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A Closed-Form Feedback Controller for Stabilization of Linearized Navier-Stokes Equations: The 2D Poisseuille Flow

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***Abstract*— We present a formula for a boundary control law which stabilizes the parabolic profile of an infinite channel flow, which is linearly unstable for high Reynolds numbers. Also known as the Poisseuille flow, this problem is frequently cited as a paradigm for transition to turbulence, whose stabilization for arbitrary Reynolds numbers, without using discretization, has so far been an open problem. Our result achieves exponential stability in the L^2 norm for the linearized Navier-Stokes equations, guaranteeing local stability for the fully nonlinear system. Explicit solutions are obtained for the closed loop system. This is the first time explicit formulae are produced for solutions of the Navier-Stokes equations. The result is presented for the 2D case for clarity of exposition. An extension to 3D is available and will be presented in a future publication.**

I. Introduction

We present an explicit boundary control law which stabilizes a benchmark 2D linearized Navier-Stokes system, the channel flow. Despite the deceptive simplicity of the geometry, there is a number of complex issues underlying this problem [13], making it extremely hard to solve.

Controllability and stabilizability results for the Navier-Stokes equations are already available for very general geometries; for example, see [9], [10], [12] and references therein. The drawback of these results is that they do not provide any means of computing a feedback controller.

Many efforts in design of feedback controllers for the Navier-Stokes system employ in-domain actuation, using optimal control methods [7] or model reduction techniques [4]. For boundary feedback control, optimal control theory has also been developed [16], and specialized to specific geometries, like cylinder wake [15]. There are also new techniques arising for specific flow control problems, like separation control [3].

Optimal control theory has been so far the most successful technique addressing channel flow stabilization [11], in a periodic setting, by using a discretized version of the equations and employing high-dimensional algebraic Riccati equations for computation of gains. The computational complexity is formidable if a very fine grid is necessary in the discretizations, for example if the Reynolds number is very large. Using Lyapunov methods, another control design was able to address the (periodic) channel flow stabilization problem; the design was explicit and did not rely on discretization, but it was restricted to low Reynolds numbers [1], [5].

Our approach is the first result that provides an explicit control law (with symbolically computed gains) for stabilization at an arbitrarily high Reynolds number in non-discretized Navier-Stokes equations, it is applicable to both infinite and periodic channel flow with arbitrary periodic box size, and also extends to 3D. Thanks to the explicitness of the controller, we are able to obtain approximate analytical solutions for Navier-Stokes equations. Exponential stability in the L^2 norm is proved for the linearized Stokes system around the Pouiseuille profile, therefore local stability is achieved for the full nonlinear Navier-Stokes system.

The method we use for solving this problem is based on the recently developed backstepping technique for parabolic systems [20], which has been successfully applied to flow control problems, for example the vortex shedding problem [2] and feedback stabilization of an unstable convection loop [23].

We start the paper by stating, in Section II, the mathematical model of the problem, which are the linearized Navier-Stokes equations for the velocity fluctuation around the (Pouisseuille) equilibrium profile. In Section III, we introduce the control law that stabilizes the equilibrium profile. Explicit solutions for the closed loop system are then stated in Section IV along with the main results of the paper. Sections V deals with the proof of L^2 stability of the closed loop system. A Fourier transform approach allows separate analysis for each wave number. For certain wave numbers, a normal velocity controller puts the system into a form where a linear Volterra operator, combined with boundary feedback, can transform the original normal velocity PDE into a stable heat equation. For the rest of wave numbers the system is proved to be open loop exponentially stable, and is left uncontrolled. These two results are combined to prove stability of the closed loop system for all wave numbers and in the physical space. In Section VI, we finish the paper with a discussion of the results.

II. Model

Consider a 2D incompressible channel flow evolving in a semi-infinite rectangle $(x, y) \in (-\infty, \infty) \times [0, 1]$ as in Figure 1. The dimensionless velocity field is governed by the Navier-Stokes equations

$$U_t = \frac{1}{Re}(U_{xx} + U_{yy}) - UU_x - VU_y - P_x\,, \quad (1)$$

$$V_t = \frac{1}{Re}(V_{xx} + V_{yy}) - UV_x - VV_y - P_y\,, \quad (2)$$

and the continuity equation

$$U_x + V_y = 0, \quad (3)$$

where U denotes the streamwise velocity, V the wall-normal velocity, P the pressure, and Re is the Reynolds number. The boundary conditions for the velocity field are the no-penetration, no-slip boundary conditions for the uncontrolled case, i.e., $V(x, 0) = V(x, 1) = U(x, 0) = U(x, 1) = 0$.

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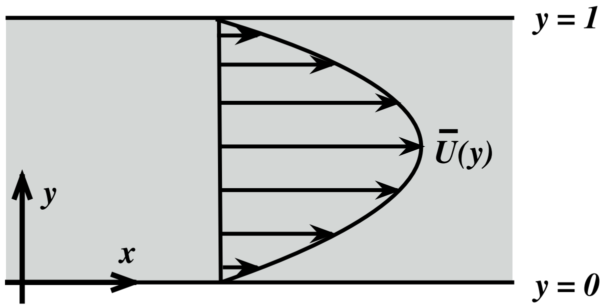

Fig. 1. 2D channel flow and equilibrium profile. Actuation is on the top wall.

Instead of using (3) we derive a Poisson equation that P verifies, combining (1), (2) and (3)

$$P_{xx} + P_{yy} = -2(V_y)^2 - 2V_x U_y, \tag{4}$$

with boundary conditions $P_y(x,0) = (1/Re)V_{yy}(x,0)$ and $P_y(x,1) = (1/Re)V_{yy}(x,1)$, which are obtained evaluating (2) at $y = 0, 1$.

