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Sampling and Parameter Estimation for a Second Order Linear System with a Fractional Brownian Motion*

T. E. Duncan and B. Pasik-Duncan
Department of Mathematics
University of Kansas
Lawrence, KS, 66045
duncan@ku.edu
bozenna@ku.edu

***Abstract*— A scalar input-scalar output linear second order system has a fractional Gaussian noise input. The fractional Gaussian noise is the formal derivative of a fractional Brownian motion with the Hurst parameter in the interval $(1/2,1)$. A family of estimators for a linear system with a fractional Gaussian noise and some unknown parameters that is obtained from the continuous time least square equations for the corresponding linear system with a white Gaussian noise is known to be strongly consistent. In this paper, it is shown that the asymptotic behaviour of this family of estimators obtained from the discretized least squares equations with only samples of the system output do not converge to the asymptotic behavior of the family of estimators of the continuous time least squares equations. This phenomenon is important for the numerical computation of parameter estimates.**

I. INTRODUCTION

In the theory of parameter estimation for continuous time stochastic linear systems, it is typically assumed that the observations of the state are also continuous. It is natural to expect that the consistency of the family of estimators is approximately valid if the output observations are discrete and the sampling interval is sufficiently small, more precisely, that there is a continuity for the asymptotic estimators as the sampling interval tends to zero. However, it is known for the parameter estimation for linear systems with white Gaussian noise that this continuity property is not satisfied. This occurrence has been related to the nonzero quadratic variation of Brownian motion [1], [2], [3], [4].

In this paper, a parameter estimation problem is described for a two-dimensional stochastic linear system with a scalar fractional Gaussian noise. It has been shown that under some conditions the family of estimators obtained by solving the least squares equations for a white Gaussian noise is strongly consistent where the white Gaussian noise is replaced by a fractional Gaussian noise [5]. This aforementioned lack of continuity of the asymptotic estimators for a white Gaussian noise input as the sampling intervals tend to zero also is shown to occur for a fractional Gaussian noise input where it is assumed that only samples of the output are available. In the latter case, this occurs even though these fractional Brownian motions have zero quadratic variation. The verification of this lack of continuity is more complicated for a fractional Brownian motion with the Hurst parameter in the interval $(1/2,1)$ because, for example, a stochastic calculus for these fractional Brownian motions is more complicated and less developed that the stochastic calculus for a Brownian motion ([6], [7]).

This numerical analysis of parameter estimation for a linear system with a fractional Brownian motion seems to be the initial work on this topic. It demonstrates the need for a careful analysis when continuous time least squares algorithms for fractional Brownian motion are discretized for numerical computation.

II. PRELIMINARIES

A two-dimensional (or second order) linear stochastic differential equation with a fractional Brownian motion is used to describe a stochastic process for a parameter estimation problem where the unknown parameters occur in the state transition matrix.

Let $(\mathbb{X}(t), t \geq 0)$ be the solution of the following stochastic differential equation

$$\begin{aligned} d\mathbb{X}(t) &= A(\alpha_0)\mathbb{X}(t)\,dt + C\,dB^H(t) \\ \mathbb{X}(0) &= \mathbb{X}_0 \end{aligned} \tag{1}$$

where $\mathbb{X}(t) = (X(t), X^{(1)}(t)) \in \mathbb{R}^2, (B^H(t), t \geq 0)$ is a real-valued standard fractional Brownian motion with the Hurst parameter $H \in (1/2,1)$, $\alpha_0 = (\alpha_0^1, \alpha_0^2)$, $\alpha_0 \in \mathscr{A} \subset \mathbb{R}^2$, $C = [0,1]^T$

$$\begin{aligned} A(\alpha_0) &= A_0 + \alpha_0^1 A_1 + \alpha_0^2 A_2 \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_0^1 \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \alpha_0^2 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned} \tag{2}$$

The solution $(\mathbb{X}(t), t \geq 0)$ of (1) is given explicitly ([8]) as

$$\mathbb{X}(t) = e^{A(\alpha_0)t}\mathbb{X}_0 + \int_0^t e^{A(\alpha_0)(t-s)}\,dB^H(s). \tag{3}$$

