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Necessary Conditions Under Mangasarian-Fromowitz Type Assumptions For Mixed Constrained Control Problems

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Abstract— **Unmaximized Hamiltonian Inclusion (UHI) type conditions are derived for optimal control problems with mixed constraints. For such problems different necessary conditions of optimality, including UHI type conditions, have previously been proved assuming that the Jacobian of the mixed constraint functional with respect to the control variable have full rank. Here we show that such requirement can be replaced by possibly weaker conditions, namely, by Mangasarian Fromowitz type conditions. Notably we consider problems in which the dynamics is nonsmooth.**

The Unmaximized Hamiltonian Inclusion type conditions we present here are written as a weak version of nonsmooth maximum principle stated in terms of a joint Clarke subdifferential. They are of interest since, in contrast to more traditional nonsmooth maximum principles, they give sufficiency for normal linear convex problems.

I. INTRODUCTION

There is a growing appreciation for the importance of constrained optimal control problems in areas like robotics, economics and process systems engineering. As the number of applications increases so does the need to broaden the scope of optimality conditions to cover larger classes of problems. Consequently one has witnessed several attempts to validate necessary conditions of optimality under assumptions which may be viewed, in some sense, as minimal (see, for example, [1], [2], [3]).

In applications optimal control problems with mixed constraints are of particular relevance. One area of application of optimality conditions for such problems is to the control of devices modelled by differential algebraic equations (DAE systems). DAE models are nowadays widespread in chemical process engineering and economics([4], [5] and references therein).

Various aspects of problems with mixed constraints, including those of the present paper, can be captured as special cases of the following problem:

$$P \quad \left\{ \begin{array}{l} \text{Minimize } l(x(0), x(1)) \\ \text{subject to} \\ \begin{array}{rcll} \dot{x}(t) & = & f(t, x(t), u(t)) & \text{a.e.} \\ 0 & = & b(t, x(t), u(t)) & \text{a.e.} \\ 0 & \geq & g(t, x(t), u(t)) & \text{a.e.} \\ (x(0), x(1)) & \in & C \end{array} \end{array} \right.$$

with data the functions $l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$, $b : [0,1] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{m_b}$, $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{m_g}$ and a closed set $C \subset \mathbb{R}^n \times \mathbb{R}^n$. We set $m = m_b + m_g$ and throughout this paper we assume that $k \geq m$. Usually one has $m_b \geq 1$ and $m_g \geq 1$. However we allow for $m_b = 0$ (no equality constraints) or $m_g = 0$ (no inequality constraints).

The distinguishing aspects of (P) are the equality mixed constraints $b(t, x, u) = 0$ and inequality mixed constraints $g(t, x, u) \leq 0$.

The subject of necessary conditions in the form of maximum principles for (P) have been addressed by a number of authors; see for example [6], [7], [8], [9], [10], to name but a few. Weak maximum principles covering problems with nonsmooth dynamics have been considered in [8] and [11]. For nonsmooth problems, strong maximum principles have also received some attention ([12] and [4]).

A common requirement in the aforementioned papers is the assumption that the Jacobian of the mixed constraint functional with respect to the control variable have full rank. Exceptions are to be found in [9], where necessary conditions for *smooth* problems are derived under assumptions that may be viewed as analogous to the so called Mangasarian-Fromowitz conditions in mathematical programming (defined further down) and in [12], where convex and non autonomous problems are treated.

In this paper we seek necessary conditions of optimality for problem (P) under hypotheses: (i) closely related to those adopted in [9], (ii) directly verifiable by the data of the problem. In contrast to [9] we treat problems with possibly nonsmooth dynamics. It is also worth mentioning that throughout this paper we assume the data of (P) to be merely measurable with respect to t.

The necessary conditions of optimality we derive for (P) are a generalization of a weak nonsmooth maximum principle, known as Unmaximized Hamiltonian Inclusion type conditions (denote in what follows simply as UHI), previously obtained for standard optimal control problems in [13].

This paper is structured as follows. After the Preliminaries in section II we introduce and discuss UHI-type conditions for standard optimal control problems. In section IV we briefly discuss various regularity assumptions under which necessary conditions of optimality have previously been derived for (P). Finally in sections V and VI we present our main result and a sketch of its proof.

II. PRELIMINARIES

Here and throughout, B represents the closed unit ball centered at the origin, $|\cdot|$ the Euclidean norm, and $|\cdot|$ the

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induced matrix norm on $\mathbb{R}^{m\times k}$. The notation $r \geq 0$ means that each component r_i of $r \in \mathbb{R}^r$ is nonnegative.