The equilibrium solution of (1)–(3) is the parabolic Poiseuille profile

$$\begin{aligned} \bar{U} &= 4y(1-y), && (5)\\ \bar{V} &= 0, && (6)\\ \bar{P} &= P_0 - \frac{8}{Re}x, && (7)\end{aligned}$$

shown in Figure 1. This equilibrium is unstable for high Reynolds numbers [19]. Defining the fluctuation variables $u = U - \bar{U}$ and $p = P - \bar{P}$, and linearizing around the equilibrium profile (5)–(7), the plant equations become the Stokes equations

$$\begin{aligned} u_t &= \frac{1}{Re}\left(u_{xx} + u_{yy}\right) + 4y(y-1)u_x \\ &\quad + 4(2y-1)V - p_x, && (8)\\ V_t &= \frac{1}{Re}\left(V_{xx} + V_{yy}\right) + 4y(y-1)V_x \\ &\quad - p_y, && (9)\\ p_{xx} + p_{yy} &= 8(2y-1)V_x, && (10)\end{aligned}$$

with boundary conditions

$$\begin{aligned} u(x,0) &= 0, && (11)\\ u(x,1) &= U_c(x), && (12)\\ V(x,0) &= 0, && (13)\\ V(x,1) &= V_c(x), && (14)\\ p_y(x,0) &= \frac{V_{yy}(x,0)}{Re}, && (15)\\ p_y(x,1) &= \frac{V_{yy}(x,1) + (V_c)_{xx}(x)}{Re} - (V_c)_t(x). && (16)\end{aligned}$$

The continuity equation is still verified

$$u_x + V_y = 0. \tag{17}$$

We have added in (12) and (14) the actuation variables $U_c(x)$ and $V_c(x)$, respectively for streamwise and normal velocity boundary control. The actuators are placed along the top wall, $y = 1$, and we assume they can be independently actuated for all $x \in \mathbb{R}$. No actuation is done inside the channel or at the bottom wall.

Taking Laplacian in equation (9) and using (10), we get an autonomous equation for the normal velocity, the well-known Orr-Sommerfeld equation,

$$\triangle V_t = \frac{1}{Re}\triangle^2 V + 4y(y-1)\triangle V_x - 8V_x, \tag{18}$$

with boundary conditions (13)–(14), as well as $V_y(x,0) = 0$, $V_y(x,1) = -(U_c)_x$, derived from (11)–(12) and (17). This equation is numerically studied in hydrodynamic theory to determine stability of the channel flow [17].

Defining $Y = -V_y$, it is possible to partially solve (18) and obtain an evolution equation for Y

$$\begin{aligned} Y_t = &\ \frac{1}{Re}\left(Y_{xx} + Y_{yy}\right) + 4y(y-1)Y_x \\ &+ \int_0^y \int_{-\infty}^{\infty} Y(\xi,\eta) \int_{-\infty}^{\infty} 16\pi k \mathrm{e}^{2\pi i k(x-\xi)} \\ &\times \left[\pi k(2y-1) - 2\sinh\left(2\pi k(y-\eta)\right)\right. \\ &\left. -2\pi k(2\eta-1)\cosh\left(2\pi k(y-\eta)\right)\right] dk d\xi d\eta \\ &+ \int_0^1 \int_{-\infty}^{\infty} Y(\xi,\eta) \int_{-\infty}^{\infty} 32\pi k \mathrm{e}^{2\pi i k(x-\xi)} \\ &\times \frac{\cosh\left(2\pi k y\right)}{\sinh\left(2\pi k\right)} \left[\cosh\left(2\pi k(1-\eta)\right)\right. \\ &\left. +\pi k(2\eta-1)\sinh\left(2\pi k(1-\eta)\right)\right] dk d\xi d\eta \\ &+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{Y_y(\xi,1) - (V_c)_{xx}(\xi)}{Re} + (V_c)_t(\xi)\right) \\ &\times 2\pi k \mathrm{e}^{2\pi i k(x-\xi)} \frac{\cosh\left(2\pi k y\right)}{\sinh\left(2\pi k\right)} dk d\xi \\ &- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{Y_y(\xi,0)}{Re} \\ &\times 2\pi k \mathrm{e}^{2\pi i k(x-\xi)} \frac{\cosh\left(2\pi k(1-y)\right)}{\sinh\left(2\pi k\right)} dk d\xi, \end{aligned} \tag{19}$$

with boundary conditions $Y_y(x,0) = 0$ and $Y(x,1) = (U_c)_x$. Equation (19) governs the channel flow, since from Y and using (17), we recover both components of the velocity field:

$$\begin{aligned} V(x,y) &= -\int_0^y Y(x,\eta)d\eta, && (20)\\ u(x,y) &= \int_{-\infty}^x Y(\xi,y)d\xi. && (21)\end{aligned}$$

Equation (19) displays the full complexity of the Navier-Stokes dynamics, which the PDE system (8)–(10) conceals through the presence of the pressure equation (10), and the Orr-Sommerfeld equation (18) conceals through the use of fourth order derivatives. Besides being unstable (for high Reynolds numbers), the Y system incorporates (on its right-hand side) the components of $Y(x,y)$ from everywhere in the domain. This is the main source of difficulty for both controlling and solving the Navier-Stokes equations. A perturbation somewhere in the flow is instantaneously felt everywhere—a consequence of the incompressible nature of the flow. Our approach to overcoming this obstacle is to use one of the two control variables (normal velocity $V_c(x)$, which is incorporated explicitly inside the equation) to prevent perturbations from propagating in the direction from the controlled boundary towards the uncontrolled boundary. This is a sort of "spatial causality" on y, which in the nonlinear control literature is referred to as the 'strict-feedback structure' [14].

III. CONTROLLER

The explicit control law consists of two parts—the normal velocity controller $V_c(x)$ and the streamwise velocity controller $U_c(x)$. $V_c(x)$ makes the integral operator in the fifth to ninth lines of (19) spatially causal in y,[1] which is a necessary structure for the application of a "backstepping" boundary controller for stabilization of spatially causal partial integro-differential equations [20]. $U_c(x)$ is a backstepping controller which stabilizes the spatially causal structure imposed by $V_c(x)$. The expressions for the control laws are

$$\begin{aligned} U_c(t,x) &= \int_0^1 \int_{-\infty}^{\infty} Q_u(x-\xi,\eta)u(t,\xi,\eta)d\xi d\eta, &(22)\\ V_c(t,x) &= h(t,x), &(23)\end{aligned}$$

