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Some Martingales from a Fractional Brownian Motion and Applications*

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Abstract— **In this paper, some continuous martingales are constructed from a fractional Brownian motion with the Hurst parameter in the interval $(1/2,1)$, and some applications are made. These processes are obtained using a stochastic calculus for a fractional Brownian motion. Square integrable, continuous martingales are exhibited as stochastic integrals with respect to a fractional Brownian motion and the associated increasing processes are given. These martingales are used to construct Radon-Nikodym derivatives (likelihood functions) for some measures that are absolutely continuous with respect to the measure of a fractional Brownian motion. A Radon-Nikodym derivative is used to relate a mutual information between a stochastic signal and this signal plus a fractional Gaussian noise to an estimation error.**

I. INTRODUCTION

Martingales from a Brownian motion have been important in the solution of many problems in stochastic system theory. These problems include estimation, filtering and mutual information. For the modeling of a physical phenomenon by a stochastic system, it has been demonstrated empirically in many cases that a Brownian motion is inadequate to capture some important properties of the phenomenon, and that a fractional Brownian motion with the Hurst parameter H in the interval $(1/2,1)$ is more appropriate. A stochastic calculus has been developed for these fractional Brownian motions (e.g. [1], [2], [3]), but more applications are required to demonstrate its usefulness. In this paper, some developments are described for a fractional Brownian motion that have been important for a Brownian motion. A family of continuous square integrable martingales that are obtained by stochastic integration with respect to a fractional Brownian motion (with $H \in (1/2,1)$) are given as well as their associated increasing processes. These martingales are used to construct nonnegative (local) martingales that integrate to one, or, equivalently, Radon-Nikodym derivatives. A special case of these Radon-Nikodym derivatives is used to relate the mutual information between a stochastic signal and this signal plus a fractional Gaussian noise to an estimation error. This mutual information result generalizes a result for a similar problem with white Gaussian noise [4], [5] which is related to a result in [6].

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II. PRELIMINARIES

A standard fractional Brownian motion $(B^H(t), t \geq 0)$, with the Hurst parameter $H \in (0,1)$ is a Gaussian process with continuous sample paths such that $\mathbb{E}[B^H(t)] = 0$ and

$$\mathbb{E}\left[B^H(s)B^H(t)\right] = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right] \tag{1}$$

for all $s,t \in \mathbb{R}_+$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space for $(B^H(t), t \geq 0)$ for a fixed $H \in (0,1)$ where $\Omega = C(\mathbb{R}_+, \mathbb{R})$ with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$, $\mathscr{F}$ is the P-completion of the associated Borel σ-algebra and P is the Gaussian measure. Let $\mathscr{F}_t$ be the P-completion of $\sigma(B^H(u), 0 \leq u \leq t)$ for $t \in \mathbb{R}_+$. For $H \in (1/2,1)$, the covariance function (1) can be computed as

$$\mathbb{E}\left[B^H(s)B^H(t)\right] = \int_0^t \int_0^s \phi_H(u-v)\,du\,dv \tag{2}$$

where

$$\phi_H(u) = H(2H-1)|u|^{2H-2}. \tag{3}$$

For $H \in (1/2,1)$, an important function for the construction of the martingales given here, as well as for some other applications of a fractional Brownian motion, is the solution of the following integral equation

$$\int_0^t g_f^t(u)\phi_H(v-u)\,du = f(v) \tag{4}$$

for $t > 0$, $v \in [0,t]$ and f a suitable deterministic function or stochastic process. The integral equation (4) can be solved explicitly as follows

$$g_f^t(s) = \frac{-s^{-1/2-H}\cos 2(\pi(1-H))}{H(2H-1)\pi 2} \cdot \frac{\Gamma\left(H-\frac{1}{2}\right)\Gamma(2-2H)}{\Gamma\left(\frac{3}{2}-H\right)} \cdot \frac{d}{ds}\int_s^t w^{2H-1}(w-s)^{1/2-H} \cdot \frac{d}{dw}\int_0^w f(z)z^{1/2-H}(w-z)^{1/2-H}\,dz\,dw.$$

To define stochastic integrals for deterministic integrands and a fractional Brownian motion integrator with $H \in (1/2,1)$ a Hilbert space of distributions, $\tilde{L}_H^2$, is basic. The distribution $F : [0,T] \to \mathbb{R}$ is an element of $\tilde{L}_H^2([0,T])$ if

$u_{1/2-H} I_{T-}^{H-1/2}(u_{H-1/2}F)$ is (Lebesgue) square integrable, that is,

$$\int_0^T \left(u_{1/2-H}(s) I_{T-}^{H-1/2}(u_{H-1/2}F)(s) \right)^2 ds < \infty$$

where $u_a(s) = s^a$ for $a \in \mathbb{R}$ and $I_{T-}^{H-1/2}(F)$ is the $(H-1/2)$ fractional integral of F, that is,

$$I_{T-}^{H-1/2}(F)(s) = \frac{1}{\Gamma\left(H-\frac{1}{2}\right)} \int_s^T \frac{F(t)}{(t-s)^{3/2-H}} dt.$$