We make use of various concepts from nonsmooth analysis, among those the *limiting normal cone to* a set C, N_C, the *limiting subdifferential* of f, $\partial f(x)$, and the Clarke subdifferential. The concepts of limiting normal cone and limiting subdifferential as well as the full calculus for these constructions in finite dimensions is described in [14] and [15].

In the case that a function f is Lipschitz continuous near x, the convex hull of the limiting subdifferential, $\text{co}\,\partial f$, coincides with the *Clarke subdifferential* which may be defined directly. Properties of generalized subdifferentials (upper semi-continuity, sum rules, etc.), are described in [16] and [1].

For (P) a process is a pair (x, u) comprising a measurable function $u : [0, 1] \to \mathbb{R}^k$ and $x \in W^{1,1}([0, 1]; \mathbb{R}^n)$ satisfying the constraints of (P). Here $W^{1,1}([0, 1]; \mathbb{R}^n)$ denotes the space of absolutely continuous $\mathbb{R}^n$-valued functions on $[0, 1]$.

Definition 2.1: A process $(\bar{x}, \bar{u})$ is a *local minimizer* if there exists some $\varepsilon > 0$, such that it minimizes the cost over all processes (x, u) of (P) which satisfy $x(t) \in \bar{x}(t) + \varepsilon B$ and it is a *weak local minimizer* if there exists some $\varepsilon > 0$, such that it minimizes the cost over all processes (x, u) of (P) satisfying

$$(x(t), u(t)) \in (\bar{x}(t), \bar{u}(t)) + \varepsilon B \text{ a.e. } t \in [0, 1] \tag{1}$$

For much of the analysis we shall denote by $(\bar{x}, \bar{u})$ the optimal solution of (P), $\bar{\phi}(t)$ the evaluation of a function ϕ at $(t, \bar{x}(t), \bar{u}(t))$, where ϕ may be f, b, g or its derivatives. Moreover, the set $\mathcal{I}_a(t)$ denotes the set of indexes of the *active constraints*, i.e.,

$$\mathcal{I}_a(t) = \big\{ i \in \{1, \ldots, m_g\} \;\;\big|\;\; g_i(t, \bar{x}(t), \bar{u}(t)) = 0 \big\}$$

and its complement, the set of indexes of the *inactive constraints*, $\mathcal{I}_c(t)$, $q_a(t)$ denotes the cardinal of $\mathcal{I}_a(t)$ and $q_c(t) = m_g - q_a(t)$, the cardinal of $\mathcal{I}_c(t)$. Also

$$\nabla_u g^{\mathcal{I}_a(t)}\big(t, \bar{x}(t), \bar{u}(t)\big) \in \mathbb{R}^{q_a(t)\times k} \tag{2}$$

(if $q_a(t) = 0$, then the latter holds vacuously) is the matrix we obtain after removing from $\nabla_u g(t, \bar{x}(t), \bar{u}(t))$ all the rows of index $i \in \mathcal{I}_c(t)$.

III. UHI-TYPE CONDITIONS FOR OPTIMAL CONTROL

Consider a standard optimal control problem

$$\textbf{(S)} \quad \begin{cases} \text{Minimize } l(x(0), x(1)) \\ \text{subject to} \\ \qquad \dot{x}(t) \;=\; f(t, x(t), w(t)) \quad \text{a.e.} \\ \qquad w(t) \;\in\; W(t) \quad \text{a.e.} \\ \qquad (x(0), x(1)) \;\in\; C \end{cases}$$

where l, f, and C are as defined above for (P) and $W : [0, 1] \to \mathbb{R}^k$ is a given multifunction. Assume that $(\bar{x}, \bar{w})$ is a reference process of (S) and $\varepsilon > 0$ a parameter.

The nonsmooth maximum principle (see e.g. [16]) for (S), which we shall refer as "nonsmooth" maximum principle ("strong" form), asserts that for a local minimizer $(\bar{x}, \bar{w})$ there exist a function $p \in W^{1,1}$ and a scalar $\lambda \geq 0$ such that

$$||p||_{L_\infty} + \lambda > 0,$$

$$-\dot{p}(t) \in \text{ co } \partial_x H(t, \bar{x}(t), p(t), \bar{w}(t)) \text{ a.e.}, \tag{3}$$

$$\max_{w\in W(t)} H(t, \bar{x}(t), p(t), w) = H(t, \bar{x}(t), p(t), \bar{w}(t)) \text{ a.e.}, \tag{4}$$

$$(p(0), -p(1)) \in \lambda\partial l(\bar{x}(0), \bar{x}(1)) + N_C(\bar{x}(0), \bar{x}(1)),$$

where H is the *Hamiltonian* for (S),

$$H(t, x, p, w) = p \cdot f(t, x, w) \tag{5}$$

In the above conditions ∂l denotes the *limiting subdifferential* of l with respect to its arguments, N_C denotes the *limiting normal cone* to C, and $\text{co}\,\partial_x H$ denotes the *Clarke subdifferential* of H with respect to x.

