t) + \tilde{g}_1(t)\varepsilon_{12}^k + \tilde{g}_3(t)\varepsilon_{13}^k)\, ds \tag{24}$$

$$c_{k,1}^{\pm} = \int_0^T e^{\lambda_{1,k}^{\pm} t}(\tilde{M}(t)\varepsilon_{21}^k + \tilde{g}_1(t) + \tilde{g}_3(t)\varepsilon_{23}^k)\, ds \tag{25}$$

$$c_{k,3}^{\pm} = \int_0^T e^{\lambda_{3,k}^{\pm} t}(\tilde{M}(t)\varepsilon_{31}^k + \tilde{g}_1(t)\varepsilon_{32}^k + \tilde{g}_3(t))\, ds \tag{26}$$

for $k \in \mathbb{N}$. Also for $k = 0$ we have four additional equations corresponding to the nullvector (14), the generalized nullvector (15), and the eigenvectors described in (16):

$$c_{0,0} = \int_0^T 0\, dt, \;\; c_{0,1} = \int_0^T \tilde{g}_1 + \tilde{g}_3\, dt, \tag{27}$$

$$c_{0,3}^{\pm} = \int_0^T e^{\lambda_{0,3}^{\pm} t}(h_3\rho_3\tilde{g}_1 - h_1\rho_3\tilde{g}_3)\, dt \tag{28}$$

B. Identical wave speeds

In the case of *identical wave speeds*, explicitly we have

$$h_\lambda = \tag{29}$$
$$(-1)^k[\tilde{M} + \frac{Nh_2}{2}(\tilde{g}_1 - \tilde{g}_3)] + \text{ l.o.t if } j = 0$$
$$(-1)^k(\tilde{g}_1 + \tilde{g}_3) + \text{ l.o.t if } j = 1$$
$$(-1)^k[-Nh_2\tilde{M} + \frac{\tilde{g}_1\alpha}{h_2h_1\rho_1} - \frac{\tilde{g}_3\alpha}{h_2h_3\rho_3}] + \text{ l.o.t if } j = 3.$$

where

$$\text{l.o.t} = (M(t) + g_1(t) + g_3(t))\mathcal{O}(k^{-1}).$$

We can define the new controls $\{f_0, f_1, f_3\}$ so that the above (with $\lambda = \lambda_{k,j}^{\pm}$) becomes ,

$$\begin{aligned} h_\lambda &= \left\langle \begin{pmatrix} 0 \\ \mathcal{B}\{M, \mathbf{g}_{\mathcal{O}}\} \end{pmatrix}, Y_\lambda^* \right\rangle_e \\ &= (-1)^k f_j + (f_0(t), f_1(t), f_3(t))\vec{\mathcal{O}}(k^{-1}). \end{aligned}$$

Hence, for $k \in \mathbb{N}$, we obtain the same system (24)-(26), but with $\{\tilde{M}, \tilde{g}_1, \tilde{g}_3\}$ replaced by $\{f_0, f_1, f_3\}$. The four equations in (27),(28) remain unchanged in the case of equal wave speeds.

C. Solution of Moment Problem

Remark 1. As is easy to see from (27),(28), it is not possible to steer a solution of (1) from the origin to the state corresponding to the null vector solution $w = 0$, $\mathbf{v}_{\mathcal{O}} = (1,1)^T$.

By Ingham's theorem (see [9]), for k sufficiently large and j fixed, either 0, 1, or 3, there exists a control on $[0,T]$ ($T > \tau$, τ is given in (4)) that solves the jth moment problem (ignoring the other two) if the perturbations ε_{ij}^k are taken to be zero. In Hansen and Rajaram[6] it is shown (using a fixed-point approach) that in the case of distinct wave speeds, all three branches $j = 0,1,3$ can be solved on the interval $[0,\tau]$ provided the ε_{ij}^k terms are k-square summable with sufficiently small l^2 norm (which is the case from the eigenvector estimates). The same proof works for the case of identical wave speeds. Hence we have

Theorem 3. *Given any $\{c_\lambda\} \in \ell^2$ there exists functions $M(t)$ and $\mathbf{g}_{\mathcal{O}}(t)$ in $L^2(0,T)$ which solve the three moment problems (24)–(26) for all $k > K$, where K is sufficiently large in any time $T > \tau$, where the control time τ is given in (4).*