A dense set in $\tilde{L}_H^2([0,T])$ can be described by functions F such that

$$\int_0^T \int_0^T |F(u)||F(v)| \phi_H(u-v) \, du \, dv < \infty. \tag{5}$$

For $H \in (1/2,1)$ and $T > 0$ the linear space $L_H^2([0,T])$ is the family of real-valued generalized processes on $(\Omega, \mathscr{F}, P)$ such that $X \in L_H^2$ if it is measurable on $\mathscr{B}([0,T]) \otimes \mathscr{F}$ and

$$|X|_{L_H^2}^2 = \mathbb{E} \int_0^T \left(u_{1/2-H}(s) I_{T-}^{H-1/2}(u_{H-1/2}X)(s) \right)^2 ds < \infty.$$

Let $\mathscr{S}$ be the family of smooth, cylindrical random variables on $(\Omega, \mathscr{F}, P)$, so $F \in \mathscr{S}$ has the form

$$F = \sum_{j=1}^n f_j \left(\int_0^T \gamma_{1j} \, dB^H, \ldots, \int_0^T \gamma_{n_j j} \, dB^H \right),$$

where $\gamma_{kj} \in \tilde{L}_H^2([0,T])$, $f_j \in C_p^\infty(\mathbb{R}^{n_j})$ for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, n_j\}$ and $C_p^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \in C^\infty$ and f and all of its derivatives have polynomial growth$\}$. If $F \in \mathscr{S}$ has the above form, then the derivative $D : \mathscr{S} \to L_H^2$ is given as

$$D_t F = \sum_{j=1}^n \sum_{i=1}^{n_j} \frac{\partial f_j}{\partial x_i} \left(\int_0^T \gamma_{1j} \, dB^H, \ldots, \int_0 1 \gamma_{n_j j} \, dB^H \right) \gamma_{ij}(t).$$

This derivative operator is a closed operator that can be extended to $D_H^{1,2} = \mathrm{Dom}(D)$ such that $D : \mathscr{S} \to L_H^2$ ([3]).

The stochastic integral is defined as a dual of D. This approach to the definition of stochastic integral uses an approach of Malliavin calculus for a Brownian motion. Another equivalent approach using the Wick product is given in [2].

Definition 1: Let $X \in L_H^2$. The generalized process X is integrable with respect to B^H if $F \mapsto \langle X, DF \rangle_{L_H^2}$ is continuous on $\mathscr{S}$ with the $L^2(\Omega)$ norm topology. The stochastic integral $\int_0^T X \, dB^H$ is a zero mean random variable such that

$$\langle X, DF \rangle_{L_H^2} = \mathbb{E}\left[F \int_0^T X dB^H \right]$$

for each $F \in \mathscr{S}$.

III. Main Results

Initially, the stochastic calculus that is described in Section II is used to define some martingales that are obtained by stochastic integration with a fractional Brownian motion for $H \in (1/2,1)$.

Theorem 1: Let $T > 0$ and $H \in (1/2,1)$ be fixed. If $(X(t), t \in [0,T])$ is the process given by

$$X(t) = \int_0^t g_Y^t \, dB^H \tag{6}$$

where $g_\cdot^t$ is the solution of the integral equation (4) and $(Y(t), t \in [0,T])$ is a process adapted to $(\mathscr{F}_t, t \in [0,T])$ and $g_Y^T \in L_H^2$, then $(X(t), \mathscr{F}_t, t \in [0,T])$ is a continuous, square integrable martingale.

It is well known that there is a (unique) increasing process associated with each continuous square integrable martingale. If $(M(t), t \geq 0)$ is such a martingale, then the associated increasing process $(\langle M, M \rangle(t), t \geq 0)$ can be defined as the increasing process such that $(M^2(t) - \langle M, M \rangle(t), t \geq 0)$ is a continuous martingale. The increasing process associated with the martingale (6) is given in the following theorem.

Theorem 2: Let the assumptions in Theorem 1 be satisfied, and let g_Y^T satisfy the analogue of (5) for L_H^2. If $(Z(t), t \in [0,T])$ is the process given by

$$Z(t) = \left(\int_0^t g_Y^t \, dB^H \right)^2 - \int_0^t \int_0^t g_Y^t(p) g_Y^t(q) \phi_H(p-q) \, dp \, dq \tag{7}$$

then $(Z(t), \mathscr{F}_t, t \in [0,T])$ is a continuous martingale.

