Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Remarks on Monotone Control Systems with Multi-Valued Input-State Characteristics

Patrick De Leenheer and Michael Malisoff

Abstract— **We announce a new global small-gain theorem for feedback interconnections of monotone input-output systems with multi-valued input-state characteristics. This extends a small-gain theorem of Angeli and Sontag for monotone systems with singleton-valued characteristics. The proof of our theorem is based on Thieme's convergence theory for asymptotically autonomous systems. We also provide an illustrative example.**

Index Terms— **Monotone control systems, asymptotic equilibria, set-valued input-state characteristics**

I. Introduction

The recent extension [1] of monotone dynamical systems theory to input-output (i/o) control systems has been used to analyze the global behavior of a large variety of important dynamics; see for example [1], [2], [3], and Section II below for the corresponding definitions. (See also [7] for an extensive discussion of monotone dynamical systems.) Of particular interest in the monotone control systems literature are feedback interconnections of monotone subsystems–or "modules"–that possess a unique globally asymptotically stable equilibrium, depending on the particular constant input applied. This led to the development of the notion of *input-state (i/s) characteristics*, which assign to each constant input value the equilibrium point to which all solutions converge. In many applications, i/s characteristics can be computed from experimental data (e.g., gene expression levels, for instance). Monotonicity, however, may be considered as a structural or *qualitative* property of an i/o system; see the graphical tests for monotonicity in [2] for example. Monotonicity of the subsystems and existence of characteristics are key ingredients for proving small-gain theorems [1], [2], [3].

In practice, on the other hand, a monotone i/o system subject to a constant input may possess *several* equilibria, with all solutions converging to one of them, depending on their initial values. Such systems are sometimes called *multi-stable*. In fact, as monotone i/o systems subject to constant inputs are monotone dynamical systems, this type of global behavior is to be expected (see [7]). This suggests that the concept of i/s characteristics should be generalized to *multi-valued* maps which assign to each constant input value the set of all the equilibria to which solutions converge.

De Leenheer (Corresponding Author): Tel.: +1 352 392 0281 ext. 240; Fax: +1 352 392 8357; Department of Mathematics; University of Florida; 411 Little Hall; PO Box 118105; Gainesville, FL 32611–8105 USA; deleenhe@math.ufl.edu. Supported by NSF/DMS Grant 0500861.

Malisoff: Department of Mathematics; Louisiana State University; 304 Lockett Hall; Baton Rouge, LA 70803-4918 USA; malisoff@lsu.edu. Supported by NSF/DMS Grant 0424011.

This naturally suggests the question of whether the small-gain theorem for monotone i/o systems in [1] remains valid if instead of the original notion of i/s characteristics, one assumes there are multi-valued characteristics for the subsystems. In this note, we explain how such an extension is possible. In our main theorem, we find that a negative feedback interconnection of monotone i/o subsystems with multi-valued characteristics is itself multi-stable, as long as all solutions of a particular discrete-time inclusion (which is typically of significantly lower dimension than the subsystems) converge.

Our result provides a significant extension of the Angeli-Sontag monotone control systems theory from [1] because [1] requires singleton-valued characteristics and thus globally asymptotically stable equilibria. For other approaches to proving multi-stability, see [6] (which is based on density functions and concludes convergence for *almost all* initial values) and [2] (where *positive* feedback interconnections of monotone i/o subsystems are considered and the trajectories also converge for almost all initial values). This earlier work does not include our result because, for example, (a) we do not require any regularity such as singleton-valuedness, differentiability, or non-degeneracy for the i/s characteristics, (b) our results provide global stabilization from all initial values, and (c) our results are intrinsic in that we make no use of density or Lyapunov functions. For an alternative approach to negative interconnections of monotone systems based on a symmetric embedding into a higher dimensional space, leading to an alternative small gain theorem under an additional injectiveness assumption on the the output map and a more restrictive boundedness assumption on the trajectories of the embedded system, see [5].

This paper is organized as follows. In Section II, we provide the relevant background and definitions for monotone control systems, asymptotically autonomous systems, multi-valued characteristics, and weakly non-decreasing set-valued maps. In Section III, we announce our small-gain theorem and discuss how it extends the small-gain theorems in [1], [2], [3]. In Section IV, we sketch the proof our theorem and we illustrate our theorem in Section V. We close in Section VI with ideas for future research. While our discussions in this note will be mainly conceptual, complete proofs of all the results below are in [4].

II. Background and Motivation

A. Monotonicity and Characteristics

We start with the relevant definitions for monotone control systems and input-state characteristics. Our monotonicity definitions follow [1], but our treatment of characteristics is apparently new because we allow unstable equilibria and discontinuous multi-valued characteristics. Our general setting is that of an input-output (i/o) system

$$\dot{x} = f(x,u), \;\; y = h(x), \;\; x \in \mathcal{X}, \;\; u \in \mathcal{U}, \;\; y \in \mathcal{Y} \qquad (1)$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is the closure of its interior and partially ordered, f and h are locally Lipschitz on some open set X containing $\mathcal{X}$, and $\mathcal{U}$ and $\mathcal{Y}$ are subsets of partially ordered Euclidean spaces $\mathcal{B}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{Y}}$ respectively. We call $\mathcal{X}$ the *state space* of (1), $\mathcal{U}$ its *input space*, and $\mathcal{Y}$ its *output space*. In general, $\mathcal{X}$ is not a linear space, since for example we often choose $\mathcal{X} = \mathbb{R}^n_{\geq 0} := \{x \in \mathbb{R}^n : x_i \geq 0\, \forall i\}$. We use $\preceq$ to denote the partial orders on our spaces, although the partial orders on our various spaces could in principle differ.