The process $(B^H(t), t \geq 0)$ is a standard fractional Brownian motion with the Hurst parameter $H \in (1/2,1)$, that is,

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$(B^H(t), t \geq 0)$ is a Gaussian process with continuous sample paths such that $\mathbb{E}[B^H(t)] = 0$ and

$$\mathbb{E}\left[B^H(s)B^H(t)\right] = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right]$$
$$= \int_0^t \int_0^s \phi_H(u-v)\, du\, dv$$

where $\phi_H(u) = H(2H-1)|u|^{2H-2}$. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a complete probability space for $(B^H(t), t \geq 0)$ for a fixed $H \in (1/2, 1)$ where $\Omega = C(\mathbb{R}_+, \mathbb{R})$ with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$, $\mathscr{F}$ is the $\mathscr{P}$-completion of the associated Borel σ-algebra and $\mathscr{P}$ is the Gaussian measure.

It is also useful to describe (1) in component form.

$$\begin{bmatrix} dX(t) \\ dX^{(1)}(t) \end{bmatrix} = \begin{bmatrix} X^{(1)}(t)\, dt \\ -(\alpha_0^2 X^{(1)}(t) + \alpha_0^1 X(t))\, dt + dB^H(t) \end{bmatrix} \tag{4}$$

The process $(X(t), t \geq 0)$ can be considered as the observation or output of the linear system (1) with the fractional Gaussian noise input dB^H/dt. In this form, the system is a scalar input-scalar output linear system with the transfer function T given by

$$T(s) = \frac{1}{s^2 + \alpha_0^2 s + \alpha_0^1}$$

so that the parameters (α_0^1, α_0^2) are the natural parameters of the transfer function or (equivalently) the characteristic polynomial. This type of transfer function describes the simplest nontrivial family of second order linear systems.

A family of estimators $(\hat{\alpha}(t), t > 0)$ are obtained for the unknown parameter α_0 in (1) by solving the linear equations that arise from the least squares estimation for (1) with a Brownian motion replacing the fractional Brownian motion. The resulting family of estimators is called a family of pseudo least squares estimators because they are not determined by the least squares equations for a fractional Brownian motion with $H \neq 1/2$. Nonetheless, it can be shown that under suitable assumptions, this family of estimators is strongly consistent and they are simpler to describe than the true least squares estimators [5].

The pseudo least squares estimator $\hat{\alpha}(t) = (\hat{\alpha}_1(t), \hat{\alpha}_2(t))$ of (α_0^1, α_0^2) satisfies the following linear equations

$$\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} \hat{\alpha}_1(t) \\ \hat{\alpha}_2(t) \end{bmatrix} = \begin{bmatrix} \int_0^t -X^{(1)}(s)\, dX^{(1)}(s) \\ -\int_0^t -X^{(1)}(s)\, dX^{(1)}(s) \end{bmatrix} \tag{5}$$

where

$$a_{ij}(t) = \int_0^t \langle A_i \mathbb{X}(s), A_j \mathbb{X}(s) \rangle\, ds.$$

Instead of the complete observations of $(\mathbb{X}(t), t \geq 0)$ that are required for the equations for the pseudo least squares estimator $\hat{\alpha}(t)$ in (5), it is assumed that only some regularly sampled values of the first component of $(\mathbb{X}(t), t \geq 0)$, that is, $(X(n\delta), n \in \mathbb{N})$ where $\delta > 0$ is fixed, are available to determine a family of estimators.

It is necessary to determine a system of equations that depend only on $(X(n\delta), n \in \mathbb{N})$ for a new family of estimators. Initially, a natural discretization of the family (5) is made. Since the stochastic integral term in (5) is defined from Riemann sums using the Wick product [7], the (Itô) stochastic integral in (5) is replaced by a Stratonovich stochastic integral plus an additive (correction) term [7]. The Stratonovich integral is defined as a limit of Riemann sums using the usual product [7].