Using the martingale (6), a nonnegative local martingale can be given as the following theorem describes. A special case of this result is given in [7]. The general result follows by the same method of proof.

Theorem 3: Let $T > 0$ be fixed, $H \in (1/2,1)$, and let $(X(t), t \in [0,T])$ be the continuous square integrable martingale given by (6). Let $(M(t), t \in [0,T])$ be the process given by

$$M(t) = \exp\left[X(t) - \frac{1}{2} \langle X, X \rangle(t) \right] \tag{8}$$

where $X(t)$ is given by (6) and $\langle X, X \rangle(t)$ is the term in (7) that is subtracted from $X^2(t)$. Then $(M(t), t \in [0,T])$ is a nonnegative continuous local martingale. If $\mathbb{E}[M(T)] = 1$, then there is a measure μ_X on $\mathscr{F}_T$ such that

$$\frac{d\mu_X}{d\mu_{B^H}} = M(T)$$

where μ_{B^H} is the measure for $(B^H(t), t \in [0,T])$.

Now, some processes are introduced to describe a model for a communication channel with an additive fractional Gaussian noise and a stochastic signal with a parameter whose square can be used to describe the relative signal power or the signal to noise ratio. This model has been extensively investigated in [4]. A generalization was given in [5].

Fix $T > 0$. Let $(X(t), t \in [0,T])$ be a real-valued jointly measurable process that is independent of $(B^H(t), t \geq 0)$ such that $g_X^T \in L_H^2$ and satisfies (5), where g_X^T is the solution of the integral equation (4).

For $\rho \in (0,\infty)$, let $(Y_\rho(t), t \in [0,T])$ be the process given by

$$\begin{aligned} dY_\rho(t) &= \rho X(t)\,dt + dB^H(t) \\ Y_\rho(0) &= 0 \end{aligned} \tag{9}$$

and let $I(X,Y_\rho)$ be the mutual information between $(X(t), t \in [0,T])$ and $(Y_\rho(t), t \in [0,T])$, that is,

$$I(X,Y_\rho) = \int N(T;\rho)\log N(T;\rho)\,d\mu_X\,d\mu_{Y_\rho} \tag{10}$$

and

$$N(T;\rho) = \frac{d\mu_{XY_\rho}}{d(\mu_X \otimes \mu_{Y_\rho})} = \frac{d\mu_{XY_\rho}}{d(\mu_X \otimes \mu_B)} \cdot \frac{d\mu_B}{d\mu_{Y_\rho}} \tag{11}$$

where μ_B, μ_X, μ_{Y_ρ} are the measures for $(B^H(t), t \in [0,T])$, $(X(t), t \in [0,T])$ and $(Y_\rho(t), t \in [0,T])$ respectively, and μ_{XY_ρ} is the joint measure for X and Y_ρ.

It can be shown using a result in [2] that

$$\begin{aligned} \frac{d\mu_{XY_\rho}}{d(\mu_X \otimes \mu_B)} = \exp\Big[&\int_0^T g^t_{\rho X}\,dB^H \\ &- \frac{1}{2}\int_0^T\int_0^T g^t_{\rho X}(u) g^t_{\rho X}(v)\phi_H(u-v)\,du\,dv\Big] \end{aligned}$$

and thus

$$\frac{d\mu_{Y_\rho}}{d\mu_B} = \mathbb{E}_X\left[\frac{d\mu_{XY_\rho}}{d(\mu_X \otimes \mu_B)}\right]$$

where $\mathbb{E}_X$ denotes integration with respect to μ_X.

The following result gives an explicit expression for the rate of change of $I(X,Y_\rho)$ with respect to ρ.

Theorem 4: Let $I(X,Y_\rho)$ be the mutual information between the processes $(X(t), t \in [0,T])$ and $(Y_\rho(t), t \in [0,T])$ that satisfy (9) with the assumptions made there. Then the following equality is satisfied

$$\begin{aligned} \frac{dI(X,Y_\rho)}{d\rho} &= \rho\mathbb{E}\Big[\int_0^T\int_0^T g^T_X(u) g^T_X(v)\phi_H(u-v)\,du\,dv \\ &\qquad - \int_0^T\int_0^T \hat{g}^T_{X,\rho}(u)\hat{g}^T_{X,\rho}(v)\phi_H(u-v)\,du\,dv\Big] \\ &= \rho\|g^T_X - \hat{g}^T_{X,\rho}\|^2_{L^2_H} \end{aligned} \tag{12}$$

where $\hat{g}^T_{X,\rho} = \mathbb{E}\left[g^T_X \mid Y_\rho(t), t \in [0,T]\right]$ and g^T_X is the solution of the integral equation (4).

The proofs of the Theorems will be given elsewhere. These results should contribute to the usefulness of fractional Brownian motions for stochastic models.

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