The set of *control functions* (which are also called *inputs*) for (1), which we denote by $\mathcal{U}_\infty$, consists of all locally essentially bounded (Lebesgue) measurable functions $\mathbf{u} : \mathbb{R} \to \mathcal{U}$. We let $t \mapsto \phi(t, x_o, \mathbf{u})$ denote the trajectory of (1) for a given initial value $x_o \in \mathcal{X}$ and $\mathbf{u} \in \mathcal{U}_\infty$. We always assume f is *forward complete* and *$\mathcal{X}$-invariant*, which means $\phi(\cdot, x_o, \mathbf{u})$ is defined on $[0,\infty)$ and valued in $\mathcal{X}$ for all $x_o \in \mathcal{X}$ and $\mathbf{u} \in \mathcal{U}_\infty$. When we consider more than one dynamic, we often use sub- or superscripts to emphasize the state variable or dynamic, so for example ϕ^f is the flow map for the dynamic f and $\mathcal{Y}_z$ is the output space for an i/o system with state variable z.

We always assume our partial orders $\preceq$ are induced by closed nonempty sets K (called *ordering cones*) and sometimes write $K_{\mathcal{X}}$ to indicate the cone inducing the partial order on the state space $\mathcal{X}$ and similarly for the other partial orders. We always assume K is a *pointed convex cone*, meaning,

$$aK \subseteq K \;\; \forall a \geq 0, \quad K + K \subseteq K, \quad K \cap (-K) = \{0\}.$$

When we say a cone K induces a partial order $\preceq$, we mean: $x \preceq y$ if and only if $y - x \in K$. This induces a partial order on $\mathcal{U}_\infty$ as follows: $\mathbf{u} \preceq \mathbf{v}$ if and only if $\mathbf{u}(t) \preceq \mathbf{v}(t)$ for Lebesgue almost all (a.a) $t \geq 0$. A function g mapping a partially ordered space into another partially ordered space is *monotone* provided: $x \preceq y$ implies $g(x) \preceq g(y)$. We call (1) *single-input single-output (SISO)* provided $\mathcal{B}_{\mathcal{U}} = \mathcal{B}_{\mathcal{Y}} = \mathbb{R}$, taken with the usual order, which is the order induced by the cone $K = [0,\infty)$.

Definition 1: We say that (1) is *monotone* provided h is monotone and

$$(p \preceq q \text{ and } \mathbf{u} \preceq \mathbf{v}) \implies (\phi(t,p,\mathbf{u}) \preceq \phi(t,q,\mathbf{v}) \;\forall t \geq 0)$$

holds for all $p, q \in \mathcal{X}$ and $\mathbf{u}, \mathbf{v} \in \mathcal{U}_\infty$.

We let $\mathrm{Equil}(f)$ denote the set of all equilibrium pairs for our dynamic f, meaning, the set of all input-state pairs $(\bar{u}, \bar{x})$ such that $f(\bar{x}, \bar{u}) = 0$. For each $(\bar{u}, \bar{x}) \in \mathrm{Equil}(f)$, we let $\mathcal{D}^f(\bar{u}, \bar{x})$ denote the *domain of attraction* of $\dot{x} = f(x, \bar{u})$ to $\bar{x}$, which is the set of all $p \in \mathcal{X}$ for which $\phi(t, p, \bar{u}) \to \bar{x}$ as $t \to +\infty$, where ϕ denotes the flow map for f. Since we are not assuming our equilibria are stable, the sets $\mathcal{D}^f(\bar{u}, \bar{x})$ are not necessarily open and could in principle be singletons. Given $(\bar{u}, \bar{x}) \in \mathrm{Equil}(f)$, we say that f is *static Lyapunov stable at* $(\bar{u}, \bar{x})$, and write $f \in \mathrm{SLS}(\bar{u}, \bar{x})$, provided the following condition holds for all $\varepsilon > 0$: There exists $\delta = \delta(\bar{u}, \bar{x}, \varepsilon) > 0$ such that for all $x_o \in \mathcal{D}^f(\bar{u}, \bar{x}) \cap \mathcal{B}_\delta(\bar{x})$(= radius δ open ball centered at $\bar{x}$), we have $|\phi(t, x_o, \bar{u}) - \bar{x}| \leq \varepsilon$ for all $t \geq 0$. The stipulation in the SLS definition that $x_o \in \mathcal{D}^f(\bar{u}, \bar{x}) \cap \mathcal{B}_\delta(\bar{x})$ is motivated by the fact that our domains of attraction $\mathcal{D}^f(\bar{u}, \bar{x})$ may or may not be open, even for systems with no controls.

Recall the following notions from [9], where we let $f^{\bar{u}}$ denote the constant input system $f(\cdot, \bar{u})$ for each $\bar{u} \in \mathcal{U}$. Given $\bar{u} \in \mathcal{U}$, we say that two nonempty (but not necessarily distinct) sets $M_1, M_2 \subseteq \mathcal{X}$ are *$f^{\bar{u}}$-chained* provided there exist a value $y \in \mathcal{X} \setminus (M_1 \cup M_2)$ and a trajectory $x : \mathbb{R} \to \mathcal{X}$ for $f^{\bar{u}}$ satisfying $x(0) = y$ whose *α-limit set*