Under mild hypotheses on the data, which include merely Lipschitz continuity of f with respect to (x, w), a common form of a nonsmooth weak maximum principle is obtained when (4) is replaced by

$$\xi(t) \in \text{co}\,\partial_w H(t, \bar{x}(t), p(t), \bar{w}(t)) \quad \text{a.e.}, \tag{6}$$

and

$$\xi(t) \in \text{co}\, N_{W(t)}(\bar{w}(t)) \quad \text{a.e.} \tag{7}$$

It has been highlighted that the normal form of the nonsmooth maximum principle ("strong" or weak) fails to provide sufficiency for linear-convex problems, in contrast to the analogous maximum principle applicable to problems with differentiable data.

In [13] a weak nonsmooth maximum principle (we present further down) is proposed for *standard* optimal control problems which provides, in the normal form, sufficiency to linear-convex nonsmooth problems. It is formulated as an *Unmaximized Hamiltonian Inclusion* type condition involving the joint subdifferential of the Hamiltonian in the (x, w) variables

$$(-\dot{p}(t), \xi(t)) \in \text{co}\,\partial H(t, \bar{x}(t), p(t), \bar{w}(t)) \quad \text{a.e.} \tag{8}$$

together with the inclusion (7).

It is a well known fact that, for nonsmooth problems, (3) and (6) are not equivalent to (8). Inclusion (8) can give more information in situations when

$$\text{co}\,\partial H \neq \text{co}\,\partial_x H \times \text{co}\,\partial_w H.$$

Notably, the joint subdifferential inclusion (8) in UHI-type conditions gives sufficiency for the linear convex normal case. For discussion in this respect we refer the reader to [13].

Let us consider the following hypotheses, which make reference to a parameter $\varepsilon > 0$ and a reference process $(\bar{x}, \bar{w})$:

H1 The function $t \to f(t, x, w)$ is Lebesgue measurable for each pair (x, w) and there exists a function K_f in L^1 such that

$$\big|f(t, x, w) - f(t, x', w')\big| \\ \leq K(t)[|x - x'|^2 + |w - w'|^2]^{1/2}$$

for $x, x' \in \bar{x}(t) + \varepsilon B$, and $w, w' \in \bar{w}(t) + \varepsilon B$ a.e. $t \in [0,1]$.

H2 The multifunction W has Borel measurable graph and $W_\delta(t) := (\bar{w}(t) + \varepsilon B) \cap W(t)$ is closed for almost all $t \in [0,1]$.

H3 The endpoint constraint set C is closed and l is locally Lipschitz in a neighbourhood of $(\bar{x}(0), \bar{x}(1))$.

The aforementioned UHI-type conditions for (S), derived in [13], are as follows.

Proposition 3.1: Let $(\bar{x}, \bar{w})$ denote a weak local minimizer for (S). If H1–H3 are satisfied and $H(t,x,p,w) = p \cdot f(t,x,w)$ defines the Hamiltonian, then there exist $\lambda \geq 0$, $p \in W^{1,1}([0,1];\mathbb{R}^n)$ and $\zeta \in L^1([0,1];\mathbb{R}^k)$ such that, for almost every $t \in [0,1]$,

(i) $\lambda + \| p \|_{L^\infty} = 1$

(ii) $(-\dot{p}(t), \zeta(t)) \in \text{co}\ \partial H(t, \bar{x}(t), p(t), \bar{w}(t))$

(iii) $\zeta(t) \in \text{co}\ N_{W(t)}(\bar{w}(t))$

(iv) $(p(0), -p(1)) \in N_C(\bar{x}(0), \bar{x}(1)) + \lambda \partial l(\bar{x}(0), \bar{x}(1))$,

where ∂H denotes the limiting subdifferential in the (x,w) variables.

Observe that if $W(t) \equiv \mathbb{R}^k$, then the multiplier ζ above is 0 and (iii) is superfluous.