Remark 2. Keeping Remark 1 in mind, one can ask whether it is possible to solve (24)–(26) for all k together with (27),(28). If possible, then exact controllability of (1)-(3) holds modulo the one dimensional uncontrollable quotient described in Remark 1. A sufficient condition for this result is that the eigenvalues grow (are not repeated) along each branch. (This insures that the minimum gap condition holds for each branch, $j = 0,1,3$.) It is not hard to show that this will hold if the coupling between the equations is sufficiently small and $\left\{\frac{K\sigma_k^4}{m + \alpha\sigma_k^2}\right\}_{k=1}^{\infty}$, ($\sigma_k = \frac{k\pi}{L}$) is a sequence of distinct numbers. Indeed for sufficiently small coupling, the l^2 norms of the coupling constants $\{\varepsilon_{ij}^k\}$ in the proof of Theorem 3 can be made arbitrarily small so that Theorem 3 is valid for the moment problems given by (24)–(26) for all k together with (28) and the second equation in (27).

D. Controllability results

The fact that we can obtain $\{\tilde{M}, \tilde{g}_1, \tilde{g}_3\} \in (L^2(0,T))^3\}$ for $T > \tau$ which solves the moment problem for $k \ge K$, implies that a corresponding $\{M, \tilde{\mathbf{g}}_{\mathcal{O}}\}$ exists that "exactly controls" the high frequency portion of the state space. More precisely, let $\mathcal{P}_\infty$ denote the spectral projection operator defined on $X_1 \times X_0$ by

$$\mathcal{P}_\infty(\sum_{k=1}^{\infty}\sum_{\lambda \in S_k} c_\lambda Y_\lambda) = \sum_{k \ge K}\sum_{\lambda \in S_k} c_\lambda Y_\lambda, \tag{30}$$

where K is the integer defined in Theorem 3.

Theorem 4. *Given any initial data $Y_0 \in X_1 \times X_0$ and $T > \tau$ (τ as defined in (4)), there exists $\{M, \tilde{\mathbf{g}}_{\mathcal{O}}\} \in (L^2(0,T))^3$ such that the solution $Y(t)$ of (7) satisfies $Y(t) \in C([0,T]; X_1 \times X_0)$ and $\mathcal{P}_\infty Y(t) = 0,\ \forall t \ge T$.*

In view of Remark 2, we also have the following corollary.

Corollary 2. *If G_2 and $\tilde{G}_2$ are sufficiently small and $\left\{\frac{K\sigma_k^4}{m + \alpha\sigma_k^2}\right\}_{k=1}^{\infty}$, $\sigma_k = \frac{k\pi}{L}$ is a sequence of distinct numbers, then for $T > \tau$ Equation (1) is exactly controllable in the quotient space $(X_0 \times X_1)/(0,1,1)^T \times (0,0,0)^T$.*

V. Simultaneous Controllability

A. Distinct wave speeds

Theorem 5. *Assume the wave speeds $\sqrt{\frac{E_j}{\rho_j}}, j = 1,3$ are distinct and the numbers $\{\rho_1 h_1, \rho_3 h_3, \frac{\alpha}{(h_1+h_2+h_3)}\}$ are distinct.*

Then the eigenvalues $\lambda^{\pm}_{k,j}$ have the following asymptotic form:

$$\lambda^{+}_{k,j} = -a_j + i\mu_j\sigma_k + \mathcal{O}(k^{-1}), \qquad (31)$$

as $k \to \infty$, where $\mu_0 = \sqrt{\frac{K}{\alpha}}, \mu_j = \sqrt{\frac{E_j}{\rho_j}}, j = 1, 3.$, and a_0, a_1, a_3 are distinct non-negative numbers. Furthermore, there exists a control $u(t)$ of the form (6) that solves all but finitely many of the equations in (24)-(26) with $T > \tau$ where

$$\tau = (2L(\frac{1}{\mu_0} + \frac{1}{\mu_1} + \frac{1}{\mu_3})).$$

Idea of the proof: Using controls of the form (6), the moment problem (24)-(26) can be rewritten using $\tilde{u} = u(T-t)$ as follows:

$$c^{\pm}_{k,0} = \int_0^T e^{\lambda^{\pm}_{0,k}t} A_k \tilde{u}(t)\, ds \qquad (32)$$

$$c^{\pm}_{k,1} = \int_0^T e^{\lambda^{\pm}_{1,k}t} B_k \tilde{u}(t)\, ds \qquad (33)$$