$$\alpha(x) := \bigcap \{\overline{x((-\infty, -t])} : t \geq 0\}$$

lies in M_1 and whose *ω-limit set*

$$\omega(x) := \bigcap \{\overline{x([t, +\infty))} : t \geq 0\}$$

lies in M_2. We say that a finite collection of nonempty sets $M_1, M_2, \ldots, M_r \subseteq \mathcal{X}$ is *$f^{\bar{u}}$-cyclically chained* provided the following holds: If $r = 1$, then M_1 is $f^{\bar{u}}$-chained to itself; and if $r > 1$, then M_i is $f^{\bar{u}}$-chained to M_{i+1} for $i = 1, 2, \ldots, r-1$ and M_r is $f^{\bar{u}}$-chained to M_1. In this situation, we call $\{M_i\}$ an *$f^{\bar{u}}$-cycle*. An *$f^{\bar{u}}$-equilibrium* is any point $\bar{x} \in \mathcal{X}$ such that $f(\bar{x}, \bar{u}) = 0$. A set $M \subseteq \mathcal{X}$ is called *$f^{\bar{u}}$-invariant* provided the flow map ϕ for f satisfies $M = \{\phi(t, x, \bar{u}) : t \geq 0, x \in M\}$. A compact $f^{\bar{u}}$-invariant set $M \subseteq \mathcal{X}$ is called *$f^{\bar{u}}$-isolated compact invariant* provided there exists an open set $\mathcal{O} \subseteq \mathcal{X}$ such that there is no compact $f^{\bar{u}}$-invariant subset $\tilde{M} \subseteq \mathcal{X}$ satisfying $M \subseteq \tilde{M} \subseteq \mathcal{O}$ except M. We use the symbol $\rightrightarrows$ to denote a *set-valued map* (also called a *multifunction*), e.g., $F : \mathcal{Z}_1 \rightrightarrows \mathcal{Z}_2$ means that F assigns each $p \in \mathcal{Z}_1$ a (nonempty) set $F(p) \subseteq \mathcal{Z}_2$.

Definition 2: We say that (1) is endowed with a *static input-state (i/s) characteristic* $k_x : \mathcal{U} \rightrightarrows \mathcal{X}$ provided:

1) $\mathrm{Graph}(k_x) = \mathrm{Equil}(f)$;
2) $\cup\{\mathcal{D}^f(\bar{u}, \bar{x}) : \bar{x} \in k_x(\bar{u})\} = \mathcal{X}$ for all $\bar{u} \in \mathcal{U}$;
3) $f \in \mathrm{SLS}(\bar{u}, \bar{x})$ for all $(\bar{u}, \bar{x}) \in \mathrm{Equil}(f)$; and
4) For each $\bar{u} \in \mathcal{U}$, $k_x(\bar{u})$ consists of $f^{\bar{u}}$-isolated compact invariant $f^{\bar{u}}$-equilibria and contains no $f^{\bar{u}}$-cycles.

In this case, we also call $k_y := h \circ k_x$ an *input-output (i/o) characteristic* for (1).

This definition reduces to the standard singleton-valued i/s characteristic definition in [1] when $\mathrm{Card}\{k_x(\bar{u})\} = 1$ for all $\bar{u} \in \mathcal{U}$. See [4] for a sufficient condition for the no-cycles requirement in 4) in terms of an ordering of the equilibria. While we do not use the SLS property below, we include it to make our definition of i/s characteristics include the singleton-valued characteristic definition in [1]. Condition 2 in the definition says for each initial state and each $\bar{u} \in \mathcal{U}$, the corresponding $f^{\bar{u}}$-trajectory asymptotically approaches some state $\bar{x} \in k_x(\bar{u})$ (where $\bar{x}$ can in principle depend on the initial state of the trajectory).

B. Weakly Non-Decreasing Set-Valued Maps

One basic property of singleton-valued i/s characteristics k_x is that they are non-decreasing in the relevant partial orders, by which we mean the following holds for all $u, v \in \mathcal{U}_x$: $u \preceq v$ implies $k_x(u) \preceq k_x(v)$; see [1] for the elementary proof. It is natural to ask whether set-valued i/s characteristics enjoy some analogous (but more general) order-preserving property. This motivates the following definition and lemma:

Definition 3: Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be partially ordered Euclidean spaces and $F : \mathcal{Z}_1 \rightrightarrows \mathcal{Z}_2$ be any set-valued map. We say that F is *weakly non-decreasing* provided the following holds for all $p, q \in \mathcal{Z}_1$ such that $p \preceq q$: For each $k_p \in F(p)$ and $k_q \in F(q)$, there exist $r_p \in F(p)$ and $r_q \in F(q)$ such that $r_p \preceq k_q$ and $k_p \preceq r_q$.

Lemma 4: If k_x is an i/s characteristic for (1) and (1) is monotone, then k_x is weakly non-decreasing.

Proof: Let p, q, k_p, and k_q be as in the hypotheses of Definition 3 and ϕ be the flow map of f. The corresponding trajectories for the constant inputs then satisfy $\phi(t, k_q, p) \preceq \phi(t, k_q, q) = k_q$ for all $t \geq 0$, and $\phi(t, k_q, p) \to r_p$ for some $r_p \in k_x(p)$ as $t \to +\infty$, so $r_p \preceq k_q$, because ordering cones are closed. The other order inequality is proved in an analogous way. ■

Definition 3 reduces to non-decreasingness in the relevant orders when F is singleton-valued. We are especially interested in solution sequences w_k satisfying discrete set-valued inclusions $w_{k+1} \in F(w_k)$ for all $k \in \mathbb{N}$ where F is weakly non-decreasing. To further motivate our study of weakly non-decreasing multifunctions, let us first assume that $F : [0,1] \to [0,1]$ is a singleton-valued and non-decreasing map in the usual orders (that is, $F(x) \leq F(y)$ when $x \leq y$). Then it is obvious that every solution of $x_{k+1} = F(x_k)$ converges. Indeed, either $x_0 \leq F(x_0)$ and then $x_0 \leq F(x_0) \leq F^2(x_0) \leq \cdots \leq F^k(x_0)$ for all $k \in \mathbb{N}$, so the sequence $\{F^k(x_0)\}$ must converge since it is bounded above by 1; or else $F(x_0) \leq x_0$, which leads to a non-increasing sequence $\{F^k(x_0)\}$. That converges as well since it is bounded below by 0. On the other hand, this simple dynamical behavior will not occur in general for *multi-valued*, weakly non-decreasing maps.

