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Measurable signal decoupling through self-bounded controlled invariants: minimal unassignable dynamics of feedforward units for pre-stabilized systems

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Abstract— **A dynamic feedforward scheme allows measurable signal decoupling to be solved independently of other problems simultaneously present in the design of an actual control system, like plant pre-stabilization, robustness with respect to uncertainties, insensitivity to inaccessible disturbances etc. The synthesis procedure, based on the properties of self-bounded controlled invariant subspaces, ensures the minimal complexity of the dynamic feedforward unit in terms of the minimal unassignable dynamics in the case of left-invertible systems and, on specific conditions, also in the case of non-left-invertible systems. The output dynamic feedback loop in charge of pre-stabilization, or, more generally, ensuring some robustness or insensitivity properties, does not affect the complexity of the dynamic feedforward unit. In fact, the particular layout where the feedback unit receives an input directly from the precompensator preserves the set of the internal unassignable eigenvalues of the minimal self-bounded controlled invariant. Hence, it maintains the unassignable dynamics of the precompensator.**

I. INTRODUCTION

Signal decoupling, i.e. the problem of making the output of a dynamic system totally insensitive to an exogenous input, has been extensively studied, particularly in the geometric approach context [1], [2]. As for decoupling of signals accessible for measurement, the necessary and sufficient condition for the structural solution was given in [3], while the necessary and sufficient condition for the solution with stability can be found in [2], where an algebraic, mixed feedback-feedforward scheme is proposed. However, the more and more sophisticated control systems and the ever increasing variety of issues to be handled (e.g., robustness with respect to parameter uncertainties, insensitivity to inaccessible and unpredictable disturbances, fault tolerance, etc) induce the designer to consider more flexible control schemes, allowing the different questions to be treated separately and intricacy of the devices aiming at satisfying each specific requirement to be minimized whenever possible [4], [5]. In this work, in particular, the measurable signal decoupling problem is solved through a dynamic feedforward scheme, while the problem of pre-stabilizing the plant (which, more generally, could also be a problem of robustness or insensitivity to disturbances) is devolved to a output dynamic feedback scheme. The synthesis of the dynamic feedforward unit is based on the properties of the minimal controlled invariant self-bounded with respect to the kernel of the output matrix satisfying the structural condition. Hence, the dynamic feedforward unit is found to have the minimal unassignable dynamics (and also the minimal dynamic order) if the plant is left-invertible with respect to the control input. The synthesis procedure is first presented for left-invertible systems and then extended to non-left-invertible systems by means of an original squaring-down technique. In the case of non-left-invertible systems, it is shown that the minimal unassignable dynamics is still guaranteed if there is no intersection between the image of the exogenous input matrix and the intersection between the maximal controlled invariant contained in the kernel of the output matrix and the image of the control input matrix. As to the output dynamic feedback loop designed to guarantee plant stability, it is shown that the proposed scheme, including a direct input from the precompensator, does not affect the complexity of the dynamic feedforward unit, since the set of the internal unassignable eigenvalues of the minimal self-bounded controlled invariant is preserved in the extended system.

The notation is assumed as in [2].

II. GEOMETRIC APPROACH BACKGROUND

The discrete-time linear system

$$x(t+1) = A\,x(t) + B\,u(t) + H\,h(t), \qquad (1)$$

$$y(t) = C\,x(t), \qquad (2)$$

is considered, where $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathbb{R}^p$, $h \in \mathbb{R}^s$, $y \in \mathbb{R}^q$ respectively denote the state, the control input, the exogenous input, the controlled output. The set of all admissible control input sequences and that of all admissible exogenous input sequences are defined as the sets $\mathcal{U}_f$ and $\mathcal{H}_f$ of all bounded sequences with values in $\mathbb{R}^p$ and $\mathbb{R}^s$, respectively. The matrices B, H, C are full-rank. The symbols $\mathcal{B}$, $\mathcal{H}$, $\mathcal{C}$ are used for $\operatorname{im} B$, $\operatorname{im} H$, $\ker C$, respectively. The notation $\mathcal{R} = \min \mathcal{J}(A, \mathcal{B})$ is used for the minimal A-invariant containing $\mathcal{B}$. The notation $\mathcal{V}^* = \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$ is used for the maximal $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$, $\mathcal{S}^* = \min \mathcal{S}(A, \mathcal{C}, \mathcal{B})$ for the minimal $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B}$, $\mathcal{R}_{\mathcal{V}^*}$ for the reachability subspace on $\mathcal{V}^*$. Let $\mathcal{V} \subseteq \mathcal{X}$ be an $(A, \mathcal{B})$-controlled invariant, F any real matrix such that $(A+BF)\,\mathcal{V} \subseteq \mathcal{V}$, and $\mathcal{R}_{\mathcal{V}} = \mathcal{V} \cap \min \mathcal{S}(A, \mathcal{V}, \mathcal{B})$. The assignable and the unassignable internal eigenvalues of $\mathcal{V}$ are respectively defined as $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}}})$ and $\sigma((A+BF)|_{\mathcal{V}/\mathcal{R}_{\mathcal{V}}})$, where $\sigma(\cdot)$ denotes the spectrum. The assignable and the

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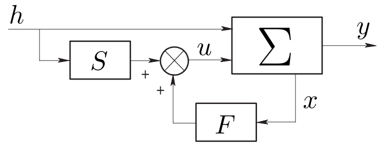

Fig. 1. Block diagram for measurable signal decoupling.

unassignable external eigenvalues of $\mathcal{V}$ are respectively defined as $\sigma((A+BF)|_{(\mathcal{V}+\mathcal{R})/\mathcal{V}})$ and $\sigma((A+BF)|_{\mathcal{X}/(\mathcal{V}+\mathcal{R})})$. Hence, $\mathcal{V}$ is an internally stabilizable $(A,\mathcal{B})$-controlled invariant iff there exists at least one real matrix F such that $(A+BF)\,\mathcal{V}\subseteq\mathcal{V}$ and $\sigma((A+BF)|_{\mathcal{V}})\subset\mathbb{C}^{\odot}$, where $\mathbb{C}^{\odot}$ denotes the set of complex numbers inside the unit circle. Likewise, $\mathcal{V}$ is an externally stabilizable $(A,\mathcal{B})$-controlled invariant iff there exists at least one real matrix F such that $(A+BF)\,\mathcal{V}\subseteq\mathcal{V}$ and $\sigma((A+BF)|_{\mathcal{X}/\mathcal{V}})\subset\mathbb{C}^{\odot}$. If (A,B) is stabilizable, any $(A,\mathcal{B})$-controlled invariant is externally stabilizable. The unassignable internal eigenvalues of $\mathcal{V}^*$ are the invariant zeros of the triple (A,B,C), i.e. $\mathcal{Z}(A,B,C)$. Let $\mathcal{V}\subseteq\mathcal{X}$ be an $(A,\mathcal{B})$-controlled invariant contained in $\mathcal{C}$, $\mathcal{V}$ is said to be self-bounded with respect to $\mathcal{C}$ if $\mathcal{V}\supseteq\mathcal{V}^*\cap\mathcal{B}$. The set of all $(A,\mathcal{B})$-controlled invariants self-bounded with respect to $\mathcal{C}$ is a non-distributive lattice with respect to $\subseteq$, $+$, $\cap$.