The UHI-type conditions given by Proposition 3.1 above have been used as an intermediate step to establish maximum principles for problems with differential algebraic equations (DAE's) ([17]) and for different classes of problem (P) ([4]). Recently they have been extended to cover optimal control problems with pure state constraints ([18]). Generalizations of Proposition 3.1 to problems with mixed constraints were obtained in [19] and [11]. It is worth mentioning that the main features of UHI-type conditions are retained when generalized to cover problems with mixed constraints and/or with pure state constraints (see [11] and [18]). Of foremost importance is the fact that they are also sufficient for optimality when applied to normal linear convex problems with mixed constraints and/or with pure state constraints.

IV. MIXED CONSTRAINTS

We now focus on optimal control problem with mixed constraints in the form of (P). Although the maximum principle is not in general valid for (P), there are classes of problems for which some form of the maximum principle holds. Usually such are those with the data satisfying regularity conditions on the mixed constraints. Derivation of optimality conditions for problems with nonregular mixed constraints remains a largely unexplored area (see [9] for references on nonregular problems).

In the literature regularity assumptions imposed on the mixed constraints vary. For smooth problems with *data continuous with respect to* t necessary conditions for (P) have previously been derived under the full rank assumption that

$$\det F(t)F(t)^T \neq 0 \quad \text{a.e. } t \in [0,1] \tag{9}$$

where F is a matrix closely related to the derivative with respect to the control variable of the functions defining the mixed constraints. Condition (9) ensures that the row vectors of matrix F are *linearly independent* and it is of interest since it permits the association of (P) with an auxiliary problem, (P_{aux_1}) where the inequality constraint $g(t,x(t),u(t)) \leq 0$ is replaced by equality constraints by considering

$$g\big(t, x(t), u(t)\big) + \varpi^2(t) = 0$$

where ϖ is a new control. Appealing to implicit function theorems (P_{aux_1}) can further be associated with a new problem (P_{aux_2}) *without* mixed constraints ((P_{aux_2}) being a problem in the form of (S)). In this respect see, for example, [6] and [10] where F is chosen to be

$$F(t) = \left[\begin{array}{c} \nabla_u b(t, \bar{x}(t), \bar{u}(t)) \\ \nabla u g^{\mathcal{I}_a(t)}(t, \bar{x}(t), \bar{u}(t)) \end{array} \right]. \tag{10}$$

When *measurability* of the data with respect to t is assumed (a standard assumption for nonsmooth problems) approaches similar to the one described above are possible when the full rank condition (9) is replaced by an "uniform" full rank condition of the form

$$\det F(t)F(t)^T > K \quad \text{a.e. } t \in [0,1], \tag{11}$$

for some $K > 0$, and, instead of classical implicit function theorems other implicit function theorems are used (see, for example, [20] and [19]). We refer the reader to [19] for a discussion about the "uniform" full rank condition (11) on various choices of matrices for problem (P).

To simplify the forthcoming discussion let us consider the following hypothesis:

H* There exists $K > 0$ such that, for almost every $t \in [0,1]$, $\det F(t)F(t)^T > K$ where F is as defined in (10).

Full rank assumptions like (9) or H^* are of relevance since necessary conditions for (P) are such that the derivative with respect to u of $\bar{g}_i$, for $i \notin \mathcal{I}_a(t)$, does not take any part in the determination of the multipliers.

In this paper we derive UHI-type conditions for (P) under the following alternative assumption:

HMF There exist constants $K_1 > 0$, $\kappa_i > 0$ with $i \in \{1, \ldots, m_g\}$, L^∞ functions $h : [0,1] \longrightarrow \mathbb{R}^k$, $\delta : [0,1] \longrightarrow \mathbb{R}^{m_g}$, $\delta(t) = (\delta_1(t), \ldots, \delta_{m_g}(t))$, such that, for almost every $t \in [0,1]$

(i) $\delta_i(t) \geq \kappa_i$ for $i \in \mathcal{I}_a(t)$,

(ii) $| h(t) | = 1$,

(iii) $\nabla_u \bar{b}(t) \cdot h(t) = 0$ and $\nabla_u \bar{g}(t) \cdot h(t) = -\delta(t)$,

(iv) $\det \nabla_u \bar{b}(t) \nabla_u \bar{b}(t)^T \geq K_1$.

Assumption HMF is an "uniform" Mangasarian-Fromowitz type condition (also known in the literature as positive linear independence condition) and it is an adaptation of condition (23) proposed in [21]. The term "uniform" here is used to emphasize the fact that inequalities are assumed to be bounded away from the origin, uniformly in t.

Assumption HMF above mainly differs from assumptions previously considered in [9] since it only needs to be checked along the optimal solution. It is also worth mentioning that the requirement that

$$\nabla_u \bar{g}_i(t) \cdot h(t) \leq -\kappa_i < 0$$

(HMF–(i)) is merely imposed on the active constraints (i.e., for $i \in \mathcal{I}_a(t)$). It is of special interest for problems for which the set $\mathcal{I}_a(t)$ is known for almost every $t \in [0,1]$ à priori.