$$c^{\pm}_{k,3} = \int_0^T e^{\lambda^{\pm}_{3,k}t} C_k \tilde{u}(t)\, ds, \qquad (34)$$

where

$$A_k = \frac{h_1 + h_3}{2K} + \frac{\varepsilon^k_{12}}{h_1 E_1} - \frac{\varepsilon^k_{13}}{h_3 E_3}, \qquad (35)$$

$$B_k = \frac{(h_1 + h_3)\varepsilon^k_{21}}{2K} + \frac{1}{h_1 E_1} - \frac{\varepsilon^k_{23}}{h_3 E_3}, \qquad (36)$$

$$C_k = \frac{(h_1 + h_3)\varepsilon^k_{31}}{2K} + \frac{\varepsilon^k_{32}}{h_1 E_1} - \frac{1}{h_3 E_3}. \qquad (37)$$

The constants A_k, B_k and C_k defined above are bounded and bounded away from zero if $k \geq K$ where K is sufficiently large. Hence dividing, (32)-(34) by A_k, B_k, C_k respectively, we get

$$d^{\pm}_{k,j} = \int_0^T e^{\lambda^{\pm}_{k,j}t} \tilde{u}(t) dt, \qquad (38)$$

where $\{d^{\pm}_{k,j}\} \in l^2$. Under the assumptions of the theorem, for $k \geq K$ sufficiently large, there exists a $\delta > 0$ such that

$$|\lambda^{m_0}_{k_0,j_0} - \lambda^{m_1}_{k_1,j_1}| \geq \delta \text{ for } (m_0, k_0, j_0) \neq (m_1, k_1, j_1). \qquad (39)$$

To solve the moment problem (38), we need the following general Proposition.

Proposition 1. *Assume that $\mu_j > 0, j = 1, ..., n,, 0, a_1 < a_2 < ... < a_n$, $\lambda^{\pm}_{k,j} = -a_j \pm i\mu_j k + z^{\pm}_{k,j}, j = 1, ..., n, k \in \mathbb{N}$, $z^{\pm}_{k,j} \in l^2$, and $\{\lambda^{\pm}_{k,j}\}$ are pairwise distinct. Let $T > \sum_{j=1}^n \frac{2\pi}{\mu_j}$. Then $\{e^{\lambda^{\pm}_{k,j}t}\}$ forms a Riesz basis for its closed span in $L^2(0, T)$.*

The proof of Proposition (1) relies on some ideas from [5] along with some standard perturbation techniques from the theory of non-harmonic Fourier series [13]. We omit the proof here. Since $\{\lambda^{\pm}_{k,j}\}$ in (38) satisfy the conditions of Proposition (1) for $j = 0, 1, 3$, and k sufficiently large, there exists a $\tilde{u}(t)$ that solves (38) for $k \geq K$, for K large and hence Theorem (5) follows from Proposition 1.

B. Identical wave speeds

The eigenvalue estimate in (13) or the case of identical wave speeds implies that the minimum gap condition fails in (32)-(34) since two branches of eigenvalues are asymptotically the same. Hence a single L^2 control input cannot solve the moment problem (32)-(34) without some restrictions on the coefficients on the left hand sides. (See [8], [12] for some studies of solution of moment problems when the gap condition fails.)

Let $\mu := \sqrt{E_1/\rho_1} = \sqrt{E_3/\rho_3}$. It follows from the definition of the physical constants that $\mu = \sqrt{K/\alpha}$ also. We can rewrite the moment problem for identical wave speeds using (23), equations in (29) and controls in (5) as follows:

$$c^{\pm}_{k,0} = (-1)^k \int_0^T e^{\lambda^{\pm}_{k,0}t} [A\tilde{g}_1(t) - B\tilde{g}_3(t)]\, dt + \text{ l.o.t}, \quad (40)$$

$$c^{\pm}_{k,1} = (-1)^k \int_0^T e^{\lambda^{\pm}_{k,1}t} [\tilde{g}_1(t) + \tilde{g}_3(t)]\, dt + \text{ l.o.t}, \quad (41)$$

$$c^{\pm}_{k,3} = (-1)^k \int_0^T e^{\lambda^{\pm}_{k,3}t} [C\tilde{g}_1(t) - D\tilde{g}_3(t)]\, dt + \text{ l.o.t}, \quad (42)$$

where

$$A = (\frac{Nh_2}{2} - \frac{h_1}{2}), B = (\frac{Nh_2}{2} - \frac{h_3}{2}) \qquad (43)$$
$$C = (\frac{Nh_2 h_1}{2} + \frac{\alpha}{h_2 h_1 \rho_1}), D = (\frac{Nh_2 h_3}{2} + \frac{\alpha}{h_2 h_3 \rho_3})$$