Internal stabilizability and self-boundedness of $(A,\mathcal{B})$-controlled invariants are notions of primary importance in the statement of the necessary and sufficient constructive condition for measurable signal decoupling with stability [2], [6], [7].

Lemma 1: Let $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B}$. If the minimal $(A,\mathcal{B}+\mathcal{H})$-controlled invariant self-bounded with respect to $\mathcal{C}$, i.e. $\mathcal{V}_m=\mathcal{V}^*\cap\min\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{H})$, is not internally stabilizable, no internally stabilizable $(A,\mathcal{B})$-controlled invariant $\mathcal{V}$ exists, which satisfies both $\mathcal{V}\subseteq\mathcal{C}$ and $\mathcal{H}\subseteq\mathcal{V}+\mathcal{B}$.

Problem 1: Refer to Fig. 1. Let Σ be ruled by (1), (2) with $x(0)=0$. Find a linear algebraic state feedback matrix F and a linear algebraic feedforward matrix S such that $\sigma(A+BF)\subset\mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t\geq 0)$, $y(t)=0$ for all $t\geq 0$.

Theorem 1: Consider the system (1), (2). Let (A,B) be stabilizable. Problem 1 is solvable iff (i) $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B}$, (ii) $\mathcal{V}_m$ is internally stabilizable.

In Theorem 1, on the assumption that (i) holds, the stabilizability condition is checked by considering $\mathcal{V}_m$, i.e. the minimal $(A,\mathcal{B})$-controlled invariant self-bounded with respect to $\mathcal{C}$ such that $\mathcal{H}\subseteq\mathcal{V}_m+\mathcal{B}$ [2]. However, the necessary and sufficient condition for measurable signal decoupling with stability is often expressed by the compact condition $\mathcal{H}\subseteq\mathcal{V}_g^*+\mathcal{B}$, where $\mathcal{V}_g^*$ is the maximal internally stabilizable $(A,\mathcal{B})$-controlled invariant contained in $\mathcal{C}$ [1]. Since both $\mathcal{V}_m$ and $\mathcal{V}_g^*$ are $(A,\mathcal{B})$-controlled invariants self-bounded with respect to $\mathcal{C}$, for any F such that $(A+BF)\mathcal{V}^*\subseteq\mathcal{V}^*$, also $(A+BF)\mathcal{V}_g^*\subseteq\mathcal{V}_g^*$ and $(A+BF)\mathcal{V}_m\subseteq\mathcal{V}_m$ hold. Moreover, if $\mathcal{V}_m$ is internally stabilizable, since $\mathcal{V}_m$ is the minimal $(A,\mathcal{B})$-controlled invariant self-bounded with respect to $\mathcal{C}$ such that $\mathcal{H}\subseteq\mathcal{V}_m+\mathcal{B}$, $\mathcal{V}_m\subseteq\mathcal{V}_g^*$ holds. The property of $(A+BF)$-invariance of both $\mathcal{V}_m$ and $\mathcal{V}_g^*$ for any F friend of $\mathcal{V}^*$ and the inclusion $\mathcal{V}_m\subseteq\mathcal{V}_g^*$ imply that, if $\mathcal{V}_m$ is internally stabilizable, the set of the internal unassignable eigenvalues of $\mathcal{V}_m$ is a subset of the set of the internal unassignable eigenvalues of $\mathcal{V}_g^*$, namely $\sigma((A+BF)|_{\mathcal{V}_m/\mathcal{R}_{\mathcal{V}^*}})\subseteq\sigma((A+BF)|_{\mathcal{V}_g^*/\mathcal{R}_{\mathcal{V}^*}})$. In the light of the inclusion above, the synthesis procedure that will be presented in Sections III, IV directly yields a dynamic unit with the minimal internal unassignable dynamics if the system is left-invertible (and even if it is not on conditions that will be specified), since this dynamics corresponds to the set of the internal unassignable eigenvalues of the subspace which is assumed as the resolvent, $\mathcal{V}_m$ in the specific case.

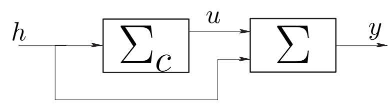

Fig. 2. Block diagram for measurable signal decoupling by dynamic feedforward.

III. MEASURABLE SIGNAL DECOUPLING BY DYNAMIC FEEDFORWARD: STABLE SYSTEMS

Throughout this work, the mixed feedback-feedforward algebraic solution shown in Fig. 1 is replaced by the dynamic feedforward solution shown in Fig. 2, where the precompesator Σ_c is defined by the quadruple (A_c,B_c,C_c,D_c). To this aim, the plant Σ is assumed to be stable. As mentioned in the Introduction, the convenience of choosing the dynamic feedforward option needs to be discussed in connection with the output dynamic feedback inner loop that will be presented in detail in Section IV. In fact, considering a dynamic feedforward unit to guarantee insensitivity of the output to measurable input signals and devolving to an output dynamic feedback inner loop the stabilization of the plant (but also the questions of insensitivity to inaccessible and unpredictable disturbances, of robustness with respect to model uncertainties and so forth) allows the designer to carry out separate synthesis procedures, with different targets, thus achieving a more complicated, but potentially better performing, overall control system (Fig. 3). As to the synthesis of the dynamic feedforward unit attaining decoupling of the measurable input, it is worth noting that the minimal unassignable dynamics of the precompensator is preserved also in the presence of the output dynamic feedback inner loop presented in Section IV, since, as it will be shown, that configuration maintains the internal unassignable eigenvalues of the resolvent subspace in the

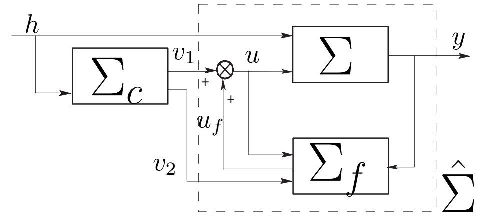

Fig. 3. Block diagram for measurable signal decoupling by dynamic feedforward with an output dynamic feedback inner loop.

state space extended to include the state of the feedback unit.