When the number of controls k of problem (P) is strictly greater than the number of equality constraints m_b validation of necessary conditions under HMF as opposed to H^* is of foremost relevance. Indeed when $k > m_b$ it is a simple matter to see that H^* implies HMF (see example (5.2) below) while the opposite is not in general true. In particular, necessary conditions under merely HMF broadens the scope of application of optimality conditions in situations when equality constraints are absence ($m_b = 0$).

V. Main Results

We impose the following additional hypotheses on (P):

H4 $[b,g](\cdot,x,u)$ is measurable for each (x,u) and $t \to g(t,\bar{x}(t),\bar{u}(t))$ is L^∞. There exists L^1 functions $L_{b,g}$ such that, for almost every $t \in [0,1]$, $[b,g](t,\cdot,\cdot)$ is continuously differentiable with Lipschitz constant $L_{b,g}(t)$ on

$$(\bar{x}(t),\bar{u}(t)) + \varepsilon B.$$

There exists a constant $K_{b,g} > 0$ such that, for almost every $t \in [0,1]$,

$$\left|\nabla_x \overline{[b,g]}(t)\right| + \left|\nabla_u \overline{[b,g]}(t)\right| \leq K_{b,g}.$$

H5 There exists an increasing function $\tilde{\theta} : \mathbb{R}^+ \to \mathbb{R}^+$, $\tilde{\theta}(s) \downarrow 0$ as $s \downarrow 0$, such that, for all (x',u'), $(x,u) \in (\bar{x}(t),\bar{u}(t)) + \varepsilon B$ and for almost every $t \in [0,1]$,

$$\begin{aligned}&|\nabla_{x,u}[b,g](t,x',u') - \nabla_{x,u}[b,g](t,x,u)| \\ &\leq \ \tilde{\theta}\Big(\ |\ (x',u') - (x,u)\ |\ \Big).\end{aligned}$$

Our main result (Theorem 5.1 below) is not proved by the traditional approach described in the previous section. Nevertheless an implicit function theorem plays a crucial role in its proof. Because the measurability with respect to t prevents application of classical implicit function theorems (see [6]) we use a sharpened variant of the Implicit Function Theorem, an *Uniform* Implicit Function Theorem previously obtained in [17], to a function defined in terms of b and g (see next section). In this respect H4, H5 and HMF–(vi) are essentials.

Hypotheses H4 and H5 mainly states that the derivatives of b and g with respect to state and control must be uniformly continuous on a tube around the optimal solution and be bounded along the optimal solution.

We are now in position to state the main result of this paper.

Theorem 5.1: Let $(\bar{x},\bar{u})$ be a weak local minimizer for problem (P). If, for some $\varepsilon > 0$, hypotheses H1, H2, H4, H5 and HMF are satisfied and

$$H(t,x,p,q,r,u) := p \cdot f(t,x,u) + q \cdot b(t,x,u) + r \cdot g(t,x,u)$$

defines the Hamiltonian, then there exist functions $p \in W^{1,1}$, q, $r \in L^1$ and a scalar $\lambda \geq 0$ such that

(i) $\|\ p\ \|_{L_\infty} + \lambda \neq 0$

(ii) $(-\dot{p}(t),0) \in \mathrm{co}\,\partial_{x,u} H(t,\bar{x}(t),p(t),q(t),r(t),\bar{u}(t))$,

(iii) $r(t) \cdot g(t,\bar{x}(t),\bar{u}(t)) = 0$ and $r(t) \leq 0$,

(iv) $(p(0),-p(1)) \in N_C(\bar{x}(0),\bar{x}(1)) + \lambda\,\partial l(\bar{x}(0),\bar{x}(1))$.

Furthermore, there exist integrable functions B_b and B_g such that

$$|q(t)| \ \leq \ B_b(t)|p(t)| \qquad \text{for a.e. } t \in [0,1], \qquad (12)$$

$$|r(t)| \ \leq \ B_g(t)|p(t)| \qquad \text{for a.e. } t \in [0,1]. \qquad (13)$$

In situations when the number of controls k is equal to the number of equality constraints m_b and when the measure of the set

$$\{t \in [0,1] : \mathcal{I}_a(t) = \emptyset\}$$

is zero (i.e., when inequality constraints are *inactive* almost everywhere) the conclusions of Theorem 5.1 remain valid when only HMF–(iv) is satisfied (see [19]).