A natural discretization of (5), where the Itô stochastic integral is replaced by a Stratonovich stochastic integral plus a correction term yields the following equations that determine a sequence of estimators $(\beta(n\delta), n \in \mathbb{N})$.

$$\begin{aligned} \delta \sum_{m=0}^{N-1} M(m\delta) M^T(m\delta) \beta(N\delta) = \\ \sum_{m=0}^{N-1} M(m\delta) \left(X^{(1)}((m+1)\delta) - X^{(1)}(m\delta) \right) \\ - \sum_{m=0}^{N-1} \sum_{n=0}^{m} e^{A(m\delta - n\delta)} C \phi_H(m\delta - n\delta) \delta^2 \end{aligned} \tag{6}$$

where $\phi_H(u) = H(2H-1)|u|^{2H-2}$ and $M(m\delta) = A_1 \mathbb{X}(m\delta) + A_2 \mathbb{X}(m\delta)$.

Since only samples of $(X(t), t \geq 0)$ are available, the required samples for $(X^{(1)}(t), t \geq 0)$ in (6) are replaced by the (Euler) forward difference approximation to the derivative of $(X(t), t \geq 0)$ as

$$\hat{X}^{(1)}(n\delta) = \frac{1}{\delta} \left(X((n+1)\delta) - X(n\delta) \right). \tag{7}$$

Replacing $(X^{(1)}(n\delta), n \in \mathbb{N})$ in (6) by the approximations $(\hat{X}^{(1)}(n\delta), n \in \mathbb{N})$ yields the equations

$$\begin{aligned} &\begin{bmatrix} \delta \sum_{m=0}^{N-1} \left[\left(\hat{X}^{(1)}(m\delta) \right)^2 \hat{\beta}_1(N\delta) + X(m\delta) \hat{X}^{(1)}(m\delta) \hat{\beta}_2(N\delta) \right] \\ \delta \sum_{m=0}^{N-1} \left[\left(\hat{X}^{(1)}(m\delta) \right) X(m\delta) \hat{\beta}_1(N\delta) + X^2(m\delta) \hat{\beta}_2(N\delta) \right] \end{bmatrix} \\ &= \begin{bmatrix} \sum_{m=0}^{N-1} -\hat{X}^{(1)}(m\delta) \left(\hat{X}^{(1)}((m+1)\delta) - \hat{X}^{(1)}(m\delta) \right) \\ -X(m\delta) \left(\hat{X}^{(1)}((m+1)\delta) - \hat{X}^{(1)}(m\delta) \right) \end{bmatrix} \\ &- \sum_{m=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{m} e^{A(m\delta - n\delta)} C \phi_H(m\delta - n\delta) \delta^2 \end{aligned} \tag{8}$$

for a sequence of estimators $(\hat{\beta}(n\delta), n \in \mathbb{N})$.

In the subsequent investigation of the behavior of the family of estimators $(\hat{\beta}(n\delta), n \in \mathbb{N})$ as $\delta \downarrow 0$, the asymptotic behavior of some covariance terms is important. So, some covariance expressions are given now. It is assumed that $A = A(\alpha_0)$ satisfies $\mathrm{Re}\lambda(A) < 0$, that is, $A(\alpha_0)$ is a stable matrix.

Let $0 \le s \le t$ and let $\mathbb{X}(0) = 0$.

$$\begin{aligned} R(s,t) &= \mathbb{E}\left[\mathbb{X}(s)\mathbb{X}^T(t)\right] \\ &= \mathbb{E}\left[\mathbb{X}(s)\left(e^{A(t-s)}\mathbb{X}(s) + \int_s^t e^{A(t-u)} C\, dB^H(u)\right)^T\right] \\ &= R(s,s)e^{A^T(t-s)} \\ &\quad + \int_0^s \int_s^t e^{A(s-u)} CC^T e^{A^T(t-v)} \phi_H(u-v)\, dv\, du \end{aligned} \tag{9}$$

and

$$\begin{aligned} R(t,s) &= \mathbb{E}\left[\mathbb{X}(t)\mathbb{X}^T(s)\right] \\ &= e^{A(t-s)} R(s,s) \\ &\quad + \int_s^t \int_0^s e^{A(t-u)} CC^T e^{A^T(s-v)} \phi_H(u-v)\, du\, dv. \end{aligned} \tag{10}$$