To see why, consider the following simple example. Assume that $F : [0,1] \rightrightarrows [0,1]$ is a multi-valued map whose graph consists of the union of three straight line segments: one connecting $A = (0,0)$ with $B = (1/2, 1/4)$, a second connecting B to $C = (1/4, 1/2)$ (of slope -1), and a third connecting C with $D = (1,1)$. This "inverted Zorro map" is illustrated in Figure 1 below and is weakly non-decreasing in the usual orders. Then the inclusion $x_{k+1} \in F(x_k)$ has periodic points of period 2. For instance, the periodic sequence $\{1/2, 1/4, 1/2, 1/4, ...\}$ is a solution of the inclusion. In fact, to every initial condition $x_0 \in [1/4, 1/2]$ corresponds a periodic sequence of period 2 satisfying the inclusion, namely $\{x_0, 3/4 - x_0, x_0, 3/4 - x_0, ...\}$ (since $3/4 - x \in F(x)$ for all $x \in [1/4, 1/2]$).

These periodic sequences are caused by the fact that the slope of the middle line segment of the graph of F is -1.

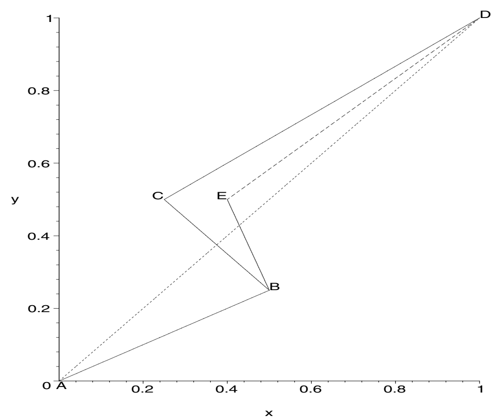

Fig. 1. The inverted Zorro map F (ABCD) and its perturbation F_ϵ with $\epsilon = 1.5$ (ABED) from Section II-B.

Any slight decrease of this slope will destroy the periodic points and leads to solutions that converge to one of the fixed points. For example, for arbitrary $\epsilon > 0$ we can define F_ϵ as the map whose graph consists of three straight line segments connecting A to B, B to $E = ((1+2\epsilon)/(4+4\epsilon), 1/2)$ (so the slope of this line segment is $-1-\epsilon$), and E to D. Then every solution of the inclusion $x_{k+1} \in F_\epsilon(x_k)$ will converge to one of the three fixed points of F. In fact, each solution sequence of this inclusion converges to either 0 or 1, except for the constant sequence at the middle fixed point $\tilde{x} = (3+2\epsilon)/(4(2+\epsilon))$. To see why, notice that if $x_o > 1/2$, then $(x_k, F_\epsilon(x_k))$ remains on the segment $\overline{ED}$, so $x_k \uparrow 1$ by the argument for the singleton-valued case. Similarly, if $x_o < (1+2\epsilon)/(4+4\epsilon)$, then $(x_k, F_\epsilon(x_k))$ remains on $\overline{AB}$ so $x_k \downarrow 0$ again by the singleton-valued case; while if x_k stays in $[(1+2\epsilon)/(4+4\epsilon), 1/2]$, then $x_{k+1} = -(1+\epsilon)x_k + \frac{3}{4} + \frac{\epsilon}{2}$ for all k. Then either $x_k \equiv \tilde{x}$, or else $|x_{k+1} - x_k| = (1+\epsilon)^k |x_1 - x_o| \to +\infty$ as $k \to +\infty$ which is impossible. Therefore, either x_k stays at $\tilde{x}$, or else x_k exits $[(1+2\epsilon)/(4+4\epsilon), 1/2]$ and then converges to either 0 or 1, as claimed.

C. Asymptotically Autonomous Systems

Recall the following "Converging-Input Converging-State" (CICS) Property, which was shown in [8]:

Lemma 5: Let $\bar{u} \in \mathcal{U}$, and let $\bar{x}$ be an asymptotically stable equilibrium point for $f^{\bar{u}}$. Let $\mathcal{K}$ be a compact subset of $\mathcal{D}^f(\bar{u}, \bar{x})$. If $x : [0, \infty) \to \mathcal{X}$ is a $\mathcal{K}$-recurrent trajectory of f for some continuous input $u : [0, \infty) \to \mathcal{U}$, and if $u(t) \to \bar{u}$ as $t \to +\infty$, then $x(t) \to \bar{x}$ as $t \to +\infty$.

Here $\mathcal{K}$-recurrence of $x(t)$ means for each $T > 0$, there exists $t > T$ such that $x(t) \in \mathcal{K}$. By asymptotic stability of $\bar{x}$, we mean in particular that the following holds (in addition to attractivity): For each $\varepsilon > 0$, there exists $\delta > 0$ such that $|\phi(t, \xi, \bar{u}) - \bar{x}| \leq \varepsilon$ for all $\xi \in \mathcal{B}_\delta(\bar{x})$ and $t \geq 0$. The proof of the CICS property in [8] uses the fact that $\mathcal{D}^f(\bar{u}, \bar{x})$ is open, which holds because $\bar{x}$ is a *stable* equilibrium.

However, in our more general situation where the i/s characteristics are multi-valued, the domains of attraction will not necessarily be open, so the CICS property does not apply. Instead, we use the theory of asymptotically autonomous systems of Thieme [9]. To this end, we invoke the following equilibrium condition (EC) from [9]:

(EC) For each $\bar{u} \in \mathcal{U}$, the ω-limit set of any precompact $f^{\bar{u}}$-trajectory on $[0, \infty)$ consists of an $f^{\bar{u}}$-equilibrium.