Problem 2: Refer to Fig. 2. Let Σ be ruled by (1), (2) with $x(0)=0$. Let $\sigma(A)\subset\mathbb{C}^{\odot}$. Find a linear dynamic feedforward compensator $\Sigma_c\equiv(A_c,B_c,C_c,D_c)$, such that $\sigma(A_c)\subset\mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t\geq 0)$, $y(t)=0$ for all $t\geq 0$.

On the assumption that the necessary and sufficient condition stated in Theorem 1 holds, the synthesis of the precompensator will be first considered for systems which are left-invertible with respect to the control input, i.e. $\mathcal{V}^*\cap\mathcal{B}=\{0\}$. If the system is left-invertible, the precompensator designed on the basis of $\mathcal{V}_m$ not only has the minimal internal unassignable dynamics, but it also is the precompensator of minimal dynamic order. In fact, since $\mathcal{V}^*\cap\mathcal{B}=\{0\}$, $\mathcal{V}_m$ is the minimal internally stabilizable $(A,\mathcal{B})$-controlled invariant satisfying $\mathcal{V}_m\subseteq\mathcal{C}$ and $\mathcal{H}\subseteq\mathcal{V}_m+\mathcal{B}$.

Lemma 2: Consider the system (1), (2). Let $\mathcal{V}^*\cap\mathcal{B}=\{0\}$ and $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B}$. Let F be any real matrix such that $(A+BF)\,\mathcal{V}_m\subseteq\mathcal{V}_m$. Denote by V_m a basis matrix of $\mathcal{V}_m$ and assume B as basis matrix of $\mathcal{B}$. Perform the similarity transformation $T=[\,T_1\ T_2\ T_3\,]$, with $T_1=V_m$ and $T_2=B$. The matrices A'_F, B', H', C', respectively corresponding to $A+BF$, B, H, C in the new basis, partitioned according to T have the structures

$$A'_F=\begin{bmatrix} A'_{11} & A'_{12} & A'_{13}\\ O & A'_{F22} & A'_{F23}\\ O & A'_{32} & A'_{33}\end{bmatrix},\quad B'=\begin{bmatrix} O\\ B'_{21}\\ O\end{bmatrix},$$
$$H'=\begin{bmatrix} H'_{11}\\ H'_{21}\\ O\end{bmatrix},\quad C'=\begin{bmatrix} O & C'_{12} & C'_{13}\end{bmatrix},$$

where $A'_{F2j}=A'_{2j}+B'_{21}F'_{1j}$, $j=2,3$, and $A'_{F21}=A'_{21}+B'_{21}F'_{11}$ has been set to 0 with $F'_{11}=-(B'_{21})^{-1}A'_{21}$.

Proof: The structure of B' is implied by $\mathcal{V}_m\cap\mathcal{B}=\{0\}$, which follows from $\mathcal{V}^*\cap\mathcal{B}=\{0\}$ and $\mathcal{V}_m\subseteq\mathcal{V}^*$. The structure of H' depends on $\mathcal{H}\subseteq\mathcal{V}_m+\mathcal{B}$, which is implied by $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B}$. The structure of C' depends on $\mathcal{V}_m\subseteq\mathcal{C}$. The zero submatrices in the first column of A'_F are due to $(A+BF)$-invariance of $\mathcal{V}_m$. ■

Theorem 2: Consider the system (1), (2). Let $\sigma(A)\subset\mathbb{C}^{\odot}$, $\mathcal{V}^*\cap\mathcal{B}=\{0\}$, $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B}$. Then, $\Sigma_c\equiv(A_c,B_c,C_c,D_c)$ solves Problem 2 if $A_c=A'_{11}$, $B_c=H'_{11}$, $C_c=F'_{11}$, $D_c=-H'_{21}$, where A'_{11}, H'_{11}, F'_{11}, H'_{21} are defined as in Lemma 2, with F any real matrix such that $(A+BF)\,\mathcal{V}_m\subseteq\mathcal{V}_m$ and $\sigma(A+BF)\subset\mathbb{C}^{\odot}$.

Proof: Let A'_{11}, H'_{11}, F'_{11}, H'_{21} be defined as in Lemma 2, with F any real matrix such that $(A+BF)\,\mathcal{V}_m\subseteq\mathcal{V}_m$ and $\sigma(A+BF)\subset\mathbb{C}^{\odot}$. Let Σ_c be

$$\begin{aligned} z(t+1) &= A'_{11}\,z(t)+H'_{11}\,h(t),\\ u(t) &= F'_{11}\,z(t)-H'_{21}\,h(t),\end{aligned}$$

with $z(0)=0$. First, it is shown that, for any admissible $h(t)$ $(t\geq 0)$, the corresponding $x(t)$ $(t\geq 0)$, starting from $x(0)=0$, lies on $\mathcal{V}_m$, since $x(t)=V_m\,z(t)$ for all $t\geq 0$. In fact, $x(0)=V_m\,z(0)$, due to the assumptions on the initial conditions. Moreover, for any $t\geq 0$, the assumption $x(t)=V_m\,z(t)$ implies

$$\begin{aligned} x(t+1) &= A\,x(t)+BF'_{11}\,z(t)+V_mH'_{11}\,h(t)\\ &= (A+BF)\,V_m\,z(t)+V_mH'_{11}\,h(t)\\ &= V_mA'_{11}\,z(t)+V_mH'_{11}\,h(t)\\ &= V_m\,z(t+1),\end{aligned}$$

where $H=V_mH'_{11}+BH'_{21}$, $F'_{11}=FV_m$, $(A+BF)\,V_m=V_mA'_{11}$ have been considered. Then, stability of Σ_c is implied by $\sigma(A_c)=\sigma(A'_{11})\subseteq\sigma(A+BF)\subset\mathbb{C}^{\odot}$, due to the block-diagonal structure of A'_F. ■

If the plant is non-left-invertible with respect to the control input, the synthesis of the dynamic feedforward unit is more intricate, since it requires that a squaring-down technique is applied to the original system and that the synthesis procedure previously described is performed on the new, left-invertible, system. As for the evaluation of the complexity of the dynamic feedforward compensator thus obtained with respect to the original problem, the minimal internal unassignable dynamics is guaranteed if the original system satisfies the condition $\mathcal{H}\cap(\mathcal{V}^*\cap\mathcal{B})=\{0\}$. In fact, as it will be shown in the following, the squaring-down technique herein proposed preserves $\mathcal{V}^*$ as the maximal controlled invariant contained in the null space of the output and suppresses those column vectors of B whose image belongs to $\mathcal{V}^*$. The following Lemma 3 and Theorem 3 introduce the squaring-down technique. Corollary 1 relates the invariant zeros of the new, left-invertible, triple to the invariant zeros of the original triple and the eigenvalues of the controllability subspace $\mathcal{R}_{\mathcal{V}^*}$ of the original triple. Finally, Theorems 4 and 5 enable the dynamic feedforward compensator for the original system to be retrieved from that designed for the new system.