where h verifies the equation

$$h_t = h_{xx} + g(t,x), \tag{24}$$

where

$$\begin{aligned} g = &\int_0^1 \int_{-\infty}^{\infty} Q_V(x-\xi,\eta)V(t,\xi,\eta)d\xi d\eta \\ &+\int_{-\infty}^{\infty} Q_0(x-\xi)\left(u_y(t,\xi,0)-u_y(t,\xi,1)\right)d\xi, \end{aligned} \tag{25}$$

and the kernels Q_u, Q_V and Q_0 are defined as

$$\begin{aligned} Q_u &= \int_{-\infty}^{\infty} \chi(k)K(k,1,\eta)\mathrm{e}^{2\pi ik(x-\xi)}dk, &(26)\\ Q_V &= \int_{-\infty}^{\infty} \chi(k)16\pi ki(2\eta-1)\cosh\left(2\pi k(1-\eta)\right) \\ &\quad\times \mathrm{e}^{2\pi ik(x-\xi)}dk, &(27)\\ Q_0 &= \int_{-\infty}^{\infty} \chi(k)\frac{2\pi ki}{Re}\mathrm{e}^{2\pi ik(x-\xi)}dk. &(28)\end{aligned}$$

In expressions (26)–(28), $\chi(k)$ is a truncating function in the wave number space whose definition is

$$\chi(k) = \begin{cases} 1, & m<|k|<M \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

where m and M are respectively the low and high cut-off wave numbers, two design parameters which can be conservatively chosen as $m \leq \frac{1}{32\pi Re}$ and $M \geq \frac{1}{\pi}\sqrt{\frac{Re}{2}}$. The function $K(k,y,\eta)$ appearing in (26) is a (complex valued) gain kernel defined as

$$K(k,y,\eta) = \lim_{n\to\infty} K_n(k,y,\eta), \tag{30}$$

where K_n is recursively defined as [2]

$$\begin{aligned} K_0 = &-\frac{Re}{3}\pi ik\eta\left(21y^2-6y(3+4\eta)+\eta(12+7\eta)\right) \\ &-2\pi k\frac{\cosh\left(2\pi k(1-y+\eta)\right)-\cosh\left(2\pi k(y-\eta)\right)}{\sinh\left(2\pi k\right)} \\ &+4iRe\eta(\eta-1)\sinh\left(2\pi k(y-\eta)\right) \\ &-6\eta i\frac{Re}{\pi k}(1-\cosh\left(2\pi k(y-\eta)\right)), &(31)\\ K_n = &K_{n-1} \\ &-4\pi kiRe\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}\int_{-\delta}^{\delta}\Bigg\{\frac{\sinh\left(\pi k(\xi+\delta)\right)}{\pi k} \\ &-(2\xi-1)+2(\gamma-\delta-1)\cosh\left(\pi k(\xi+\delta)\right)\Bigg\} \\ &\times K_{n-1}\left(k,\frac{\gamma+\delta}{2},\frac{\gamma+\xi}{2}\right)d\xi d\delta d\gamma \\ &+\frac{Re}{2}\pi ik\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}(\gamma-\delta)(\gamma-\delta-2) \\ &\times K_{n-1}\left(k,\frac{\gamma+\delta}{2},\frac{\gamma-\delta}{2}\right)d\delta d\gamma \\ &+2\pi k\int_0^{y-\eta}\frac{\cosh\left(2\pi k(1-\delta)\right)-\cosh\left(2\pi k\delta\right)}{\sinh\left(2\pi k\right)} \\ &\times K_{n-1}\left(k,y-\eta,\delta\right)d\delta. &(32)\end{aligned}$$

The terms of this series can be computed symbolically as they only involve integration of polynomials and exponentials. In implementation, a few terms are sufficient to obtain a highly accurate approximation because the series is rapidly convergent [20].

Remark 3.1: The controller (23) also has the property of applying zero net mass flux, that is, that $\int_{-\infty}^{\infty} V_c(\xi)d\xi = 0$. The details are shown in the appendix.

Remark 3.2: Since the controllers are defined as convolutions in the x direction, they are spatially invariant[3] in the x direction in the sense of [6]. What is a little harder to see is that the feedback kernels in the controller decay as the difference $x-\xi$ grows—a property that allows to truncate the integrals with respect to ξ to the vicinity of x, which allows sensing to be restricted just to a neighborhood (in the x direction) of the actuator. This decay is at least of the order of $1/(x-\xi)$, as is shown in the appendix.

Remark 3.3: (23) is a dynamic controller whose magnitude is determined by the variable $h(t,x)$, which evolves according to (24). We use an initial condition $h(0,x) \equiv 0$. The stabilization result remains valid for $h(0,x) \neq 0$, however they require additional routine effort to account for the exponentially stable effect of the compensator internal dynamics (which are of heat equation type).

IV. MAIN RESULTS

Due to the explicit form of the controller, the solution of the closed loop system is also obtained in the explicit form,

$$\begin{aligned} u(t,x,y) &= u^*(t,x,y)+\epsilon_u(t,x,y), &(33)\\ V(t,x,y) &= V^*(t,x,y)+\epsilon_V(t,x,y), &(34)\end{aligned}$$

[1] The second to fourth and tenth to eleventh lines are already spatially causal in y.

[2] This infinite sequence is convergent, smooth, uniformly bounded over $(y,\eta)\in[0,1]^2$, and analytic in k.

[3] The feedback operator commutes with the translation in the x direction.

where

$$
\begin{aligned}
u^* &= 2\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi(k)\mathrm{e}^{-t\frac{4k^2\pi^2+\pi^2j^2}{Re}+2\pi ik(x-\xi)} \\
&\quad\times\left[\sin(\pi jy)+\int_0^y L(k,y,\eta)\sin(\pi j\eta)\,d\eta\right] \\
&\quad\times\int_0^1\left[\sin(\pi j\eta)-\int_\eta^1 K(k,\sigma,\eta)\sin(\pi j\sigma)\,d\sigma\right] \\
&\quad\times u(0,\xi,\eta)d\eta d\xi dk\,, \qquad (35)\\
V^* &= -2\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi(k)\mathrm{e}^{-t\frac{4k^2\pi^2+\pi^2j^2}{Re}+2\pi ik(x-\xi)} \\
&\quad\times\Bigg[\int_0^y\left(\int_\eta^y L(k,\sigma,\eta)d\sigma\right)\sin(\pi j\eta)\,d\eta \\
&\quad+\frac{1-\cos(\pi jy)}{\pi j}\Bigg]\int_0^1\Bigg[\pi j\cos(\pi j\eta) \\
&\quad+K(k,\eta,\eta)\sin(\pi j\eta)-\int_\eta^1 K_\eta(k,\sigma,\eta) \\
&\quad\times\sin(\pi j\sigma)\,d\sigma\Bigg]V(0,\xi,\eta)d\eta d\xi dk. \qquad (36)
\end{aligned}
$$