An *asymptotically autonomous system* is defined to be a system $\dot{x} = H(t, x)$ that admits a second dynamic $\dot{x} = \bar{H}(x)$ (called a *limiting dynamic*) such that $H(t, x) \to \bar{H}(x)$ as $t \to +\infty$ locally uniformly in x. For example, if $u \in \mathcal{U}_\infty$ is continuous and $\bar{u} \in \mathcal{U}$ is such that $u(t) \to \bar{u}$ as $t \to +\infty$, then for our dynamic f, $\dot{x} = H(t, x) := f(x, u(t))$ is asymptotically autonomous with limiting dynamic $\dot{x} = \bar{H}(x) := f(x, \bar{u})$. Notice that monotone i/o systems with i/s characteristics satisfy (EC), since all of their trajectories for constant inputs converge. Using this observation, the following lemma is immediate from [9, Corollary 4.3] and our definitions:

Lemma 6: Assume (1) is endowed with an i/s characteristic. Let $\bar{u} \in \mathcal{U}$ and $u : [0, \infty) \to \mathcal{U}$ be any locally Lipschitz function for which $u(t) \to \bar{u}$ as $t \to +\infty$. Let $x : [0, \infty) \to \mathcal{X}$ be any bounded trajectory for (1) and this input $u(t)$. Then $x(t)$ converges towards an $f^{\bar{u}}$-equilibrium as $t \to +\infty$.

III. Small-Gain Theorem

We next state our small-gain theorem, which extends [1, Theorem 3]. The main novelty of our result is that it applies to cases where one of the interconnected subsystems has a multi-valued i/s characteristic, but see Remark 10 for the extension to interconnections where *both* subsystems have multi-valued i/s characteristics. In what follows, an *equilibrium of a discrete inclusion* $w_{k+1} \in F(w_k)$ is any value $\bar{w}$ such that $\bar{w} \in F(\bar{w})$; the set of all equilibria for this inclusion is denoted by $\mathcal{E}(F)$. A multi-function F is *locally bounded* provided it maps bounded sets into bounded sets. A dynamics F has a *pointwise globally attractive set* S provided each maximal trajectory for F asymptotically approaches *some point* in S, which generally depends on the specific trajectory. We prove the following theorem in [4]:

Theorem 7: Consider the following interconnection of two SISO dynamic systems:

$$\begin{array}{ll} \dot{x} = f_x(x, w), & y = h_x(x) \\ \dot{z} = f_z(z, y), & w = h_z(z) \end{array} \tag{2}$$

with $\mathcal{U}_x = \mathcal{Y}_z$ and $\mathcal{U}_z = \mathcal{Y}_x$. Assume the following:

1) The first system is monotone when its input w and output y are ordered by the "standard order" induced by the positive real semi-axis.
2) The second system is monotone when its input y is ordered by the standard order and its output w is ordered by the opposite order (induced by the negative real semi-axis).
3) The respective static i/s characteristics k_x and k_z exist with k_x singleton-valued and k_z locally bounded.
4) Each trajectory of (2) is bounded; and each solution sequence $\{v_k\}$ of $v_{k+1} \in (k_y \circ k_w)(v_k)$ converges.

Then $\cup\{\{k_x(\bar{w})\} \times (k_z \circ k_y)(\bar{w}) : \bar{w} \in \mathcal{E}(k_w \circ k_y)\}$ is the pointwise globally attractive set for (2).

Here $k_y = h_x \circ k_x$ and $k_w = h_z \circ k_z$. Theorem 7 differs from the small-gain theorem [1, Theorem 3] primarily in that (a) we replaced the discrete system $w_{k+1} = (k_w \circ k_y)(w_k)$ from [1] with a discrete inclusion, (b) we replaced the single valuedness of k_z with local boundedness of k_z, and (c) we conclude that (2) is attracted to a set of equilibrium points instead of to a single point as in [1]. Moreover, unlike [2], our theorem gives *global* convergence of the interconnection from all initial values.

Remark 8: Assumption 4 in Theorem 7 is equivalent to the following:

4′ Each trajectory of (2) is bounded; and the sequence $\{k_y(w_k)\}$ converges for each solution sequence $\{w_k\}$ of $w_{k+1} \in (k_w \circ k_y)(w_k)$.

In fact, if Assumption 4 holds and w_k is a solution of $w_{k+1} \in (k_w \circ k_y)(w_k)$, then $k_y(w_k)$ converges because the sequence $v_k = k_y(w_k)$ is a solution for $v_{k+1} \in (k_y \circ k_w)(v_k)$. Conversely, if Assumption 4′ holds, and if $\{v_k\}$ is any solution of $v_{k+1} \in (k_y \circ k_w)(v_k)$, then we may inductively find a new sequence r_k such that $v_{k+1} \equiv k_y(r_k)$ and $r_{k+1} \in (k_w \circ k_y)(r_k)$ for all k. Thus, v_k converges. However, it could be that Assumption 4 holds but that there exists a *divergent* solution $\{w_k\}$ for $w_{k+1} \in (k_w \circ k_y)(w_k)$ [4]. On the other hand, if the trajectories of (2) are bounded, and if each solution sequence of $w_{k+1} \in (k_w \circ k_y)(w_k)$ converges, then Assumption 4′ (and equivalently Assumption 4) holds because k_y is continuous (by arguments from [1, Proposition V.5] and our assumption that k_x is singleton valued).

IV. Discussion of Proof of Theorem 7

We next sketch the proof of our small-gain theorem. For a complete proof, see [4]. The following key lemma generalizes [1, Proposition V.8] to systems with multi-valued characteristics. In it, we set $u_{\inf} := \liminf_{t \to +\infty} u(t)$ and $u_{\sup} := \limsup_{t \to +\infty} u(t)$ for any continuous scalar function u on $[0, \infty)$.

