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Observer Design for a Class of Switched Systems

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Abstract— **The main contribution of this paper is a new approach to the design of state observers for a class of switched systems. This class of systems allows for switching among Linear Parameter Varying (LPV) systems affected by unmeasurable input/output disturbances. The proposed observers have guaranteed performance even during the switching transient, and "mitigate" the effect of unmeasured disturbance. Moreover, it is shown that observer design can be done using linear programming. Finally, a numerical example is presented which shows the efficacy of the proposed approach.**

I. Introduction

Recently, considerable effort has been put in the study of switched systems and, hence, the problem of obtaining full state information for control and diagnostic purposes has become an active research area. In this paper, an observer design methodology is presented which is applicable to a wide class of switched systems, which we refer to as *superdetectable systems*.

When Luenberger proposed a state observer design method in 1960s, observer design was only aimed at Linear Time Invariant (LTI) systems with full knowledge of the inputs and outputs [1]. However, in many instances, the plants depend on varying parameters and, at the same time, are affected by unknown disturbance and noise. To address these problems, several alternative approaches have been proposed. For example, in [2], an observer design method was proposed for LPV systems. Also, a switching observer was designed for switched systems in [3]. Moreover, the approaches taken in [4] and [5] addressed the problem of observer design for uncertain systems with persistent disturbances. Nevertheless, previous results on observer design for switched systems have never addressed the case where one has linear parameter varying plants subjected to unmeasurable disturbance and noise.

The approach taken in this paper addresses the problems of parameter uncertainty, system switching and disturbance attenuation simultaneously. It has its basis on the concept of *superstability* introduced in [6] and [7]. A superstable system has many "interesting" properties. For example, given any bounded input and bounded initial conditions, the output will be constrained within a certain bound, which can be computed beforehand. Moreover, this property still holds for LPV systems, provided that all "frozen" LTI systems are superstable [8]. These "nice" features provide the motivation for the approach taken in this paper.

The paper is organized as follows: In Section II, the notion of superstability and some preliminary results are given; The notion of superdetectable system is introduced in Section III; The structure of switched system and the corresponding switching observer are derived in Section IV; The performance of switching observers is discussed in Section V; Section VI develops an LP approach to observer design; Simulation results are shown in Section VII; Conclusions and possible directions for future work are presented in Section VIII.

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II. Notation and Preliminary Results

In the sequel, $\|\cdot\|_1$ denotes the 1-norm of a matrix, i.e., given a matrix $A = ((a_{ij}))_{n\times m} \in \mathbf{R}^{n\times m}$, $\|A\|_1 = \max_{1\le i\le n} \sum_{j=1}^{m} |a_{ij}|$. The ∞-norm of a vector is denoted by $\|\cdot\|_\infty$, i.e., given a vector $\mathbf{x} = ((x_i))_{n\times 1} \in \mathbf{R}^{n\times 1}$, $\|\mathbf{x}\|_\infty = \max_{1\le i\le n}(|x_i|)$. Moreover, $A(i,:)$ denotes the i-th row of matrix A. And given a real number c, let $(c)_+ = \max(0,c)$.

A. Superstability

A matrix $A = ((a_{ij}))_{n\times n} \in \mathbf{R}^{n\times n}$ is said to be *superstable* if

$$q \doteq \|A\|_1 < 1$$

where $1-q$ is called *the degree of superstability*. Furthermore, a discrete time system

$$\mathbf{x}_{k+1} = A_k\mathbf{x}_k + B_k\mathbf{u}_k \tag{1}$$

is *superstable* if matrix A_k is superstable for all k (see [7] for an in depth discussion on superstability). A superstable LPV system has a number of "interesting" properties.

Theorem 1: [8] *If a discrete-time LPV system (2) is superstable,*

$$\mathbf{x}_{k+1} = A(\mathbf{w}(k))\mathbf{x}_k + B(\mathbf{w}(k))\mathbf{u}_k \tag{2}$$

where $\mathbf{x}_k \in \mathbf{R}^n$, $\mathbf{u}_k \in \mathbf{R}^m$, $\mathbf{w}(k) \in \mathscr{W}$ and A,B are matrices of appropriate dimensions, then

(a) for $\mathbf{u}_k = 0$, $k = 0,1\ldots$, and any initial conditions $\mathbf{x}_{-1}$,

$$\|\mathbf{x}_k\|_\infty \le q^{k+1}\|\mathbf{x}_{-1}\|_\infty,$$

where $q = \max_{\mathbf{w}(k)\in\mathscr{W}} \|A(\mathbf{w}(k))\|_1$;

(b) for $\|\mathbf{u}_k\|_\infty \le 1$, $k = 0,1\ldots$, and $\|\mathbf{x}_{-1}\|_\infty \le \mu$

$$\|\mathbf{x}_k\|_\infty \le \mu,$$

where $\mu = \max_{\mathbf{w}(k),i} \frac{\|B(i,:)\|_1}{1-\|A(i,:)\|_1}$ *for all* $\mathbf{w}(k) \in \mathscr{W}$ *and* $1 \le i \le n$*. Here* μ *is called equalized performance.*

The results above suggest that for system (2) with bounded input $\|\mathbf{u}_k\|_\infty \le 1$, there exists an *invariant set*

$$Q = \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x}\|_\infty \le \mu\},$$

i.e, if $\|\mathbf{u}_k\|_\infty \le 1$ for all $k > 0$, and $\mathbf{x}_{k_1} \in Q$, then $\mathbf{x}_{k_2} \in Q$ for all $k_2 \ge k_1$.