The variables $\epsilon_u(t,x,y)$ and $\epsilon_V(t,x,y)$ represent the error of approximation of the velocity field and are bounded in the following way

$$
\begin{aligned}
&||\epsilon_u(t)||_{L^2}^2+||\epsilon_V(t)||_{L^2}^2 \\
&\le \mathrm{e}^{-\frac{Re}{4}t}\left(||\epsilon_u(0)||_{L^2}^2+||\epsilon_V(0)||_{L^2}^2\right), \qquad (37)
\end{aligned}
$$

where both $\epsilon_u(0,x,y)$ and $\epsilon_V(0,x,y)$ can be written in terms of the initial conditions of the velocity field as

$$
\begin{aligned}
\epsilon_u(0,x,y) &= u(0,x,y)-\int_{-\infty}^{\infty}\frac{\sin(2\pi M\xi)-\sin(2\pi m\xi)}{\pi\xi} \\
&\quad\times u(0,x-\xi,y)d\xi, \qquad (38)\\
\epsilon_V(0,x,y) &= V(0,x,y)-\int_{-\infty}^{\infty}\frac{\sin(2\pi M\xi)-\sin(2\pi m\xi)}{\pi\xi} \\
&\quad\times V(0,x-\xi,y)d\xi, \qquad (39)
\end{aligned}
$$

The bound on the errors is proportional to the initial kinetic energy of ϵ_u and ϵ_V, which, as made explicit in the expressions (38)–(39), is in turn proportional to the kinetic energy of u and V at very small and very large length scales (the integral that we are substacting from the initial conditions represents the intermediate length scale content), and decays exponentially. Therefore, this initial energy will typically be a very small fraction of the overall kinetic energy, making the errors ϵ_u and ϵ_V very small in comparison with u^* and V^* respectively.

The kernel L in (35) is defined as a convergent, smooth sequence of fuctions

$$
L(k,y,\eta)=\lim_{n\to\infty}L_n(k,y,\eta), \qquad (40)
$$

whose terms are recursively defined as

$$
L_0 = K_0, \qquad (41)
$$

and

$$
\begin{aligned}
L_n &= L_{n-1}+4iRe\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}\int_{-\delta}^{\delta}\{2\pi k(\gamma+\xi-1) \\
&\quad\times\cosh(\pi k(\xi-\delta))+\sinh(\pi k(\xi-\delta)) \\
&\quad-\pi k(2\delta-1)\}\,L_{n-1}\left(k,\frac{\gamma+\xi}{2},\frac{\gamma-\delta}{2}\right)d\xi d\delta d\gamma \\
&\quad-\frac{Re}{2}\pi ik\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}(\gamma+\delta)(\gamma+\delta-2) \\
&\quad\times L_{n-1}\left(k,\frac{\gamma+\delta}{2},\frac{\gamma-\delta}{2}\right)d\delta d\gamma\,. \qquad (42)
\end{aligned}
$$

Control laws (22)–(32) guarantee the following result.

Theorem 4.1: The equilibrium $u(x,y)\equiv V(x,y)\equiv 0$ of system (8)–(16), (22)–(32) is exponentially stable in the L^2 sense. Moreover, the solutions for $u(t,x,y)$ and $V(t,x,y)$ are given explicitly by (33)–(42).

Remark 4.1: Theorem 4.1, stated for the linearized equations (8)–(9), is valid for the nonlinear equations (1)–(2) in a *local* sense, i.e., provided that the initial data are sufficiently close (in the appropiate norm) to the equilibrium (5)–(7).

V. Sketch of proof

As common for infinite channels, we use a Fourier transform in x. The transform pair (direct and inverse transform) has the following definition:

$$
\begin{aligned}
f(k,y) &= \int_{-\infty}^{\infty}f(x,y)\mathrm{e}^{-2\pi ikx}dx, \qquad (43)\\
f(x,y) &= \int_{-\infty}^{\infty}f(k,y)\mathrm{e}^{2\pi ikx}dx. \qquad (44)
\end{aligned}
$$

Note that we use the same symbol f for both the original $f(x,y)$ and the image $f(k,y)$. In hydrodynamics, k is referred to as the "wave number."

One property of the Fourier transform is that the L^2 norm is the same in Fourier space as in physical space, i.e.,

$$
||f||_{L^2}^2=\int_0^1\int_{-\infty}^{\infty}f^2(k,y)dkdy=\int_0^1\int_{-\infty}^{\infty}f^2(x,y)dxdy, \quad (45)
$$

allowing us to derive L^2 exponential stability in physical space from the same property in Fourier space. This result is called Parseval's formula in the literature [8].

We also define the L^2 norm of $f(k,y)$ with respect to y:

$$
||f(k)||_{\hat{L}^2}^2=\int_0^1|f(k,y)|^2dy. \qquad (46)
$$

The $\hat{L}^2$ norm as a function of k is related to the L^2 norm as

$$
||f||_{L^2}^2=\int_{-\infty}^{\infty}||f(k)||_{\hat{L}^2}^2dk \qquad (47)
$$

Equations (8)–(10) written in the Fourier domain are

$$
\begin{aligned}
u_t &= \frac{-4\pi^2k^2u+u_{yy}}{Re}+8k\pi iy(y-1)u \\
&\quad+4(2y-1)V-2\pi ikp, \qquad (48)\\
V_t &= \frac{-4\pi^2k^2V+V_{yy}}{Re}+8\pi kiy(y-1)V \\
&\quad-p_y, \qquad (49)\\
-4\pi^2k^2p+p_{yy} &= 16\pi ki(2y-1)V, \qquad (50)
\end{aligned}
$$