Thus

$$\begin{aligned} R(t,t) &= \mathbb{E}\left[\mathbb{X}(t)\mathbb{X}^T(t)\right] \\ &= \int_0^t \int_0^t e^{A(t-u)} CC^T e^{A^T(t-v)} \phi_H(u-v)\, du\, dv \\ &= \int_0^t \int_0^t e^{Au} CC^T e^{A^T v} \phi_H(u-v)\, du\, dv, \end{aligned} \tag{11}$$

$$\begin{aligned} R(\infty,\infty) &= \lim_{t\to\infty} R(t,t) \\ &= \int_0^\infty \int_0^\infty e^{Au} CC^T e^{A^T v} \phi_H(u-v)\, du\, dv, \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{dR(t,t)}{dt} &= AR + RA^T + \int_0^t CC^T e^{A^T(t-v)} \phi_H(t-v)\, dv \\ &\quad + \int_0^t e^{A(t-u)} CC^T \phi_H(t-u)\, du, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \lim_{t\to\infty} \frac{dR(t,t)}{dt} &= 0 \\ &= AR(\infty,\infty) + R(\infty,\infty)A^T \\ &\quad + \int_0^\infty CC^T e^{A^T v} \phi_H(v)\, dv \\ &\quad + \int_0^\infty e^{Au} CC^T \phi_H(u)\, du. \end{aligned} \tag{14}$$

III. Main Result

The main result shows that the sequence of estimators $(\hat{\beta}(n\delta), n \in \mathbb{N})$ determined by (8) does not determine an asymptotic family of estimators that converges to the true parameter as $\delta \downarrow 0$.

Proposition 1: Assume that $\mathrm{Re}\lambda(A(\alpha_0)) < 0$ for $A(\alpha_0)$ in (1). Let $\mathbb{X}(t) = (X(t), X^{(1)}(t))^T$ for $t \ge 0$ be the solution of (1). Let $(\hat{\beta}(n\delta), n \in \mathbb{N})$ be the sequence of estimators that is obtained from (8). Then

$$\lim_{\delta\downarrow 0} \lim_{n\to\infty} \hat{\beta}(n\delta) \ne \alpha_0 \tag{15}$$

where the limit as $n \to \infty$ denotes convergence in probability.

Proof: Equation (15) is verified by showing that the limit as $\delta \downarrow 0$ of the limit of the averaging of (8) is different from the corresponding limits of (6). Initially fix $\delta > 0$. To effect this verification, it is shown that the following limit exists and this limit is explicitly given.

$$\lim_{\delta\downarrow 0} \lim_{j\to\infty} \mathbb{E}\left[\frac{(X(j_2) - 2X((j_1)) + X(j_0))^2}{\delta^3}\right] \tag{16}$$

where $j_i = (j+i)\delta$.

Initially, the expectation in this expression is computed without the denominator term δ^3.

$$\begin{aligned} &\mathbb{E}\left[(X((j+2)\delta)) - 2X((j+1)\delta) + X(j\delta))^2\right] \\ &= \mathbb{E}\Big[X^2((j+2)\delta)) + 4X^2((j+1)\delta) + X^2(j\delta) \\ &\quad + 2X(j\delta)X((j+2)\delta) - 4X(j\delta)X((j+1)\delta) \\ &\quad - 4X((j+1)\delta)X((j+2)\delta)\Big] \end{aligned} \tag{17}$$

The term that is to be investigated in (8) is

$$\sum X^{(1)}(m\delta)\left(X^{(1)}((m+1)\delta) - X^{(1)}(m\delta)\right).$$

However, by symmetry which arises from $\int x\,dx = x^2 - \int x\,dx$ it suffices to consider a quadratic term and take one-half of the result.