Lemma 3: Consider the system (1), (2). Let $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B}$. Let F be any real matrix such that $(A+BF)\,\mathcal{V}^*\subseteq\mathcal{V}^*$. Perform the similarity transformations $T=[\,T_1\ T_2\ T_3\ T_4\,]$, with $\operatorname{im}T_1=\mathcal{R}_{\mathcal{V}^*}$, $\operatorname{im}\,[\,T_1\ T_2\,]=\mathcal{V}^*$, $\operatorname{im}\,[\,T_1\ T_3\,]=\mathcal{S}^*$, and $U=[\,U_1\ U_2\,]$, with $\operatorname{im}U_1=B^{-1}\mathcal{V}^*$, $\operatorname{im}U_2=\left(B^{-1}\mathcal{V}^*\right)^{\perp}$. The matrices A', B', H', C', respectively corresponding to A, B, H, C in the new bases, partitioned according to T and U have the structures

$$A'=\begin{bmatrix} A'_{11} & A'_{12} & A'_{13} & A'_{14}\\ O & A'_{22} & A'_{23} & A'_{24}\\ A'_{31} & A'_{32} & A'_{33} & A'_{34}\\ O & O & A'_{43} & A'_{44}\end{bmatrix},\qquad(3)$$

$$B'=\begin{bmatrix} B'_{11} & B'_{12}\\ O & O\\ O & B'_{32}\\ O & O\end{bmatrix},\quad H'=\begin{bmatrix} H'_{11}\\ H'_{21}\\ H'_{31}\\ O\end{bmatrix},\qquad(4)$$

$$C'=\begin{bmatrix} O & O & C'_{13} & C'_{14}\end{bmatrix}.\qquad(5)$$

The matrix A'_F, corresponding to $A+BF$, partitioned according to T has structure

$$A'_F=\begin{bmatrix} A'_{F11} & A'_{F12} & A'_{F13} & A'_{F14}\\ O & A'_{22} & A'_{23} & A'_{24}\\ O & O & A'_{F33} & A'_{F34}\\ O & O & A'_{43} & A'_{44}\end{bmatrix},\qquad(6)$$

where $A'_{F1j} = A'_{1j} + B'_{11}F'_{1j} + B'_{12}F'_{2j}$, with $j = 1,2,3,4$, $A'_{F3j} = A'_{3j} + B'_{32}F'_{2j}$, with $j = 3,4$, and where $A'_{F3j} = A'_{3j} + B'_{32}F'_{2j}$, with $j = 1,2$, have been set to zero by imposing $F'_{2j} = -(B'_{32})^{+}A'_{3j}$, with $j = 1,2$, respectively.

Proof: The structure of B' is due to $\mathcal{B} \subseteq \mathcal{S}^*$ and $\mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{R}_{\mathcal{V}^*}$. The structure of H' is implied by $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. The structure of C' is implied by $\mathcal{V}^* \subseteq \mathcal{C}$. The zero submatrices in the fourth row of A' are due to the structure of B' and to $(A,\mathcal{B})$-controlled invariance of $\mathcal{V}^*$. The zero submatrix in the second row of A' is due to the structure of B' and to $(A,\mathcal{B})$-controlled invariance of $\mathcal{R}_{\mathcal{V}^*}$. The zero submatrices in the third row of A'_F are due to $(A+BF)$-invariance of $\mathcal{V}^*$. ■

Theorem 3: Consider the system (1), (2). Let (A,B,C) be non-left-invertible. Let F be any real matrix such that $(A+BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$. Let $\left(B^{-1}\mathcal{V}^*\right)^{\perp} \neq \{0\}$ and U_2 be a basis matrix of $\left(B^{-1}\mathcal{V}^*\right)^{\perp}$. Set $\tilde{A} = A + BF$, $\tilde{B} = B\,U_2$. Then, $\max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\mathcal{C}) = \mathcal{V}^*$ and $(\tilde{A},\tilde{B},C)$ is left-invertible.

Proof: Consider the triple $(A+BF,B,C)$ and perform the similarity tranformations T,U defined in Lemma 3. First, note that the matrix $\tilde{B}' = B'U'_2$, corresponding to $\tilde{B} = BU_2$ in the new bases, matches the second column of B', since $U'_2 = U^{-1}U_2 = [\,O\ I\,]^{\top}$. Also note that, in the new basis, $\mathcal{V}^* = \operatorname{im}[\,T'_1\ T'_2\,]$, with $T'_1 = T^{-1}T_1 = [\,I\ O\ O\ O\,]^{\top}$ and $T'_2 = T^{-1}T_2 = [\,O\ I\ O\ O\,]^{\top}$. Then, $\mathcal{V}^* \cap \tilde{\mathcal{B}} = \{0\}$ is derived from the comparison of the basis matrices of $\mathcal{V}^*$ and $\tilde{\mathcal{B}}$ in the new coordinates, since B'_{32} is full-rank. On the other hand, $\mathcal{V}^*$, which is the maximal $(A,\mathcal{B})$-controlled invariant contained in $\mathcal{C}$, is also the maximal $(A+BF)$-invariant contained in $\mathcal{C}$. Hence, due to the particular structure of $\tilde{\mathcal{B}}$, $\mathcal{V}^*$ is also the maximal $(A+BF,\tilde{\mathcal{B}})$-controlled invariant contained in $\mathcal{C}$, i.e. $\mathcal{V}^* = \max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\mathcal{C})$. ■

Corollary 1: Consider the system (1), (2). Let (A,B,C) be non-left-invertible. Let F be any real matrix such that $(A+BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$. Let $\left(B^{-1}\mathcal{V}^*\right)^{\perp} \neq \{0\}$ and let U_2 be a basis matrix of $\left(B^{-1}\mathcal{V}^*\right)^{\perp}$. Set $\tilde{A} = A + BF$ and $\tilde{B} = B\,U_2$. Then, $\mathcal{Z}(\tilde{A},\tilde{B},C) = \mathcal{Z}(A,B,C) \uplus \sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}})$ holds.