We choose the following variables as new controls

$$u_1(t) = \frac{1}{\mu^2}(\frac{\tilde{g}_1(t)}{h_1\rho_1} - \frac{\tilde{g}_3(t)}{h_3\rho_3}), \qquad (44)$$

$$u_2(t) = \frac{1}{\mu^2}(\tilde{g}_1(t) + \tilde{g}_3(t)). \qquad (45)$$

Let $m_1 = h_1\rho_1$ and $m_3 = h_3\rho_3$. Then $\tilde{g}_1, \tilde{g}_3$ can be related to u_1, u_2 as follows:

$$\tilde{g}_1 = \frac{m_3 m_1}{m_3 + m_1}(u_1 + \frac{u_2}{m_3}) \qquad (46)$$

$$\tilde{g}_3 = \frac{m_3 m_1}{m_3 + m_1}(-u_1 + \frac{u_2}{m_3}) \qquad (47)$$

Using (44)-(47), (40)-(42) can be rewritten as follows:

$$d^{\pm}_{k,0} = \int_0^T e^{\lambda^{\pm}_{k,0}t} u_1(t) dt + \text{ l.o.t }, \qquad (48)$$

$$d^{\pm}_{k,1} = \int_0^T e^{\lambda^{\pm}_{k,1}t} u_2(t) dt + \text{ l.o.t }, \qquad (49)$$

$$d^{\pm}_{k,3} = \int_0^T e^{\lambda^{\pm}_{k,3}t} u_1(t) dt + \text{ l.o.t }, \qquad (50)$$

where

$$d^{\pm}_{k,0} = \frac{1}{(A + B)} l^{\pm}_{k,0},\ d^{\pm}_{k,1} = \frac{(-1)^k}{\mu^2} c^{\pm}_{k,1} \qquad (51)$$

$$d^{\pm}_{k,3} = \frac{1}{(C + D)} l^{\pm}_{k,3} \qquad (52)$$

$$l_{k,0}^{\pm} = \left((-1)^k \frac{m_1 m_3}{m_1+m_3} c_{k,0}^{\pm} - \int_0^T e^{\lambda_{k,0}^{\pm} t} u_2(t) \frac{(A-B)}{m_3}\right)$$
$$l_{k,3}^{\pm} = \left((-1)^k \frac{m_1 m_3}{m_1+m_3} c_{k,3}^{\pm} - \int_0^T e^{\lambda_{k,3}^{\pm} t} u_2(t) \frac{(C-D)}{m_3}\right)$$

One can check, using the definition of the physical constants that $A+B$ and $C+D$ are always nonzero in (51), (52). To obtain (48), (50) we first solve the moment problem (49) (ignoring the l.o.t.'s) for k sufficiently large. (Solvability follows from e.g., Ingham's Theorem, since $T > \frac{2L}{\mu}$ see e.g., [1]). Then $u_2(t)$ can be used to calculate $d_{k,j}^{\pm}, j = 0, 3, k \in \mathbb{N}$. By using an argument similar to the proof of Theorem 5, we can solve the moment problems (48) and (50) (for k sufficiently large using a single control $u_1(t) \in L^2(0,T)$ if $T > \frac{4L}{\mu}$. Hence if $T > \frac{4L}{\mu}$, then the moment problems given by (48)-(50) can be solved using two controls $(u_1(t), u_2(t))$ if $T > \frac{4L}{\mu}$. Hence we have the following theorem.

Theorem 6. *Assume the wave speeds* $\sqrt{\frac{E_j}{\rho_j}}, j = 1, 3$ *are identical. Then there exist controls of the form (5) that solve all but finitely many of the equations in (40)–(42) with* $T > \frac{4L}{\mu}$*, where* $\mu = \sqrt{\frac{E_j}{\rho_j}}, j = 1, 3$.