with boundary conditions

$$u(k,0) = 0, \tag{51}$$
$$u(k,1) = U_c(k), \tag{52}$$
$$V(k,0) = 0, \tag{53}$$
$$V(k,1) = V_c(k), \tag{54}$$
$$p_y(k,0) = \frac{V_{yy}(k,0)}{Re}, \tag{55}$$
$$p_y(k,1) = \frac{V_{yy}(k,1) - 4\pi^2k^2V_c(k)}{Re} - (V_c)_t(k), \tag{56}$$

and the continuity equation (17) is now

$$2\pi kiu(k,y) + V_y(k,y) = 0. \tag{57}$$

Thanks to linearity and spatial invariance, there is no coupling between different wave numbers. This allows us to consider the equations for each wave number independently. Then, the main idea behind the design of the controller is to consider two different cases depending on the wave number k. For wave numbers $m < |k| < M$, which we will refer to as *controlled* wave numbers, we design a backstepping controller that achieves stabilization, whereas for wave numbers in the range $|k| \geq M$ or $|k| \leq m$, which we will call *uncontrolled* wave numbers, the system is left without control but is exponentially stable. This is a well-known fact from hydrodynamic stability theory [19].

Estimates of m and M are found in the paper based on Lyapunov analysis and allow us to use feedback for only the wave numbers $m < |k| < M$. This is crucial because feedback over the entire infinite range of k's would not be convergent. The truncations at $k = m, M$ are truncations in Fourier space which do not result in a discontinuity in x.

We now analyze equations (48)–(50) in detail, for both controlled and uncontrolled wave numbers.

A. *Controlled wave numbers*

For $m < |k| < M$ we first solve (50), which is an ODE in y for each k, in order to eliminate the pressure. Introducing the solution into (48),

$$\begin{aligned} u_t = {} & \frac{1}{Re}\left(-4\pi^2k^2u + u_{yy}\right) + 8\pi kiy(y-1)u \\ & +4(2y-1)V + 16\pi k\int_0^y V(k,\eta) \\ & \times(2\eta-1)\sinh\left(2\pi k(y-\eta)\right)d\eta \\ & -16\pi k\frac{\cosh(2\pi ky)}{\sinh(2\pi k)}\int_0^1 V(k,\eta) \\ & \times(2\eta-1)\cosh\left(2\pi k(1-\eta)\right)d\eta \\ & +i\frac{\cosh\left(2\pi k(1-y)\right)}{\sinh(2\pi k)}\frac{V_{yy}(k,0)}{Re} \\ & -i\frac{\cosh(2\pi ky)}{\sinh(2\pi k)}\left(\frac{V_{yy}(k,1) - 4\pi^2k^2V_c(k)}{Re}\right. \\ & \left.-(V_c)_t(k)\right). \end{aligned} \tag{58}$$

We don't need to separately write and control the V equation because, by the continuity equation (57) and using the fact that $V(k,0) = 0$, we can write V in terms of u

$$V(k,y) = \int_0^y V_y(k,\eta)d\eta = -2\pi ki\int_0^y u(k,\eta)d\eta. \tag{59}$$

Introducing (59) in (58), and simplifying the resulting double integral by changing the order of integration, we reduce (58) to an autonomous equation that governs the whole velocity field. This equation is

$$\begin{aligned} u_t = {} & \frac{1}{Re}\left(-4\pi^2k^2u + u_{yy}\right) + 8\pi kiy(y-1)u \\ & +8i\int_0^y\{\pi k(2y-1) - 2\sinh\left(2\pi k(y-\eta)\right) \\ & -2\pi k(2\eta-1)\cosh\left(2\pi k(y-\eta)\right)\}u(k,\eta)d\eta \\ & +16i\frac{\cosh(2\pi ky)}{\sinh(2\pi k)}\int_0^y\{\cosh\left(2\pi k(y-\eta)\right) \\ & \pi k(2\eta-1)\sinh\left(2\pi k(y-\eta)\right)\}u(k,\eta)d\eta \\ & +\frac{2\pi k\cosh\left(2\pi k(1-y)\right)}{\sinh(2\pi k)}\frac{u_y(k,0)}{Re} \\ & +i\frac{\cosh(2\pi ky)}{\sinh(2\pi k)}\left(\frac{2\pi kiu_y(k,1) + 4\pi^2k^2V_c(k)}{Re}\right. \\ & \left.+(V_c)_t(k)\right), \end{aligned} \tag{60}$$

with boundary conditions

$$u(k,0) = 0, \tag{61}$$
$$u(k,1) = U_c(k). \tag{62}$$

Note that the relation between Y in (19) and u in (60) is that $Y(k,y) = 2\pi kiu(k,y)$.

Now, we design the controller in two steps. First, we set V_c so that (60) has a strict-feedback form in the sense previously defined:

$$\begin{aligned} (V_c)_t = {} & \frac{2\pi ki\left(u_y(k,0) - u_y(k,1)\right) - 4\pi^2k^2V_c}{Re} \\ & -16\pi ki\int_0^1(2\eta-1) \\ & \times V(k,\eta)\cosh\left(2\pi k(1-\eta)\right)d\eta. \end{aligned} \tag{63}$$

This can be integrated and explicitly stated as a dynamic controller in the Laplace domain:

$$\begin{aligned} V_c = {} & \frac{2\pi ki}{s + \frac{4\pi^2k^2}{Re}}\left[\frac{u_y(s,k,0) - u_y(s,k,1)}{Re} - 8\right. \\ & \times\int_0^1(2\eta-1)V(s,k,\eta) \\ & \left.\times\cosh\left(2\pi k(1-\eta)\right)d\eta\right]. \end{aligned} \tag{64}$$

Control law (63) can be expressed in the time domain and physical space as (23)–(25) and (27), (28), by use of the convolution theorem of the Fourier transform.