Next we present an example in which H^* is not verified and yet HMF is and Theorem 5.1 holds.

Example 5.2: Consider the problem of minimizing $x(1)$ subject to $\dot{x}(t) = u$, $0 \geq v^2 - u$, $0 \geq u - v^3$ and $x(0) = 0$. The minimizer is $(0,0,0)$ and $\mathcal{I}_a(t) = \{1,2\}$ for all $t \in [0,1]$. We have

$$\begin{gathered}H(t,x,p,r_1,r_2,u,v) = \\ p \cdot u + r_1 \cdot (v^2 - u) + r_2(u - v^3).\end{gathered}$$

For this problem the matrix (10) is

$$F(t) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}.$$

It is a simple matter to see that Theorem 5.1 holds with, for example, $p(t) = -1$, $r_1 = -1$, $r_2 = 0$ and $\lambda = 1$.

Although H^* is not satisfied in this example hypothesis HMF is with $h(t) = [\sqrt{2}/2 \ -\sqrt{2}/2]^T$ and $\kappa_1 = \kappa_2 = -\sqrt{2}/2$. Since no equality constraints are present K_1 is ignored.

Hypothesis HMF is crucial for the validity of Theorem 5.1 as the following simple but nevertheless illustrative example shows:

Example 5.3: Consider the problem of minimizing $x(1)$ subject to $\dot{x}(t) = u$, $0 \geq x - v^2$ and $x(0) = 0$. The minimizer is $(0,0,0)$. Observe that HMF is not satisfied for this problem since here we have $1 \in \mathcal{I}_a(t)$ for all $t \in [0,1]$ and

$$F(t) = \nabla_{u,v}(x - v^2)\big|_{(x\equiv 0,u\equiv 0,v\equiv 0)} = [0\ 0].$$

Applying the conditions of Theorem 5.1, we get

$$(-\dot{p}(t),0,0) = (r(t),p(t),0)$$

and $p(1) = -\lambda$, which implies that $p(t) = 0$, $r(t) = 0$ and $\lambda = 0$, contradicting the nontriviality condition (i).

Here both the full rank condition on F in the sense of (9) and H^* are not satisfied since $F(t) = [0\ 0]$.

Theorem 5.1 coincides with the main results in [19] and [11], where UHI-type conditions are validated for (P) under regular conditions on the mixed constraints in the vein of hypothesis H^*. Since, when $k > m_b$, the class of problems to which Theorem 5.1 applies is larger than those to which the results in [19] and [11] do, we conclude that Theorem 5.1 broadens the scope of necessary conditions.

VI. SKETCH OF THE PROOF

We define a sequence of optimization problems to which Ekeland's Variational Principle applies. This gives rise to a sequence of standard optimal control problems satisfying the conditions under which Proposition 3.1 is validated. Taking limits and rewriting the necessary conditions thus obtained we prove the theorem. In this last step of the proof Hypothesis HMF plays a crucial role.

The equality and inequality constraints are incorporated into the dynamics of the aforementioned sequence of optimal control problems by different means. Equality constraints are incorporated into the dynamics by appealing to an Uniform Implicit Function proved in [17]. On the other hand inequality constraints are included both into the dynamics and the cost. In this respect the definition of a $\max$ function and state and control augmentation techniques are essential.

We start the proof which breaks into steps. Let ε be as in H1, H2, H4, H5 and HMF. Define the following matrices:

$$\Gamma_u(t) = \begin{bmatrix} \nabla_u \bar{b}(t) \\ \nabla_u \bar{g}(t) \end{bmatrix}, \quad M = \begin{bmatrix} 0 & I \end{bmatrix}$$

where $M \in \mathcal{M}_{m_g \times (m_b + m_g)}$. Let $\beta : [0,1] \to \mathbb{R}^m$ be any measurable function and set $\bar{\beta}$ to be

$$\bar{\beta}(t) = -g(t, \bar{x}(t), \bar{u}(t)).$$

Step 1: We apply an Uniform Implicit Function proved in [17] to a function

$$m\big(t, (\xi, u, \beta), \zeta\big) :=$$
$$\begin{pmatrix} b\big(t, \bar{x}(t) + \xi, \bar{u}(t) + u + \Gamma_u(t)^T \zeta\big) \\ g\big(t, \bar{x}(t) + \xi, \bar{u}(t) + u + \Gamma_u(t)^T \zeta\big) + \bar{\beta}(t) + \beta + M\zeta \end{pmatrix}$$

in order to obtain an "uniform" implicit function d.