Lemma 9: Let the hypotheses of Theorem 7 hold, $(x(t), z(t))$ be any trajectory of (2), and $\zeta \in \omega(z)$. Then there exist $k_- \in k_z(y_{\inf})$ and $k_+ \in k_z(y_{\sup})$ such that $k_- \preceq \zeta \preceq k_+$.

Proof: We only prove the existence of k_-; the proof of the existence of k_+ is analogous. Set $\mu = y_{\inf}$ and let ξ be the initial value for $z(t)$. Let $t_j \to +\infty$ and $\mu_j \to \mu$ be two sequences such that $\mu_j \in \mathcal{U}_z$ and $y(t) \geq \mu_j$ for all $t \geq t_j$ and all j. Then for all $t \geq t_j$ and $j \in \mathbb{N}$,

$$\begin{aligned} z(t) \;=\; \phi(t, \xi, y) \;&=\; \phi(t - t_j, \phi(t_j, \xi, y), y(\cdot + t_j)) \\ &\succeq\; \phi(t - t_j, \phi(t_j, \xi, y), \mu_j), \end{aligned} \tag{3}$$

where ϕ is the flow map for f_z and the last inequality follows from the monotonicity of f_z. Hence, if $z(s_l) \to \zeta$ for some sequence $s_l \to +\infty$, then we can set $t = s_l$ in (3) and use the closedness of order cones to get values $v_j \in k_z(\mu_j)$ for which

$$\zeta \succeq \lim_{l \to \infty} \phi(s_l - t_j, \phi(t_j, \xi, y), \mu_j) = v_j \quad \forall j \in \mathbb{N}. \tag{4}$$

Since k_z is locally bounded with closed graph (by the continuity of the dynamic f_z in all arguments), there are $k_- \in k_z(\mu)$ such that $\zeta \succeq v_j \to k_-$, by passing to a subsequence without relabeling, which gives the desired order inequality. ■

Next notice that since w is ordered by the negative real semi-axis, and since k_z is weakly non-decreasing (by Lemma 4), it follows that

$$\max_{k_p \in k_w(p)} \min_{k_q \in k_w(q)} (k_p - k_q)(p-q) \leq 0 \quad \forall p, q \in \mathcal{U}_z. \quad (5)$$

Choose any initial value ξ for (2), and let $(x(t), z(t))$ denote the corresponding trajectory for (2) starting at ξ. This trajectory is defined on $[0, \infty)$ since our trajectories are assumed to be bounded. Set $y_+ = y_{\sup}$ and $y_- = y_{\inf}$. Using (5) and proceeding inductively, it is possible (see [4]) to construct sequences $\{s_\pm^{(r)}\}$ satisfying the following for all $j \in \mathbb{N}$:

$$\begin{aligned}(k_y \circ k_w)^{2j}(y_-) \quad &\ni \quad s_-^{(2j)} \leq y_- \leq y_+ \\ &\leq \quad s_+^{(2j)} \in (k_y \circ k_w)^{2j}(y_+) \end{aligned} \quad (6)$$

$$\begin{aligned}(k_y \circ k_w)^{2j-1}(y_+) \quad &\ni \quad s_+^{(2j-1)} \leq y_- \leq y_+ \\ &\leq \quad s_-^{(2j-1)} \in (k_y \circ k_w)^{2j-1}(y_-). \end{aligned} \quad (7)$$

This can be done in such a way that

$$s_\pm^{(j)} \in (k_y \circ k_w)^{j-1}(s_\pm^{(1)}) \ \forall j \in \mathbb{N} \quad (8)$$

so Assumption 4 from our theorem provides values $\bar{r}_\pm$ such that $s_\pm^{(j)} \to \bar{r}_\pm$ as $j \to +\infty$. Letting $j \to +\infty$ in (6) shows that $\bar{r}_- \leq \bar{r}_+$; and letting $j \to +\infty$ in (7) gives $\bar{r}_+ \leq \bar{r}_-$. Thus, $\bar{r}_+ = \bar{r}_- = y_+ = y_- =: \bar{y}$. Applying Lemma 6 to $f = f_z$ and the input $u(t) = y(t) \to \bar{y}$ gives $z(t) \to \bar{z}$ for some $\bar{z} \in k_z(\bar{y})$. As h_z is continuous, $w(t)$ converges also; i.e., $w_+ = w_- =: \bar{w}$. Thus, $\bar{w} = h_z(\bar{z}) \in k_w(\bar{y})$, and one can also show that $\bar{y} = k_y(\bar{w})$. We conclude that $\bar{w} \in (k_w \circ k_y)(\bar{w})$, so $\bar{w} \in \mathcal{E}(k_w \circ k_y)$. Therefore, Theorem 7 follows once we show that $(x(t), z(t))$ converges to some point in $\{k_x(\bar{w})\} \times (k_z \circ k_y)(\bar{w})$ as $t \to +\infty$. To this end, first note that $x(t) \to k_x(\bar{w})$ as $t \to +\infty$ as a consequence of Lemma 5 above, applied to $f = f_x$ and the input $u(t) = w(t) \to \bar{w}$, since we are assuming that k_x is singleton-valued. As $\bar{z} \in k_z(\bar{y}) = k_z(k_y(\bar{w}))$, the result follows.

Remark 10: One can extend our theorem to examples where k_x and k_z are *both* multi-valued (see [4]). In fact, our theorem remains true if we replace Assumption 3 by:

3′. *The respective i/s characteristics k_x and k_z exist and are locally bounded* .