The expectation of each of the six terms on the right-hand side of (17) is determined from (9) and (10), then the limit as $j \to \infty$ is determined and an expansion for small $\delta > 0$ is made. It follows directly that

$$\begin{aligned} \lim_{\delta\downarrow 0}\lim_{j\to\infty} \mathbb{E}\left[X^2((j+2)\delta)\right] &= \lim_{\delta\downarrow 0}\lim_{j\to\infty} \mathbb{E}\left[X^2((j+1)\delta)\right] \\ &= \lim_{\delta\downarrow 0}\lim_{j\to\infty} \mathbb{E}\left[X^2(j\delta)\right] \\ &= (R(\infty,\infty))_{11}. \end{aligned} \tag{18}$$

Using the computation of the covariance R in (9), (10), it follows that

$$\begin{aligned} &\lim_{j\to\infty} \mathbb{E}[X(j\delta)X((j+1)\delta)] \\ &= \lim_{j\to\infty} \mathbb{E}[X((j+1)\delta)X((j+2)\delta)] \\ &= \Big(R(\infty,\infty)e^{A^T\delta} \\ &\quad + \int_0^\infty \int_{-\delta}^0 e^{Au} CC^T e^{A^T(\delta+v)} \phi(u-v)\, du\, dv\Big)_{11} \\ &= \Big(e^{A\delta} R(\infty,\infty) \\ &\quad + \int_0^\infty \int_{-\delta}^0 e^{A(\delta+u)} CC^T e^{A^T v} \phi(u-v)\, du\, dv\Big)_{11} \end{aligned} \tag{19}$$

where $\phi = \phi_H$ for notational simplicity, and furthermore

$$\begin{aligned} \lim_{j\to\infty} \mathbb{E}[X(j\delta)X((j+2)\delta)] &= \Big(R(\infty,\infty)e^{2A^T\delta} \\ &\quad + \int_0^\infty \int_{-2\delta}^0 e^{Au} CC^T e^{A^T(2\delta+v)} \phi(u-v)\, du\, dv\Big)_{11}. \end{aligned} \tag{20}$$

Let $f(\delta)$ be the arithmetical average of the two expressions for the limit in (19), that is,

$$f(\delta) = \frac{1}{2}\Big(R(\infty,\infty)e^{A^T\delta} + \int_0^\infty \int_{-\delta}^0 e^{Au}CC^T e^{A^T(\delta+v)}\phi(u-v)\,dv\,du + e^{A\delta}R(\infty,\infty) + \int_0^\infty \int_{-\delta}^0 e^{A(\delta+u)}CC^T e^{A^T v}\phi(u-v)\,du\,dv\Big)_{11}. \quad (21)$$

Then the limit in (20) is $f(2\delta)$.

Now an expansion in δ of the expectation in (17) is made where for notational simplicity $R = R(\infty,\infty)$. The coefficients of δ^k, $k = 0,1,2,3$ in the expansion are considered individually. The coefficient of δ^0 is

$$((6I+2I-4I-4I)R)_{11} = 0.$$

The coefficient of δ^1 is the following, where the two expressions in (19) are used to exhibit the symmetry in the expansion via f.

$$\Big(2RA^T + 2AR - 2RA^T - 2AR - 2RA^T - 2AR + 2\int_0^\infty e^{Au}CC^T\phi(u)\,du + 2\int_0^\infty CC^T e^{A^T v}\phi(v)\,dv - 2\int_0^\infty e^{Au}CC^T\phi(u)\,du - 2\int_0^\infty CC^T e^{A^T v}\phi(v)\,dv - 2\int_0^\infty e^{Au}CC^T\phi(u)\,du - 2\int_0^\infty CC^T e^{A^T v}\phi(v)\,dv\Big)_{11} = 0$$

This equality follows from the algebraic equation for $R = R(\infty,\infty)$ in (14).