Proof: By virtue of Theorem 3 $\max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\mathcal{C}) = \mathcal{V}^*$ and all the internal eigenvalues of $\max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\mathcal{C})$ are unassignable, since the triple $(\tilde{A},\tilde{B},C)$ is left-invertible. ■

Theorem 4: Consider the system (1), (2). Let (A,B,C) be non-left-invertible. Let F be any real matrix such that $(A+BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$. Let U_2 be a basis matrix of $\left(B^{-1}\mathcal{V}^*\right)^{\perp} \neq \{0\}$. Let the system $\tilde{\Sigma}$ be ruled by

$$\tilde{x}(t+1) = \tilde{A}\,\tilde{x}(t) + \tilde{B}\,\tilde{u}(t) + H\,h(t), \tag{7}$$

$$\tilde{y}(t) = C\,\tilde{x}(t), \tag{8}$$

where $\tilde{A} = A + BF$ and $\tilde{B} = B\,U_2$. If Problem 1 stated for system (1), (2) is solvable, then Problem 1 stated for system (7), (8) is solvable.

Proof: Since $\mathcal{V}^* = \max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\mathcal{C})$ by virtue of Theorem 3 and $\mathcal{V}^* + \mathcal{B} = \mathcal{V}^* + \tilde{\mathcal{B}}$ by definition of $\tilde{B}$,

$$\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B} \iff \mathcal{H} \subseteq \max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\mathcal{C}) + \tilde{\mathcal{B}} \tag{9}$$

holds. Let the inclusions in (9) hold. The subspace $\mathcal{V}_m$, which is the minimal $(A,\mathcal{B})$-controlled invariant self-bounded with respect to $\mathcal{C}$ such that $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{B}$, is also the minimal $(A+BF)$-invariant contained in $\mathcal{C}$ and containing $\mathcal{V}^* \cap \mathcal{B}$ such that $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{B}$. Hence, $\mathcal{V}_m$ is the minimal $(A+BF,\tilde{\mathcal{B}})$-controlled invariant contained in $\mathcal{C}$ and containing $\mathcal{V}^* \cap \mathcal{B}$ such that $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{B}$. By definition of $\tilde{\mathcal{B}}$ and by virtue of the inclusions $\mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{V}_m$, it follows that $\mathcal{V}_m + \mathcal{B} = \mathcal{V}_m + \tilde{\mathcal{B}}$, which, in turn, implies that $\mathcal{V}_m$ is the minimal $(\tilde{A},\tilde{\mathcal{B}})$-controlled invariant contained in $\mathcal{C}$ and containing $\mathcal{V}^* \cap \mathcal{B}$ such that $\mathcal{H} \subseteq \mathcal{V}_m + \tilde{\mathcal{B}}$. On the other hand, $\tilde{\mathcal{V}}_m = \max\mathcal{V}(\tilde{A},\tilde{\mathcal{B}},\mathcal{C}) \cap \min\mathcal{S}(\tilde{A},\mathcal{C},\tilde{\mathcal{B}}+\mathcal{H})$ is the minimal $(\tilde{A},\tilde{\mathcal{B}})$-controlled invariant contained in $\mathcal{C}$ such that $\mathcal{H} \subseteq \tilde{\mathcal{V}}_m + \tilde{\mathcal{B}}$. Hence, $\tilde{\mathcal{V}}_m \subseteq \mathcal{V}_m$ holds. Due to this latter inclusion, internal stabilizability of $\mathcal{V}_m$ implies that of $\tilde{\mathcal{V}}_m$. Finally, the inclusion on the right-hand side of (9) and the internal stabilizability of $\tilde{\mathcal{V}}_m$ imply solvability of Problem 1 stated for system (7), (8), by virtue of Theorem 1. ■

Theorem 5: On the assumptions of Theorem 4, let $\tilde{\Sigma}_c \equiv (\tilde{A}_c,\tilde{B}_c,\tilde{C}_c,\tilde{D}_c)$ solve Problem 2 stated for system (7), (8). Then, Σ_c solving Problem 2 stated for system (1), (2) is defined by (A_c,B_c,C_c,D_c), where $A_c = \tilde{A}_c$, $B_c = \tilde{B}_c$, $C_c = F\,V_m + U_2\,\tilde{C}_c$, $D_c = U_2\,\tilde{D}_c$, with V_m denoting a basis matrix of $\mathcal{V}_m$, F any real matrix such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $\sigma((A+BF)|_{\mathcal{V}_m}) \subset \mathbb{C}^{\odot}$, and U_2 a basis matrix of $\left(B^{-1}\mathcal{V}^*\right)^{\perp} \neq \{0\}$.

Proof: The state equations of the feedforward connection of Σ_c and Σ shown in Fig. 2 are

$$\begin{cases} x(t+1) = A\,x(t) + BC_c\,z(t) + (BD_c + H)\,h(t), \\ z(t+1) = A_c\,z(t) + B_c\,h(t), \end{cases}$$

with $x(0) = 0$, $z(0) = 0$. The state equations of the corresponding feedforward connection of $\tilde{\Sigma}_c$ and $\tilde{\Sigma}$ are

$$\begin{cases} \tilde{x}(t+1) = (A+BF)\,\tilde{x}(t) + BU_2\tilde{C}_c\,\tilde{z}(t) \\ \qquad\qquad + (BU_2\tilde{D}_c + H)\,h(t), \\ \tilde{z}(t+1) = \tilde{A}_c\,\tilde{z}(t) + \tilde{B}_c\,h(t), \end{cases}$$

with the initial conditions $\tilde{x}(0) = 0$, $\tilde{z}(0) = 0$. The thesis follows by imposing $x(t) = \tilde{x}(t)$, $z(t) = \tilde{z}(t)$ for all $t \geq 0$. ■