Remark 3. (Partial exact controllability) Note that all the eigenfunctions in (21) corresponding to the $j = 1$ branch have zero for the first component. In fact, the corresponding motions described by $j = 1$ branch are undamped longitudinal motions with zero transverse displacement. Thus, if the goal is to control only the transverse beam motions, the control u_2 in (48)–(50) is entirely unnecessary. It follows that we can drive any initial state in $X_1 \times X_0$ to a state in which $w = 0$ (modulo a finite dimensional space) by using the control $u_1(t) \in L^2(0,T), T > \frac{4L}{\mu}$.

The previous remark can also be understood directly from the following decoupling that occurs in the case of equal wave speeds. Let $\mu = \sqrt{\frac{E_1}{\rho_1}} = \sqrt{\frac{E_3}{\rho_3}}$. We make the following variable substitution.

$$z = \rho_1 h_1 v_1 + \rho_3 h_3 v_3, \qquad y = v_1 - v_3.$$

Then (1)–(3) (with $\mu = \sqrt{\frac{E_1}{\rho_1}} = \sqrt{\frac{E_3}{\rho_3}}$) decouples into the following two systems valid for $(x,t) \in (0,L) \times (0,\infty)$ to get,

$$\begin{aligned}
&m\ddot{w} - \alpha D_x^2 \ddot{w} + K D_x^4 w - D_x N h_2 (G_2 \varphi + \tilde{G_2} \dot{\varphi}) = 0 \\
&\ddot{y} - \mu^2 D_x^2 y - \left(\frac{1}{\rho_1 h_1} + \frac{1}{\rho_3 h_3}\right)(G_2 \varphi + G_2 \dot{\varphi}) = 0 \\
&(\text{where } \varphi = h_2^{-1} y + N w_x) \qquad (53) \\
&w(0,t) = D_x^2 w(0,t) = D_x y(0,t) = w(L,t) = 0, \\
&D_x^2 w(0,t) = M(t), \ D_x y(L,t) = g_1(t) - g_3(t)
\end{aligned}$$

and the following wave equation,

$$\begin{aligned}
&\ddot{z} - \mu^2 D_x^2 z = 0 \qquad (54) \\
&D_x z(0,t) = 0, \, D_x z(L,t) = \rho_1 h_1 g_1(t) + \rho_3 h_3 g_3(t).
\end{aligned}$$

From Theorem 6, controls $\{u_1, u_2\}$ are sufficient to control all but finitely many dimensions of the entire state space. The control $u_2(t)$ in (45) is precisely the forcing term $\rho_1 h_1 g_1(t) + \rho_3 h_3 g_3(t)$ that appears in (54). Thus setting $u_2 = 0$ results only in a lack of controllability of the z component, which is entirely independent of w. It is easy to see that setting u_2 to zero in (48)–(50) does not affect the solvability of (48), (50). Hence with $u_1(t)$ alone, we can drive any initial state in $X_1 \times X_0$ to a state in which $w = 0$ (modulo a finite dimensional quotient space).

C. Controllability Results

Let $\mathcal{P}_\infty$ be the spectral operator defined in (30), with K as in Theorem 5. We have:

Theorem 7. *For the case of distinct wave speeds:* $\sqrt{E_1/\rho_1} \neq \sqrt{E_1/\rho_1}$. *Assume the hypothesis of Theorem 5 holds. Then given any initial data* $Y_0 \in X_1 \times X_0$ *and* $T > \tau$ *(τ as defined in Theorem 5), there exists* $u \in L^2(0,T)$ *such that the solution of (1), (2), (3) (as defined by* $Y(t)$ *of (7)) satisfies* $Y(t) \in C([0,T]; X_1 \times X_0)$ *and* $\mathcal{P}_\infty Y(t) = 0, \ \forall t \geq T$.

For the case of identical wave speeds, we have an analogous result:

Theorem 8. *For the case of identical wave speeds:* $\sqrt{E_1/\rho_1} = \sqrt{E_1/\rho_1}$. *Given any initial data* $Y_0 \in X_1 \times X_0$ *and* $T > \frac{4L}{\mu}$ *(μ as defined in Theorem 6), there exists* $u \in L^2(0,T)$ *such that the solution of (1), (2), (3) (as defined by* $Y(t)$ *of (7)) satisfies* $Y(t) \in C([0,T]; X_1 \times X_0)$ *and* $\mathcal{P}_\infty Y(t) = 0, \ \forall t \geq T$. *($K$ in the definition of* $\mathcal{P}_\infty$ *is determined in Theorem 6.)*

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