Observe that, for almost every $t \in [0,1]$, $m(t, (0,0,0), 0) = 0$ and $\dfrac{\partial m}{\partial \zeta}(t, (0,0,0), 0)$ is invertible with norm bounded away from 0. Thus Theorem 3.2 in [17] asserts the existence of $\sigma_1 \in (0, \varepsilon)$, $\delta_1 \in (0, \varepsilon)$ and an implicit map

$$d : [0,1] \times \sigma_1 B \times \sigma_1 B \times \sigma_1 B \longrightarrow \delta_1 B$$

such that $d(\cdot, \xi, u, \beta)$ is a measurable function for fixed (ξ, u, β), the functions $\{d(t, \cdot, \cdot, \cdot) |\, t \in [0,1]\}$ are Lipschitz continuous with common Lipschitz constant $K_d > 0$ and $d(t, \cdot, \cdot, \cdot)$ is continuously differentiable for fixed $t \in [0,1]$.

Step 2: Define the set

$$R^+ = \{r \in \mathbb{R}^m : r_i \geq 0,\ i \in \{1, \ldots, m\}\},$$

a function $G : [0,1] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{m_g} \to \mathbb{R}^{m_g}$ as

$$G(t,x,u,\beta) = g\Big(t, x, u + \Gamma_u(t)^T d\big(t, x - \bar{x}(t), u - \bar{u}(t)\big), \beta\Big)$$

and set

$$G^+(t,x,u,\beta) = \max\left\{0, G_1(t,x,u,\beta), \ldots, G_{m_g}(t,x,u,\beta)\right\}.$$

Define also the function

$$f_1(t,x,u,\beta) = f\Big(t, x, u + \Gamma_u(t)^T d\big(t, x - \bar{x}(t), u - \bar{u}(t)\big), \beta - \bar{\beta}(t)\Big).$$

Step 3: We define a sequence of optimization problems and check that Ekeland's Variational Principle applies to such sequence of problems.

For a conveniente choice of $\sigma \in (0, \sigma_1)$ (for details see [11]) set $\mathcal{B}_\sigma(t) = R^+ \bigcap \left\{\bar{\beta}(t) + \sigma B\right\}$. Set also

$$\bar{z}(t) = \int_0^t G^+(s, \bar{x}(s), \bar{u}(s), \bar{\beta}(s))ds.$$

Take W to be the set of all measurable functions (u, β) and all vectors $(a,b) \in \mathbb{R}^n \times \mathbb{R}^n$ such that, for almost every $t \in [0,1]$, $u(t) \in \big(\bar{u}(t) + \sigma B\big)$, $\beta(t) \in \mathcal{B}_\sigma(t)$, $(a,b) \in C$ and for which there exist absolutely continuous functions x, y and z such that

$$\begin{array}{rcll} \dot{x}(t) & = & f_1(t,x,u,\beta) & \text{a.e.} \\ \dot{y}(t) & = & 0 & \text{a.e.} \\ \dot{z}(t) & = & G^+(t,x,u,\beta) & \text{a.e.} \\ (x(t), y(t), z(t)) & \in & (\bar{x}(t), \bar{x}(1), \bar{z}(t)) + \sigma B & \text{a.e.} \\ (x(0), y(0), z(0)) & = & (a, b, 0). & \end{array}$$

Let $\{\epsilon_k\}_{k \in \mathbb{N}}$ be a sequence of positive scalars such that $\lim\limits_{k \to \infty} \epsilon_k = 0$. Define the function

$$\Psi_k(x, y, x', y', z) = \max\left\{l(x,y) - l(\bar{x}(0), \bar{x}(1)) + \epsilon_k^2,\ z + |x' - y'|\right\}.$$

To simplify the notation set $E = (a,b) \in \mathbb{R}^{2n}$. Let

$$\mid E - E' \mid = \mid a - a' \mid + \mid b - b' \mid$$

and

$$\nu((u,\beta),(u',\beta')) = \int_0^1 |u(t) - u'(t)|\, dt + \int_0^1 |\beta(t) - \beta'(t)|\, dt.$$

Define $d_W : W \times W \to \mathbb{R}$,

$$d_W((u,\beta,E),(u',\beta',E')) = \nu((u,\beta),(u',\beta')) + \mid E - E' \mid.$$

Consider the sequence of optimization problems

$$(\mathbf{R_k}) \quad \left\{ \begin{array}{ll} \text{Minimize} & J_k(u,\beta,E) \\ \text{subject to} & (u,\beta,E) \in W \end{array} \right.$$

where $J_k(u,\beta,E) = \Psi_k(x(0), y(0), x(1), y(1), z(1))$.