Introducing V_c in (60) yields

$$\begin{aligned} u_t = {} & \frac{1}{Re}\left(-4\pi^2k^2u + u_{yy}\right) + 8\pi kiy(y-1)u \\ & +8i\int_0^y\{\pi k(2y-1) - 2\sinh\left(2\pi k(y-\eta)\right) \\ & -2\pi k(2\eta-1)\cosh\left(2\pi k(y-\eta)\right)\}u(k,\eta)d\eta \\ & -2\pi k\frac{\cosh(2\pi ky) - \cosh\left(2\pi k(1-y)\right)}{\sinh(2\pi k)} \\ & \times\frac{u_y(k,0)}{Re}. \end{aligned} \tag{65}$$

Equation (65) can be stabilized using the backstepping technique for parabolic partial integro-differential equations [20]. Using backstepping, we map u, for each wave number $m < |k| < M$, into the family of heat equations

$$\begin{aligned} \alpha_t &= \frac{1}{Re}\left(-4\pi^2 k^2 \alpha + \alpha_{yy}\right), && (66)\\ \alpha(k,0) &= 0, && (67)\\ \alpha(k,1) &= 0, && (68) \end{aligned}$$

where

$$\begin{aligned} \alpha &= u - \int_0^y K(k,y,\eta)u(t,k,\eta)d\eta, && (69)\\ u &= \alpha + \int_0^y L(k,y,\eta)\alpha(t,k,\eta)d\eta, && (70) \end{aligned}$$

are respectively the direct and inverse transformation. The kernel K is found to verify the following equation

$$\begin{aligned} \frac{1}{Re}K_{yy} &= \frac{1}{Re}K_{\eta\eta} + 8\pi i k\eta(\eta-1)K \\ &\quad -8i\left\{\pi k(2y-1) - \sinh\left(2\pi k(y-\eta)\right)\right. \\ &\quad \left.-2\pi k(2\eta-1)\cosh\left(2\pi k(y-\eta)\right)\right\} \\ &\quad +8i\int_\eta^y \left\{\pi k(2\xi-1) - 2\sinh\left(2\pi k(\xi-\eta)\right)\right. \\ &\quad \left.-2\pi k(2\eta-1)\cosh\left(2\pi k(\xi-\eta)\right)\right\} \\ &\quad \times K(k,y,\xi)d\xi, && (71) \end{aligned}$$

a hyperbolic partial integro-differential equation (PIDE) in the region $\mathcal{T} = \{(y,\eta) : 0 \leq \eta \leq y \leq 1\}$ with boundary conditions:

$$\begin{aligned} K(y,y) &= -\frac{2Re}{3}\pi i k y^2(2y-3) \\ &\quad -2\pi k\frac{\cosh(2\pi k) - 1}{\sinh(2\pi k)}, && (72)\\ K(y,0) &= \frac{2\pi k}{\sinh(2\pi k)}\Bigg\{\cosh(2\pi k y) \\ &\quad -\cosh\left(2\pi k(1-y)\right) \\ &\quad +\int_0^y K(k,y,\xi)\left[\cosh\left(2\pi k(1-\xi)\right)\right. \\ &\quad \left.-\cosh(2\pi k\xi)\right]d\xi\Bigg\}. && (73) \end{aligned}$$

The equation can be transformed into an integral equation for calculating the kernel symbolically. To do this, we transform the PIDE into an integral equation and solve it explicitly via a successive approximation series. The series definition of K is (30)–(32). We skip the details, since we follow [20] exactly, with the only difference that the kernel is complex valued, which does not change the proof, that shows absolute convergence of (30)–(32). In addition, using the estimates of the proof and the fact that the terms in the series definition (31)–(32) of K are analytic in k, it can be shown that the kernel itself is also analytic as a complex function of k, for any bounded k [18], so in particular, it will be analytic in the annulus $m < |k| < M$.

From the transformation (69) and the boundary condition (61) the control law is

$$U_c = \int_0^1 K(k,1,\eta)u(t,k,\eta)d\eta. \qquad (74)$$

Using the convolution theorem of the Fourier transform we write the control law (74) back in physical space. The resulting expressions is (22).

The equation for the inverse kernel L in (70) is similar to the one of K and enjoys similar properties

$$\begin{aligned} \frac{1}{Re}L_{yy} &= \frac{1}{Re}L_{\eta\eta} - 8\pi i k y(y-1)L \\ &\quad -8i\left\{\pi k(2y-1) - 2\sinh\left(2\pi k(y-\eta)\right)\right. \\ &\quad \left.-2\pi k(2\eta-1)\cosh\left(2\pi k(y-\eta)\right)\right\} \\ &\quad -8i\int_\eta^y \left\{\pi k(2y-1) - \sinh\left(2\pi k(y-\xi)\right)\right. \\ &\quad \left.+2\pi k(2\xi-1)\cosh\left(2\pi k(y-\xi)\right)\right\} \\ &\quad \times L(k,\xi,\eta)d\xi, && (75) \end{aligned}$$

again a hyperbolic partial integro-differential equation in the region $\mathcal{T}$ with boundary conditions

$$\begin{aligned} L(y,y) &= -\frac{2Re}{3}\pi i k y^2(2y-3) \\ &\quad -2\pi k\frac{\cosh(2\pi k) - 1}{\sinh(2\pi k)}, && (76)\\ L(y,0) &= \frac{2\pi k}{\sinh(2\pi k)}\Bigg\{\cosh(2\pi k y) \\ &\quad -\cosh\left(2\pi k(1-y)\right)\Bigg\}. && (77) \end{aligned}$$

The equation can be transformed into an integral equation and calculated via the successive approximation series (41)–(42).

By using (59) and (69)–(70), V can also be expressed in terms of α

$$\begin{aligned} \alpha &= i\frac{V_y - \int_0^y K(k,y,\eta)V_y(t,k,\eta)d\eta}{2\pi k} && (78)\\ V &= -2\pi k i\int_0^y \left[1 + \int_\eta^y L(k,\eta,\sigma)d\sigma\right] \\ &\quad \times\alpha(t,k,\eta)d\eta. && (79) \end{aligned}$$

Since we can solve the heat equation (66)–(68) explicitly, the inverse transformations (70) and (79) yield the explicit solutions $u^*(t,k,y)$ and $V^*(t,k,y)$, respectively.

Moreover, since (69)–(70) map (65) into (66), stability properties of the velocity field follows from those of the α system.