IV. MEASURABLE SIGNAL DECOUPLING BY DYNAMIC FEEDFORWARD: PRE-STABILIZED SYSTEMS

The assumption of stability of the system, introduced in order to replace the mixed feedback-feedforward algebraic solution with the dynamic feedforward solution, is not restrictive with respect to those of stabilizability of (A,B) and detectability of (A,C) which are usually considered. On these assumptions, the given system can be pre-stabilized by output dynamic feedback. In this section it is shown that the scheme introduced in Fig. 3 does not affect the complexity of the dynamic feedforward unit achieving measurable signal decoupling, since the set of the invariant zeros of the original system coincides with that of the invariant zeros of the system extended to include the output dynamic feedback unit. On the assumptions of stabilizability and detectability of the original system, stability of the overall system is guaranteed

$$
\hat{A}'_F = \left[\begin{array}{cccc|cccc}
\hat{A}'_{F11} & \hat{A}'_{F12} & \hat{A}'_{F13} & \hat{A}'_{F14} & O & O & O & O \\
O & A'_{22} & A'_{23} & A'_{24} & O & O & O & O \\
O & O & \hat{A}'_{F33} & \hat{A}'_{F34} & O & O & O & O \\
O & O & A'_{43} & A'_{44} & O & O & O & O \\ \hline
O & \hat{A}'_{F52} & \hat{A}'_{F53} & \hat{A}'_{F54} & \hat{A}'_{F55} & \hat{A}'_{F56} & \hat{A}'_{F57} & \hat{A}'_{F58} \\
O & \hat{A}'_{F62} & \hat{A}'_{F63} & \hat{A}'_{F64} & \hat{A}'_{F65} & \hat{A}'_{F66} & \hat{A}'_{F67} & \hat{A}'_{F68} \\
O & \hat{A}'_{F72} & \hat{A}'_{F73} & \hat{A}'_{F74} & \hat{A}'_{F75} & \hat{A}'_{F76} & \hat{A}'_{F77} & \hat{A}'_{F78} \\
O & \hat{A}'_{F82} & \hat{A}'_{F83} & \hat{A}'_{F84} & \hat{A}'_{F85} & \hat{A}'_{F86} & \hat{A}'_{F87} & \hat{A}'_{F88}
\end{array}\right], \tag{18}
$$

by virtue of the well-known separation property, recalled without proof in Theorem 6. Instead, the result concerning the invariant zeros is proved through Theorems 7 and 8.

Refer to Fig. 3. Consider the system Σ_f, ruled by

$$
\begin{aligned}
w(t+1) &= (A+GC)\,w(t) + B\,u(t) \\
&\quad - G\,y(t) + v_2(t), && (10) \\
u_F(t) &= F\,w(t), && (11)
\end{aligned}
$$

where $w \in \mathbb{R}^n$, $v_2 \in \mathbb{R}^n$, $u_F \in \mathbb{R}^p$ respectively denote the state, the control input, the measurable output. Also consider the overall system $\hat{\Sigma}$ obtained by connecting Σ_f to Σ, ruled by (1), (2) with the additional control input $v_1 \in \mathbb{R}^n$, so that $u(t) = u_F(t) + v_1(t)$. Let the state, the control input, the measurable output of $\hat{\Sigma}$ be $\hat{x}(t) = \left[x(t)^\top \; w(t)^\top\right]^\top$, $\hat{v}(t) = \left[v_1(t)^\top \; v_2(t)^\top\right]^\top$, $\hat{y}(t) = y(t)$, respectively. Then, $\hat{\Sigma}$ is ruled by

$$
\begin{aligned}
\hat{x}(t+1) &= \hat{A}\,\hat{x}(t) + \hat{B}\,\hat{v}(t) + \hat{H}\,h(t), && (12) \\
\hat{y}(t) &= \hat{C}\,\hat{x}(t), && (13)
\end{aligned}
$$

with

$$
\hat{A} = \begin{bmatrix} A & BF \\ -GC & A+BF+GC \end{bmatrix}, \; \hat{B} = \begin{bmatrix} B & O \\ B & I \end{bmatrix}, \tag{14}
$$

$$
\hat{H} = \begin{bmatrix} H \\ O \end{bmatrix}, \; \hat{C} = \begin{bmatrix} C & O \end{bmatrix}. \tag{15}
$$

Theorem 6 (Separation Property): Consider Σ ruled by (1) ,(2), Σ_f by (10) ,(11), $\hat{\Sigma}$ by (12) ,(13) with (14) ,(15). Then, $\sigma(\hat{A}) = \sigma(A+BF) \uplus \sigma(A+GC)$.

Corollary 2: Consider Σ ruled by (1) ,(2), Σ_f by (10) ,(11), $\hat{\Sigma}$ by (12) ,(13) with (14) ,(15). Let (A, B) be stabilizable and (A, C) detectable. Then, there exist real matrices F, G such that $\sigma(\hat{A}) \subset \mathbb{C}^{\odot}$.

Theorem 7: Consider Σ ruled by (1) ,(2), Σ_f by (10) ,(11), $\hat{\Sigma}$ by (12) ,(13) with (14) ,(15). Let V^*, R_{V^*} be basis matrices of $\mathcal{V}^* = \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$, $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B})$, respectively. Then, basis matrices of $\hat{\mathcal{V}}^* = \max \mathcal{V}(\hat{A}, \hat{\mathcal{B}}, \hat{\mathcal{C}})$ and $\mathcal{R}_{\hat{\mathcal{V}}^*} = \hat{\mathcal{V}}^* \cap \min \mathcal{S}(\hat{A}, \hat{\mathcal{C}}, \hat{\mathcal{B}})$ respectively are

$$
\hat{V}^* = \begin{bmatrix} V^* & O \\ O & I \end{bmatrix}, \quad R_{\hat{V}^*} = \begin{bmatrix} R_{V^*} & O \\ O & I \end{bmatrix}. \tag{16}
$$

Proof: Let K be a basis matrix of $\mathcal{C}$. Due to the structure of $\hat{C}$, a basis matrix of $\hat{\mathcal{C}}$ is $\hat{K} = \begin{bmatrix} K & O \\ O & I \end{bmatrix}$. Then, the thesis follows from $\hat{\mathcal{R}} = \min \mathcal{J}(\hat{A}, \hat{\mathcal{B}}) \supseteq \mathrm{im}\,[O\;I]^\top$, implied by the structure of $\hat{B}$. ■

Theorem 8: Consider Σ ruled by (1) ,(2), Σ_f by (10) ,(11), $\hat{\Sigma}$ by (12) ,(13) with (14) ,(15). Then, $\mathcal{Z}(\hat{A}, \hat{B}, \hat{C}) = \mathcal{Z}(A, B, C)$.