Observe that W is nonempty since $(\bar{u}, \bar{\beta}, \bar{x}(0), \bar{x}(1)) \in W$, d_W defines a metric in W and, with respect to this metric, the set W is a complete metric space and the function $(u, \beta, E) \to J_k(u, \beta, E)$ is continuous on (W, d_W).

Setting $\bar{E} = (\bar{x}(0), \bar{x}(1))$, we get, for all $k \in \mathbb{N}$,

$$J_k(\bar{u}, \bar{\beta}, \bar{E}) = \Psi_k(\bar{x}(0), \bar{x}(1), \bar{x}(1), \bar{x}(1), \bar{z}(1)) = \epsilon_k^2.$$

It is a simple matter to see $(\bar{u}, \bar{\beta}, \bar{E})$ is an "ϵ_k^2-minimizer" for (R_k).

Step 4: Rewriting the conclusions of Ekeland's Theorem in control theoretic terms we obtain a sequence of standard optimal control problems.

Write (x_k, y_k, z_k) the trajectory corresponding to (u_k, β_k, E_k). For each $k \in \mathbb{N}$, the process $(x_k, y_k, z_k, w_1 \equiv 0, w_2 \equiv 0, u_k, \beta_k)$ solves the control problem (C_k):

$$\begin{cases} \text{Minimize } \Psi_k(x(0), y(0), x(1), y(1), z(1)) \\ \qquad\qquad +\epsilon_k \pi_k(x(0), y(0), z(1), w_1(1), w_2(1)) \\ \text{subject to} \\ \quad \dot{x}(t) = f_1(t, x, u, \beta) \\ \quad \dot{y}(t) = 0, \\ \quad \dot{z}(t) = G^+(t, x, u, \beta), \\ \quad \dot{w}_1(t) = |u(t) - u_k(t)|, \quad \dot{w}_2(t) = |\beta(t) - \beta_k(t)|, \\ \quad (x(t), y(t), z(t)) \in (\bar{x}(t), \bar{x}(1), \bar{z}(t)) + \sigma B, \\ \quad (u(t), \beta(t)) \in \big(\bar{u}(t) + \sigma B\big) \times \mathcal{B}_\sigma(t), \\ \quad (x(0), y(0), z(0)) \in C \times \{0\}, \\ \quad (w_1(0), w_2(0)) = (0, 0, 0) \end{cases}$$

where all the equalities and inclusions but the last two are to be understood in an *almost everywhere* sense and

$$\pi_k(x, y, z, w_1, w_2) = \\ |x - x_k(0)| + |y - y_k(0)| + |z(1) - z_k(1)| \\ +w_1(1) + w_2(1).$$

Observe that for k sufficiently large, it can easily be shown that $\Psi_k(x_k(0), y_k, x_k(1), y_k, z_k(1)) > 0$.

Step 5: We apply Proposition 3.1 to the sequence of standard optimal control problems obtained in the previous step.

It asserts the existence of scalars η_k and r_k, vectors q_k, $e_k \in \mathbb{R}^n$, integrable function $\zeta_k : [0, 1] \to \mathbb{R}^{m_g}$ and an absolutely continuous function $p_k \in W^{1,1}$ such that:

(a) $\eta_k + |p_k(1)| = 1$

(b) $\eta_k \in [0, 1]$, $| e_k |= 1$

(c) $(-p_k(1) - (1 - \eta_k)e_k, -q_k + (1 - \eta_k)e_k, -r_k - (1 - \eta_k)) \in \varepsilon_k\big(B \times B \times B\big)$,

(d) $(p_k(0), q_k) \in N_C(x_k(0), y_k) +\eta_k \partial\, l(x_k(0), y_k(0)) + \epsilon_k(B \times B)$,

(e) $\zeta_k(t) \in \text{co } N_{\mathcal{B}_{\sigma_1(t)}}(\beta_k(t))$a.e.

(f) $(-\dot{p}_k(t), 0, \zeta_k(t)) \in \text{co } \partial \tilde{H}(t, x_k(t), p_k(t), r_k, u_k(t), \beta_k(t)) +\varepsilon_k\big(\{0\} \times \{0\} \times \{0\} \times B \times B \times B \times B\big)$ a.e.

where

$$\tilde{H}(t, x, p, r, u, \beta) = \\ p \cdot f_1(t, x, u, \beta) + r \cdot G^+(t, x, u, \beta).$$

Step 6 : We now consider $\varepsilon_k \to 0$ and we take limits to obtain necessary conditions for (P).

In this step HMF is crucial to assert the non triviality condition (i) in Theorem 5.1.

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