The conclusion of the theorem becomes that our interconnection (2) has $\cup\{k_x(\bar{w}) \times (k_z \circ k_y)(\bar{w}) : \bar{w} \in \mathcal{E}(k_w \circ k_y)\}$ as its pointwise globally attractive set.

Remark 11: Our small gain theorem remains true even if the local boundedness condition on k_z is replaced by:

(A) For each bounded subset $S \subseteq \mathcal{X}_z$, there exist $a, b \in \mathcal{X}_z$ such that $a \preceq x \preceq b$ for all $x \in S$.

This is related to the bounded orders requirement from [1] but is more restrictive since we require a and b to be in $\mathcal{X}_z$ rather than merely in the ambient Banach space. To see why our theorem remains true under this alternative hypothesis, it suffices to prove Lemma 9 under this alternative hypothesis since that is the only place in the proof of the theorem where the local boundedness of k_z is used; see [4]. To this end, we argue exactly as in the proof of the lemma up through (4). We can assume $\{\mu_i\}$ is increasing. Pick $\underline{\phi} \in \mathcal{X}_z$ such that $\underline{\phi} \preceq \phi(t_j, \xi, y)$ for all j, which exists by Assumption (A). Then

$$\phi(s_l - t_j, \phi(t_j, \xi, y), \mu_j) \succeq \phi(s_l - t_j, \underline{\phi}, \mu_1) \ \ \forall l, j,$$

by the definition of monotonicity. Pick $\underline{v} \in \mathcal{X}_z$ such that $\phi(t, \underline{\phi}, \mu_1) \to \underline{v}$ as $t \to +\infty$. Then $\zeta \succeq v_j \succeq \underline{v}$ for all j. By our pointedness assumption $K_z \cap (-K_z) = \{0\}$, it follows that the order interval

$$[\underline{v}, \zeta] = \{x \in \mathcal{X}_z : \underline{v} \preceq x \preceq \zeta\}$$

is bounded (see [1, p. 1690]) hence compact, so v_j has a convergent subsequence. The remainder of the proof of the lemma is as before.

V. ILLUSTRATION

We next illustrate our theorem using the interconnection

$$\begin{array}{ll} \dot{x} = -x + 5 + w, & y = x \\ \dot{z} = -P(z) + y, & w = \frac{1}{1+z^2} \end{array} \quad (9)$$

evolving on $[0, \infty) \times [0, \infty)$, where $P(z) = z(2z^2 - 9z + 12)$. We order x and z by the standard cone $[0, \infty)$. The dynamic (9) satisfies Conditions 1-2 from Theorem 7. Replacing w with $\frac{1}{1+w^2}$ in (9) gives the planar positive feedback system

$$\begin{array}{ll} \dot{x} = -x + 5 + \frac{1}{1+w^2}, & y = x \\ \dot{z} = -P(z) + y, & w = z. \end{array} \quad (10)$$

Using superscripts o to label the characteristics of our original interconnection (9) and using k_x and so on to denote the characteristics of (10) (where they exist), we get

$$k_x^o\left(\frac{1}{1+w^2}\right) \equiv k_x(w)$$

and $k_z^o \equiv k_z$. Moreover, if $u_{k+1} \in (k_w^o \circ k_y^o)(u_k)$ with $u_k > 0$ for all k, then $w_{k+1} \in (k_w \circ k_y)(w_k)$ for all k when the w_k's are taken to satisfy

$$\frac{1}{1+w_k^2} = u_k$$

for all $k \in \mathbb{N}$. Moreover, since w in (9) is always positive, $(k_w^o \circ k_y^o)(0) \subseteq (0, \infty)$, so $u_k > 0$ for all $k \geq 1$ along all solution sequences $\{u_k\}$ of $u_{k+1} \in (k_w^o \circ k_y^o)(u_k)$. Thus, if each solution sequence $\{w_k\}$ for $w_{k+1} \in (k_w \circ k_y)(w_k)$ converges, then each solution sequence $\{u_k\}$ for $u_{k+1} \in (k_w^o \circ k_y^o)(u_k)$ converges as well, which gives the required convergence of solutions of $v_{k+1} \in (k_y^o \circ k_w^o)(v_k)$ by Remark 8. The fact that Condition 4 also holds for the original interconnection (9) will then follow once we show that (10) has bounded trajectories, because (9) has the same trajectories as (10).

It therefore remains to check (i) that (10) satisfies Condition 3 from our theorem, (ii) that all its trajectories are bounded, and (iii) that each solution of $w_{k+1} \in (k_w \circ k_y)(w_k)$ converges (by Remark 8). To this end, first note that since the outputs of both subsystems in (10) are their states, the i/s and i/o characteristics coincide for (10)-where they exist–so we define $k_1 = k_x = k_y$ and $k_2 = k_z = k_w$ wherever the characteristics exist. The characteristic of the first subsystem in (10) is $k_1(w) = 5 + \frac{1}{1+w^2}$ ($w \in \mathbb{R}_+$), while the characteristic for the second subsystem is multi-valued and only determined implicitly as follows:

$$k_2(y) = \{z \in \mathbb{R} : P(z) = y\}, \quad y \in \mathbb{R}_+.$$

A bifurcation analysis of $\dot{z} = -P(z) + y$, with $y \in \mathbb{R}_+$ as a bifurcation parameter, shows that $k_2(y)$ is a characteristic which is

1) single-valued if $y \in [0, 4)$ or if $y \in (5, \infty)$.
2) triple-valued if $y \in (4, 5)$.
3) double-valued if $y = 4$ (with $k_2(4) = \{1/2, 2\}$) or $y = 5$ (with $k_2(5) = \{1, 5/2\}$).