Proof: Consider Σ, ruled by (1) ,(2). Let F be any real matrix s.t. $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$. Perform the similarity transformations T, U as in Lemma 3. In the new coordinates, A', B', C', A'_F have the structures in (3), (4), (5), (6). Moreover, $\mathcal{V}^* = \mathrm{im}\, V'^* = \mathrm{im}\,[T'_1 \; T'_2]$, $T'_1 = T^{-1} T_1 = [I\;O\;O\;O]^\top$, $T'_2 = T^{-1} T_2 = [O\;I\;O\;O]^\top$ and $(A' + B'F')V'^* = V'^* X$ holds with $X = \begin{bmatrix} A'_{F11} & A'_{F12} \\ O & A'_{22} \end{bmatrix}$, which implies $\mathcal{Z}(A, B, C) = \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \sigma(A'_{22})$. Then, consider $\hat{\Sigma}$, defined according to (12) ,(13) , with (14) ,(15) in the new bases. The matrices $\hat{A}'$, $\hat{B}'$, $\hat{C}'$ have the structures

$$
\hat{A}' = \left[\begin{array}{cccc|cccc}
A'_{11} & A'_{12} & A'_{13} & A'_{14} & \hat{A}'_{15} & \hat{A}'_{16} & \hat{A}'_{17} & \hat{A}'_{18} \\
O & A'_{22} & A'_{23} & A'_{24} & O & O & O & O \\
A'_{31} & A'_{32} & A'_{33} & A'_{34} & \hat{A}'_{35} & \hat{A}'_{36} & \hat{A}'_{37} & \hat{A}'_{38} \\
O & O & A'_{43} & A'_{44} & O & O & O & O \\ \hline
O & O & \hat{A}'_{53} & \hat{A}'_{54} & \hat{A}'_{55} & \hat{A}'_{56} & \hat{A}'_{57} & \hat{A}'_{58} \\
O & O & \hat{A}'_{63} & \hat{A}'_{64} & O & \hat{A}'_{66} & \hat{A}'_{67} & \hat{A}'_{68} \\
O & O & \hat{A}'_{73} & \hat{A}'_{74} & \hat{A}'_{75} & \hat{A}'_{76} & \hat{A}'_{77} & \hat{A}'_{78} \\
O & O & \hat{A}'_{83} & \hat{A}'_{84} & O & O & \hat{A}'_{87} & \hat{A}'_{88}
\end{array}\right],
$$

$$
\hat{B}' = \left[\begin{array}{cc|cccc}
B'_{11} & B'_{12} & O & O & O & O \\
O & O & O & O & O & O \\
O & B'_{32} & O & O & O & O \\
O & O & O & O & O & O \\ \hline
B'_{11} & B'_{12} & I & O & O & O \\
O & O & O & I & O & O \\
O & B'_{32} & O & O & I & O \\
O & O & O & O & O & I
\end{array}\right],
$$

$$
\hat{C}' = \left[\begin{array}{cccc|cccc} O & O & C'_{13} & C'_{14} & O & O & O & O \end{array}\right],
$$

where the relations

$$
\begin{aligned}
\hat{A}'_{1j} &= B'_{11} F'_{1\bar{j}} + B'_{12} F'_{2\bar{j}} && \text{if } j = 5, 6, 7, 8, \\
\hat{A}'_{3j} &= B'_{32} F'_{2\bar{j}} && \text{if } j = 5, 6, 7, 8, \\
\hat{A}'_{ij} &= -G'_{\bar{i}1} C'_{1j} && \text{if } i = 5, 6, 7, 8, \\
& && \quad \text{and } j = 3, 4, \\
\hat{A}'_{5j} &= A'_{1\bar{j}} + B'_{11} F'_{1\bar{j}} + B'_{12} F'_{2\bar{j}} && \text{if } j = 5, 6,
\end{aligned}
$$

$$\begin{aligned}
\hat{A}'_{5j} &= A'_{1\bar{j}} + B'_{11}F'_{1\bar{j}} + B'_{12}F'_{2\bar{j}} \\
&\quad + G'_{11}C'_{1\bar{j}} && \text{if } j=7,8,\\
\hat{A}'_{6j} &= A'_{2\bar{j}} && \text{if } j=6,\\
\hat{A}'_{6j} &= A'_{2\bar{j}} + G'_{21}C_{1\bar{j}} && \text{if } j=7,8,\\
\hat{A}'_{7j} &= A'_{3\bar{j}} + B'_{32}F'_{2\bar{j}} && \text{if } j=5,6,\\
\hat{A}'_{7j} &= A'_{3\bar{j}} + B'_{32}F'_{2\bar{j}} + G'_{31}C'_{1\bar{j}} && \text{if } j=7,8,\\
\hat{A}'_{8j} &= A'_{4\bar{j}} + G'_{41}C'_{1\bar{j}} && \text{if } j=7,8,
\end{aligned}$$

hold, with $\bar{i}=i-4$ $(i=5,6,7,8)$ and $\bar{j}=j-4$ $(j=5,6,7,8)$. Let $\hat{F}$ be any real matrix such that $(\hat{A}+\hat{B}\hat{F})\hat{\mathcal{V}}^* \subseteq \hat{\mathcal{V}}^*$. The matrix $\hat{A}'_F = \hat{A}' + \hat{B}'\hat{F}'$ has the structure shown in (18), where the relations

$$\begin{aligned}
\hat{A}'_{F1j} &= A'_{1j} + B'_{11}\hat{F}'_{1j} \\
&\quad + B'_{12}\hat{F}'_{2j} && \text{if } j=1,2,3,4,\\
\hat{A}'_{F3j} &= A'_{3j} + B'_{32}\hat{F}'_{2j} && \text{if } j=3,4,\\
\hat{A}'_{F5j} &= B'_{11}\hat{F}'_{1j} + B'_{12}\hat{F}'_{2j} + \hat{F}'_{3j} && \text{if } j=2,\\
\hat{A}'_{F5j} &= B'_{11}\hat{F}'_{1j} + B'_{12}\hat{F}'_{2j} + \hat{F}'_{3j} \\
&\quad - G'_{11}C'_{1j} && \text{if } j=3,4,\\
\hat{A}'_{Fij} &= \hat{F}'_{\tilde{i}j} && \text{if } i=6,8 \text{ and } j=2,5,\\
\hat{A}'_{Fij} &= \hat{F}'_{\tilde{i}j} - G'_{\tilde{i}1}C'_{1j} && \text{if } i=6,8 \text{ and } j=3,4,\\
\hat{A}'_{F7j} &= B'_{32}\hat{F}'_{2j} + \hat{F}'_{5j} && \text{if } j=2,\\
\hat{A}'_{F7j} &= B'_{32}\hat{F}'_{2j} + \hat{F}'_{5j} - G'_{31}C'_{1j} && \text{if } j=3,4,\\
\hat{A}'_{Fij} &= A'_{\bar{i}\bar{j}} + \hat{F}'_{\tilde{i}j} && \text{if } i=5,7 \text{ and } j=5,6,\\
\hat{A}'_{Fij} &= A'_{\bar{i}\bar{j}} + \hat{F}'_{\tilde{i}j} + G'_{\bar{i}1}C'_{1\bar{j}} && \text{if } i=5,7 \text{ and } j=7,8,\\
\hat{A}'_{Fij} &= A'_{\bar{i}\bar{j}} + \hat{F}'_{\tilde{i}j} && \text{if } i=6 \text{ and } j=6,\\
\hat{A}'_{Fij} &= \hat{F}'_{\tilde{i}j} && \text{if } i=8 \text{ and } j=6,\\
\hat{A}'_{Fij} &= A'_{\bar{i}\bar{j}} + \hat{F}'_{\tilde{i}j} + G'_{\bar{i}1}C'_{1\bar{j}} && \text{if } i=6,8 \text{ and } j=7,8,
\end{aligned}$$