Proposition 5.1: For any k in the range $m < |k| < M$, the equilibrium $u(t,k,y) \equiv V(t,k,y) \equiv 0$ of the system (48)–(56) with feedback control laws (63), (74) is exponentially stable in the L^2 sense, i.e.,

$$\begin{aligned} &||V(t,k)||^2_{\hat{L}^2} + ||u(t,k)||^2_{\hat{L}^2} \\ \leq\ & D_0 \mathrm{e}^{\frac{-1}{2Re}t}\left(||V(0,k)||^2_{\hat{L}^2} + ||u(0,k)||^2_{\hat{L}^2}\right), && (80) \end{aligned}$$

where D_0 is defined as:

$$\begin{aligned} D_0 &= (1 + 4\pi^2 M^2) \\ &\quad \times \max_{m<|k|<M}\{(1+||L||_\infty)^2(1+||K||_\infty)^2\}. && (81) \end{aligned}$$

Proof: First, from the α equation (66) it is possible to get an L^2 estimate

$$||\alpha(t,k)||^2_{\hat{L}^2} \leq \mathrm{e}^{-\frac{1}{2Re}t}||\alpha(0,k)||^2_{\hat{L}^2}, \qquad (82)$$

then employing the direct and inverse transformations (69)–(70) and (79) we get (80)–(81). ■

Now, if we apply the feedback laws (63), (74) for *all* wave numbers $m < |k| < M$, then the control laws in physical space are given by expressions (22)–(28), where the inverse transform integrals are truncated at $k = m, M$ in (26)–(28). If we define

$$V^*(t,x,y) = \int_{-\infty}^{\infty} \chi(k) V(t,k,y) \mathrm{e}^{2\pi ikx} dk, \quad (83)$$

$$u^*(t,x,y) = \int_{-\infty}^{\infty} \chi(k) u(t,k,y) \mathrm{e}^{2\pi ikx} dk, \quad (84)$$

which are variables that contain all velocity field information for wave numbers $m < |k| < M$, the following result holds.

Proposition 5.2: Consider equations (8)–(16) with control laws (22)–(23). Then the variables $u^*(t,x,y)$ and $V^*(t,x,y)$ defined in (83)–(84) decay exponentially:

$$\begin{aligned} &||V^*(t)||^2_{L^2} + ||u^*(t)||^2_{L^2} \\ \leq\ & D_0 \mathrm{e}^{\frac{-1}{2Re}t} \left(||V^*(0)||^2_{L^2} + ||u^*(0)||^2_{L^2} \right). \end{aligned} \quad (85)$$

Proof: The Fourier transform of the star variables is, by definition, the same as the Fourier transform of the original variables for $m < |k| < M$, and zero otherwise. Therefore, applying Parseval's formula and Proposition 5.1,

$$\begin{aligned} & ||V^*(t)||^2_{L^2} + ||u^*(t)||^2_{L^2} \\ =\ & \int_{-\infty}^{\infty} \left(||V^*(t,k)||^2_{\hat{L}^2} + ||u^*(t,k)||^2_{\hat{L}^2} \right) dk \\ =\ & \int_{-\infty}^{\infty} \chi(k) \left(||V(t,k)||^2_{\hat{L}^2} + ||u(t,k)||^2_{\hat{L}^2} \right) dk \\ \leq\ & D_0 \mathrm{e}^{\frac{-1}{2Re}t} \int_{-\infty}^{\infty} \chi(k) \left(||V(0,k)||^2_{\hat{L}^2} + ||u(0,k)||^2_{\hat{L}^2} \right) dk \\ =\ & D_0 \mathrm{e}^{-\frac{1}{2Re}t} \left(||V^*(0)||^2_{L^2} + ||u^*(0)||^2_{L^2} \right), \end{aligned} \quad (86)$$

proving (85). ■

B. Uncontrolled wave number analysis

For the uncontrolled system (48)–(49), we define, for each k, the Lyapunov functional

$$\Lambda(k,t) = \frac{1}{2} \left(||V(t,k)||^2_{\hat{L}^2} + ||u(t,k)||^2_{\hat{L}^2} \right) \quad (87)$$

The time derivative of Λ is

$$\begin{aligned} \dot{\Lambda} =\ & -\frac{8\pi^2 k^2}{Re} \Lambda - \frac{1}{Re} \left(||u_y(k)||^2_{\hat{L}^2} + ||V_y(k)||^2_{\hat{L}^2} \right) \\ & +4 \int_0^1 (2y-1) \frac{u\bar{V} + \bar{u}V}{2} dy, \end{aligned} \quad (88)$$

where the bar denotes the complex conjugate, and the pressure term has disappeared using integration by parts and the continuity equation (57). The second term in the first line of (88) can also be bounded using the Poincare inequality, thanks to the Dirichlet boundary condition at $y = 0$:

$$-||u_y(k)||^2_{\hat{L}^2} - ||V_y(k)||^2_{\hat{L}^2} \leq -\frac{\Lambda}{2}. \quad (89)$$

Consider now separately the two cases $|k| \leq m$ and $|k| \geq M$. In the first case, we can bound the second line of (88) as

$$\dot{\Lambda} \leq -\frac{8\pi^2 k^2}{Re} \Lambda - \frac{1}{2Re} \Lambda + 4\Lambda, \quad (90)$$

so, if $|k| \geq \frac{1}{\pi} \sqrt{\frac{Re}{2}}$, then

$$\dot{\Lambda} \leq -\frac{1}{2Re} \Lambda. \quad (91)$$

Now, consider the case of small wave numbers. We bound the second line of (88) using the continuity equation (57)

$$\dot{\Lambda} \leq -\frac{8\pi^2 k^2}{Re} \Lambda - \frac{1}{2Re} \Lambda + 8\pi |k| \Lambda, \quad (92)$$

so, if $|k| \leq \frac{1}{32\pi Re}$, then

$$\dot{\Lambda} \leq -\frac{1}{4Re} \Lambda. \quad (93)$$

We have just proved the following result:

Proposition 5.3: If $m = \frac{1}{32\pi Re}$ and $M = \frac{1}{\pi} \sqrt{\frac{Re}{2}}$, then for both $|k| \leq m$ and $|k| \geq M$ the equilibrium $u(t,k,y) \equiv V(t,k,y) \equiv 0$ of the uncontrolled system (48)–(56) is exponentially stable in the L^2 sense:

$$\begin{aligned} & ||V(t,k)||^2_{\hat{L}^2} + ||u(t,k)||^2_{\hat{L}^2} \\ \leq\ & \mathrm{e}^{\frac{-1}{4Re}t} \left(||V(0,k)||^2_{\hat{L}^2} + ||u(0,k)||^2_{\hat{L}^2} \right). \end{aligned} \quad (94)$$

Since the decay rate in (94) is independent of k, that allows us to claim the following result for *all* uncontrolled wave numbers.