The four defining properties of a characteristic (see Definition 2) can indeed be readily verified: For each $y \in \mathbb{R}_+$, the subsystem $\dot{z} = -P(z) + y$ has a finite number of isolated compact equilibria and has no cycles (since the system is scalar), and every solution converges to one of the equilibria. It is not difficult to see that k_2 is locally bounded. To apply Theorem 7, we only need to verify that (10) satisfies Condition 4 of our theorem.

One can readily show (cf. [4]) that the trajectories of (10) (or equivalently of (9)) are bounded. This boundedness can be shown as a consequence of the following [4]: *Claim (G): If $(x(t), z(t))$ is any trajectory of (9) defined on some interval $[0, T]$, then there is a compact set D depending only on $(x(0), z(0))$ (and not on T) such that $(x(t), z(t)) \in D$ for all $t \in [0, T]$.* The boundedness of the trajectories then follows by applying Claim (G) with $T = 1, 2, 3$ and so on.

Next we consider the discrete inclusion

$$w_{k+1} \in (k_2 \circ k_1)(w_k),$$

which reduces to a discrete equation $w_{k+1} = (k_2 \circ k_1)(w_k)$ because $k_1(w) > 5$ and $k_2(y)$ is single-valued when $y > 5$. In fact, for all $w_0 \in \mathbb{R}_+$, the discrete equation gives $w_k > 5/2$ for all $k \geq 1$. Also, the interval $(5/2, \infty)$ is forward invariant for the discrete equation. Finally, since $|k_1'(w)|$ is decreasing for $w \geq 5/2$, elementary calculus gives

$$|k_2'(k_1(w))k_1'(w)| \;\leq\; \frac{5}{(1+25/4)^2}\frac{2}{9} \;<\; 1 \;\; \forall w \geq 5/2,$$

since $k_2'(y) = 1/P'(k_2(y)) \leq 2/9$ when $y \geq 5$. Hence $k_2 \circ k_1$ is a contraction map on $[5/2, \infty)$, so the discrete equation has a unique globally attractive fixed point $\bar{w}$. Therefore, Remark 8 tells us that (10) satisfies Conditions 3-4 of our theorem, as claimed. Since

$$\mathcal{E}(k_w^o \circ k_y^o) = \left\{ \frac{1}{1+\bar{w}^2} : \bar{w} \in \mathcal{E}(k_2 \circ k_1) \right\},$$

we conclude that (9) has the unique globally attractive equilibrium

$$\left\{ \left(5 + \frac{1}{1+\bar{w}^2}, k_2\left(5 + \frac{1}{1+\bar{w}^2} \right) \right) \right\}.$$

Figure 2 below illustrates this.

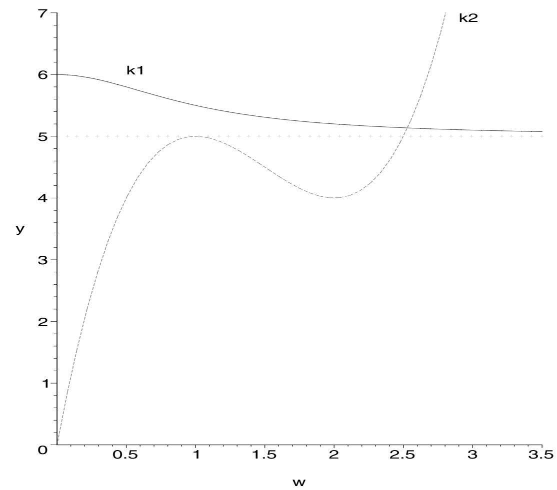

Fig. 2. Characteristics $k_1(w)$, $k_2(y)$ and $R(w)$ from Section V.

Remark 12: In the preceding example, $w_{k+1} \in (k_w \circ k_y)(w_k)$ had a unique equilibrium, but our theory applies to examples where $\mathcal{E}(k_w \circ k_y)$ has more than one element as well; see [4] for an example of this phenomenon.

VI. Conclusion

We announced a new small-gain theorem for interconnections of monotone i/o systems with set-valued i/s characteristics. This allows cases where the trajectory for any given constant input can converge to several possible equilibrium states, depending on its initial value. This extends a recent small gain theorem of Angeli and Sontag that applies to systems with singleton-valued characteristics. Our result is based on the theory of asymptotically autonomous systems, which requires that the equilibria of the subsystems contain no chains. This suggests the question of how to extend our theory to cases where the sets of equilibria of the subsystems are more general, e.g., cases where they contain chains or limit cycles. Research on these questions is in progress.

References

[1] D. Angeli, E. Sontag, *Monotone control systems*. IEEE Trans. Autom. Control **48** (2003), 1684–1698.
[2] D. Angeli, E. Sontag, *Multistability in monotone input/output systems*. Syst. & Contr. Letters **51** (2004), 185–202.
[3] D. Angeli, E. Sontag, *Interconnections of monotone systems with steady-state characteristics*. In: Optimal Control, Stabilization, and Nonsmooth Analysis. Springer-Verlag, Heidelberg, 2004, 135-154.
[4] P. De Leenheer, M. Malisoff, *A small-gain theorem for monotone systems with multi-valued input-state characteristics*. arXiv math.OC/0506508. IEEE Trans. on Autom. Control, accepted.
[5] G. Enciso, H. Smith, E. Sontag, *Non-monotone systems decomposable into monotone systems with negative feedback*. J. Diff. Eq., in press
[6] A. Rantzer, *A dual to Lyapunov's stability theorem*. Syst. & Contr. Letters **42** (2001), 161–168.
[7] H. Smith, *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems*. American Mathematical Society, Providence, RI, 1995.
[8] E. Sontag, *A remark on the converging-input converging-state property*. IEEE Trans. Autom. Control **48** (2003), 313–314.
[9] H. Thieme, *Convergence results and a Poincaré-Bendixson trichotomy for asymptotically autonomous differential equations*. J. Math. Biol. **30** (1992), 755–763.