hold with $\tilde{i}=i-2$ $(i=6,8)$, $\bar{i}=i-4$ $(i=5,6,7,8)$, $\bar{j}=j-4$ $(j=5,6,7,8)$ and where the entries

$$\begin{aligned}
\hat{A}'_{F3j} &= A'_{3j} + B'_{32}\hat{F}'_{2j} && \text{if } j=1,2,\\
\hat{A}'_{F5j} &= B'_{11}\hat{F}'_{1j} + B'_{12}\hat{F}'_{2j} + \hat{F}'_{3j} && \text{if } j=1,\\
\hat{A}'_{Fij} &= \hat{F}'_{\tilde{i}j} && \text{if } i=6,8 \text{ and } j=1,\\
\hat{A}'_{F7j} &= B'_{32}\hat{F}'_{2j} + \hat{F}'_{5j} && \text{if } j=1,\\
\hat{A}'_{F1j} &= B'_{11}(F'_{1\bar{j}} + \hat{F}'_{1j}) \\
&\quad + B'_{12}(F'_{2\bar{j}} + \hat{F}'_{2j}) && \text{if } j=5,6,7,8,\\
\hat{A}'_{F3j} &= B'_{32}(F'_{2\bar{j}} + \hat{F}'_{2j}) && \text{if } j=5,6,7,8,
\end{aligned}$$

have been set to zero by respectively imposing

$$\begin{aligned}
\hat{F}'_{2j} &= -(B'_{32})^{+}A'_{3j} && \text{if } j=1,2,\\
\hat{F}'_{3j} &= -B'_{11}\hat{F}'_{1j} - B'_{12}\hat{F}'_{2j} && \text{if } j=1,\\
\hat{F}'_{\tilde{i}j} &= O && \text{if } i=6,8 \text{ and } j=1,\\
\hat{F}'_{5j} &= -B'_{32}\hat{F}'_{2j} && \text{if } j=1,\\
\hat{F}'_{1j} &= F'_{1\bar{j}} && \text{if } j=5,6,7,8,\\
\hat{F}'_{2j} &= F'_{2\bar{j}} && \text{if } j=5,6,7,8.
\end{aligned}$$

According to Theorem 7, $\hat{\mathcal{V}}^* = \operatorname{im}\hat{V}'^* = \operatorname{im}[\,\hat{T}'_1\ \hat{T}'_2\ \hat{T}'_3\,]$ and $\mathcal{R}_{\hat{\mathcal{V}}^*} = \operatorname{im}R'_{\hat{\mathcal{V}}^*} = \operatorname{im}[\,\hat{T}'_1\ \hat{T}'_3\,]$, where $\hat{T}'_1 = [\,T'^{\top}_1 \mid O\,]^{\top}$, $\hat{T}'_2 = [\,T'^{\top}_2 \mid O\,]^{\top}$, and $\hat{T}'_3 = [\,O \mid I\,]^{\top}$. Simple algebraic computations show that $(\hat{A}' + \hat{B}'\hat{F}')\hat{V}'^* = \hat{V}'^*\hat{X}$ holds with

$$\hat{X} = \left[\begin{array}{cc|cccc}
\hat{A}'_{F11} & \hat{A}'_{F12} & O & O & O & O \\
O & A'_{22} & O & O & O & O \\
\hline
O & \hat{A}'_{F52} & \hat{A}'_{F55} & \hat{A}'_{F56} & \hat{A}'_{F57} & \hat{A}'_{F58} \\
O & \hat{A}'_{F62} & \hat{A}'_{F65} & \hat{A}'_{F66} & \hat{A}'_{F67} & \hat{A}'_{F68} \\
O & \hat{A}'_{F72} & \hat{A}'_{F75} & \hat{A}'_{F76} & \hat{A}'_{F77} & \hat{A}'_{F78} \\
O & \hat{A}'_{F82} & \hat{A}'_{F85} & \hat{A}'_{F86} & \hat{A}'_{F87} & \hat{A}'_{F88}
\end{array}\right].$$

Hence, $\mathcal{Z}(\hat{A},\hat{B},\hat{C}) = \sigma((\hat{A}+\hat{B}\hat{F})|_{\hat{\mathcal{V}}^*/\mathcal{R}_{\hat{\mathcal{V}}^*}}) = \sigma(A'_{22})$. ■

Corollary 3: Consider the system Σ, described by (1),(2), the dynamic feedback unit Σ_f, described by (10),(11), and the overall system $\hat{\Sigma}$, described by (12),(13) with (14),(15). Then, $\hat{\Sigma}$ is minimum-phase iff Σ is minimum-phase.

Proof: By virtue of Theorem 8, $\mathcal{Z}(\hat{A},\hat{B},\hat{C}) \subset \mathbb{C}^{\odot}$ iff $\mathcal{Z}(A,B,C) \subset \mathbb{C}^{\odot}$. ■

In the light of the previous results, the dynamic feedforward unit Σ_c must be designed on the basis of the original system Σ according to the procedure detailed in Section III, despite of possible instability of Σ. Then, the stability of the overall system is guaranteed by connecting Σ_c to the stabilized system $\hat{\Sigma}$ with $v_2(t) = -B\,v_1(t) - GCV_m\,z(t)$, where $v_1(t)$ and $z(t)$ respectively denote the output and the state of Σ_c. This connection ensures that the state $w(t)$ of Σ_f, starting from $w(0)=0$, is identically zero in ideal conditions, while any possible perturbation is managed by feedback.

V. CONCLUSIONS

A dynamic feedforward scheme based on the properties of the minimal self-bounded controlled invariant guarantees measurable signal decoupling with the minimal complexity of the dynamic unit for left-invertible systems (and also for non-left-invertible systems on some specified conditions). An output dynamic feedback with a direct input from the precompensator stabilizes the plant without affecting the complexity of the feedforward unit.

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