Proposition 5.4: The variables $\epsilon_u(t,x,y)$ and $\epsilon_V(t,x,y)$ defined as

$$\epsilon_u(t,x,y) = \int_{-\infty}^{\infty} (1 - \chi(k))\, u(t,k,y) \mathrm{e}^{2\pi ikx} dk, \quad (95)$$

$$\epsilon_V(t,x,y) = \int_{-\infty}^{\infty} (1 - \chi(k))\, V(t,k,y) \mathrm{e}^{2\pi ikx} dk, \quad (96)$$

decay exponentially as

$$\begin{aligned} & ||\epsilon_V(t)||^2_{L^2} + ||\epsilon_u(t)||^2_{L^2} \\ \leq\ & \mathrm{e}^{\frac{-1}{4Re}t} \left(||\epsilon_V(0)||^2_{L^2} + ||\epsilon_u(0)||^2_{L^2} \right). \end{aligned} \quad (97)$$

Proof: As in Proposition 5.2. ■

C. Analysis for the entire wave number range

Using (33)–(34),

$$\begin{aligned} ||V(t)||^2_{L^2} =\ & \int_{-\infty}^{\infty} ||V(t,k)||^2_{\hat{L}^2} dk \\ =\ & \int_0^1 \int_{-\infty}^{\infty} \left(V^*(t,k,y) + \epsilon_V(t,k,y) \right)^2 dk dy \\ =\ & \int_0^1 \int_{-\infty}^{\infty} \left((V^*)^2 + \epsilon_V^2 + 2V^* \epsilon_V \right) dk dy \\ =\ & ||V^*(t)||^2_{L^2} + ||\epsilon_V(t)||^2_{L^2}, \end{aligned} \quad (98)$$

where we have used the fact that $V^*(t,k,y)\epsilon_V(t,k,y) = \chi(k)(1-\chi(k))V(t,k,y)$ and $\chi(k)(1-\chi(k))$ is zero for all k by its definition (29).

This shows that the L^2 norm of V is the sum of the L^2 norms of $V^*(t,k,y)$ and $\epsilon_V(t,k,y)$. The same holds for u. Therefore, Theorem 4.1 follows from Propositions 5.2 and 5.4. Noting that D_0 as defined in (81) is greater than unity, we obtain the following estimate of the decay:

$$\begin{aligned} & ||V(t)||^2_{L^2} + ||u(t)||^2_{L^2} \\ \leq\ & D_0 \mathrm{e}^{\frac{-1}{4Re}t} \left(||V(0)||^2_{L^2} + ||u(0)||^2_{L^2} \right). \end{aligned} \quad (99)$$

The explicit solutions are (33)–(34), obtained by solving explicitly (66), using (70) and (79), and applying the inverse Fourier transform, whereas the error bounds are obtained from Proposition 5.4.

VI. Discussion

The result was presented in 2D for ease of notation. Since 3D channels are spatially invariant in both streamwise and spanwise direction, it is possible to extend the design to 3D, by applying the Fourier transform in both invariant directions and following the same steps, with some refinements which include actuation of the spanwise velocity at the wall. The result also trivially extends to periodic channel flow of arbitrary periodic box size, 2D or 3D, only requiring substitution of the Fourier transform by Fourier series, with all other expressions still holding.

In this paper we only show L^2 stability, but our controllers attain closed loop exponential stability in the H^1 and H^2 norms as well; the statement and proof, which were skipped due to page limit, will appear in a future publication.

Our control laws are written as state feedback, however, we have developed an observer design methodology [21] which is dual to the control methodology employed here [20]. This has allowed us to develop an observer for the channel flow, which is presented in a companion paper [22]. Both results can be combined for an output feedback design, which uses measurements of $P(x,0)$ and $u_y(x,0)$ only, and the actuation of $V(x,1)$, $u(x,1)$.

Our controller requires actuation of both velocity components at the wall. An assumption made throughout the flow control literature is that the boundary values of velocity are actuated through micro-jet actuators that perform "zero-mean" blowing and suction. Effective actuation of wall velocity at angles as low as 5° relative to the wall has been demonstrated experimentally using differentially actuated pairs of jets.

Appendix

We show some properties of the control laws stated in Remarks 3.1 and 3.2.

Remark 3.2 gave bounds on the decay rate of the kernels (26)–(28). All the kernel definitions are of the form

$$Q(x-\xi,y)=\int_{-\infty}^{\infty}\chi(k)f(k,y)\mathrm{e}^{2\pi i k(x-\xi)}dk, \qquad (100)$$

for some f analytic in k and smooth in y. Then, integrating by parts, we find that

$$\begin{aligned}|Q(x-\xi,y)| &\leq \frac{(M-m)}{\pi|x-\xi|}\max_{m<|k|<M}\left|\frac{df}{dk}(k,y)\right| \\ &\quad +\frac{2}{\pi|x-\xi|}\max_{m<|k|<M}|f(k,y)| \\ &= \frac{C}{|x-\xi|}, \qquad (101)\end{aligned}$$

showing that the kernels decay at least like $1/|x-\xi|$.

The zero net flux property, as defined in Remark 3.1, is verified if

$$\int_{-\infty}^{\infty}V_c(t,x)dx=0. \qquad (102)$$

From the definition of the Fourier transform of V_c,

$$V_c(t,k=0)=\int_{-\infty}^{\infty}V_c(t,x)dx, \qquad (103)$$

and since $k=0$ lies on the uncontrolled wave number range $-m<k<m$, $V_c(t,k=0)=0$, and the property is verified.

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