To determine the coefficient of δ^2, it should be recalled that this coefficient is one-half of $c - 8f(\delta) + 2f(2\delta) = g(\delta)$ where $c \in \mathbb{R}$ and $f(\delta)$ is given by (21). Thus

$$g''(0) = -8f''(0) + 2f''(0)2^2 = 0.$$

Now the coefficient of δ^3 is determined from the expression for f in (21). It follows that

$$f'''(0) = \frac{1}{2}\Big(A^3R + \int_0^\infty A^2CC^T e^{A^T v}\phi(v)\,dv + R\left(A^T\right)^3 + \int_0^\infty e^{Au}CC^T\left(A^T\right)^2\phi(u)\,du\Big)_{11}. \quad (22)$$

Thus the coefficient of δ^3 is

$$\frac{1}{3!2}\left(-8f'''(0) + 2f'''(0)2^3\right) = \frac{2}{3}f'''(0). \quad (23)$$

Now the asymptotic behavior of (6) is investigated assuming that both of the sequences of samples $(X(n\delta), n \in \mathbb{N})$ and $(X^{(1)}(n\delta), n \in \mathbb{N})$ are available. It is known that any evaluation of the (continuous) integrand in the intervals of the partition of the Riemann sums can be used to define the stochastic integral ([7], [9]). Using the Mean Value Theorem, the following sum is used.

$$\frac{1}{N\delta}\sum_{j=0}^{N-1}\frac{(X((j+1)\delta) - X(j\delta))}{\delta}\left(X^{(1)}((j+1)\delta) - X^{(1)}(j\delta)\right)$$

By analogy with the consideration of (16), it is necessary to determine the following limit.

$$\lim_{\delta\downarrow 0}\lim_{j\to\infty}\mathbb{E}\left[\frac{(X(j_1) - X(j_0))\left(X^{(1)}(j_1) - X^{(1)}(j_0)\right)}{\delta^2}\right] \quad (24)$$

where $j_i = (j+i)\delta$.

Initially, the expectation in this expression is determined from the individual terms without the denominator term δ^2.

$$\lim_{j\to\infty}\mathbb{E}\left[X((j+1)\delta)X^{(1)}((j+1)\delta)\right] = \lim_{j\to\infty}\mathbb{E}\left[X(j-\delta)X^{(1)}(j\delta)\right] = (R(\infty,\infty))_{21} = (R(\infty,\infty))_{12}$$

$$\lim_{j\to\infty}\mathbb{E}\left[X(j\delta)X^{(1)}((j+1)\delta) + X((j+1)\delta)X^{(1)}(j\delta)\right] = \Big(e^{A\delta}R(\infty,\infty) + \int_0^\infty\int_{-\delta}^0 e^{A(\delta+u)}CC^T e^{A^T v}\phi(u-v)\,du\,dv + R(\infty,\infty)e^{A^T\delta} + \int_0^\infty\int_{-\delta}^0 e^{Au}CC^T e^{A^T(\delta+v)}\phi(u-v)\,dv\,du\Big)_{21} \quad (25)$$

As computed above, the coefficients of δ^0 and δ^1 in the expansion of the expectation of the numerator in (25) are zero. The coefficient of δ^2 is

$$\frac{1}{2}\Big(A^2R + \int_0^\infty ACC^T e^{A^T v}\phi(v)\,dv + R(A^T)^2 + \int_0^\infty e^{Au}CC^TA^T\phi(u)\,du\Big)_{21}. \quad (26)$$

Comparing (23) and (26), it is clear that the equations for the asymptotic estimators differ, which verifies (15). ■

IV. Conclusion

In this paper it has been shown that a family of estimators obtained by the solution of a discretized version of the continuous time least squares equations, where the state component that is not observed is approximated using the Euler forward difference approximation for the derivative, does not converge to the true parameter vector as the observation time tends to infinity and the sampling interval tends to zero. Since this result has been verified only for second order systems, it is important to extend this result to nth order linear systems with a fractional Brownian motion by analogy with the investigation in [2] for linear systems with a Brownian motion. Furthermore, it is important to modify the discretization of the continuous time least squares equations or to provide other numerical differentiation schemes that do not have this asymptotic bias property for the estimators of parameters for linear systems with a fractional Brownian motion again by analogy to the result for linear systems with a Brownian motion ([2], [3]). Finally, it is important to perform some extensive numerical investigation for these algorithms because there is only a very limited amount of numerical work for linear systems with a fractional Brownian motion.

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