III. Superdetectable Systems

In this section, we define the class of systems that is addressed in this paper: The class of superdetectable systems. Consider a system of form

$$\begin{aligned} \mathbf{x}_{k+1} &= A(\mathbf{w}(k))\mathbf{x}_k + B(\mathbf{w}(k))\mathbf{u}_k \\ \mathbf{y}_k &= C(\mathbf{w}(k))\mathbf{x}_k + D(\mathbf{w}(k))\mathbf{u}_k \end{aligned} \tag{3}$$

where $\mathbf{x}_k \in \mathbf{R}^n$ is the state vector, $\mathbf{u}_k \in \mathbf{R}^m$ is the command input, $\mathbf{y}_k \in \mathbf{R}^l$ is the output, $\mathbf{w}(k) \in \mathscr{W} \subset \mathbf{R}^s$ is a time varying parameter, and $A(\mathbf{w}(k))$, $B(\mathbf{w}(k))$, $C(\mathbf{w}(k))$ and $D(\mathbf{w}(k))$ are matrices of appropriate dimensions. Consider an observer of the form

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} &= A(\mathbf{w}(k))\hat{\mathbf{x}}_k + B(\mathbf{w}(k))\mathbf{u}_k + L(\mathbf{y}_k - \hat{\mathbf{y}}_k) \\ \hat{\mathbf{y}}_k &= C(\mathbf{w}(k))\hat{\mathbf{x}}_k + D(\mathbf{w}(k))\mathbf{u}_k \end{aligned} \tag{4}$$

where L is the observer gain of appropriate dimension. Therefore the error dynamics can be described as

$$\mathbf{e}_{k+1} = (A(\mathbf{w}(k)) - LC(\mathbf{w}(k)))\mathbf{e}_k. \tag{5}$$

We are now ready to provide the definition of the so-called superdetectable systems.

Definition 1: System (3) is called superdetectable if there exists an L *such that its observer error dynamics (5) is superstable, i.e.,*

$$\|A(\mathbf{w}(k)) - LC(\mathbf{w}(k))\|_1 < 1,$$

for all $\mathbf{w}(k) \in \mathscr{W}$.

IV. Problem formulation

We are now ready to provide a precise definition of the problem addressed in this paper. Consider a class of uncertain linear systems given by:

$$\begin{aligned} \mathbf{x}_{k+1} &= A_p(\mathbf{w}(k))\mathbf{x}_k + B_p(\mathbf{w}(k))\mathbf{u}_k + E_p(\mathbf{w}(k))\mathbf{d}_k \\ \mathbf{y}_k &= C_p(\mathbf{w}(k))\mathbf{x}_k + D_p(\mathbf{w}(k))\mathbf{u}_k + F_p(\mathbf{w}(k))\mathbf{v}_k \end{aligned} \tag{6}$$

where $\mathbf{x}_k \in \mathbf{R}^n$ is the state vector, $\mathbf{u}_k \in \mathbf{R}^{m_1}$ is the command input, $\mathbf{d}_k \in \mathbf{R}^{m_2}$ is the input disturbance (or sensor error), $\mathbf{y}_k \in \mathbf{R}^l$ is the output, $\mathbf{v}_k \in \mathbf{R}^{m_3}$ is the output disturbance (or actuator error), $p \in P = \{1, 2, \ldots, c\}$ denotes the number of the subsystems, and A_p, B_p, E_p, C_p, D_p, F_p are of the form

$$\begin{aligned} A_p(\mathbf{w}(k)) &= \sum_{i=1}^{s} A_{p,i} w_i(k), \quad B_p(\mathbf{w}(k)) = \sum_{i=1}^{s} B_{p,i} w_i(k), \\ E_p(\mathbf{w}(k)) &= \sum_{i=1}^{s} E_{p,i} w_i(k), \quad C_p(\mathbf{w}(k)) = \sum_{i=1}^{s} C_{p,i} w_i(k), \\ D_p(\mathbf{w}(k)) &= \sum_{i=1}^{s} D_{p,i} w_i(k), \quad F_p(\mathbf{w}(k)) = \sum_{i=1}^{s} F_{p,i} w_i(k), \end{aligned}$$

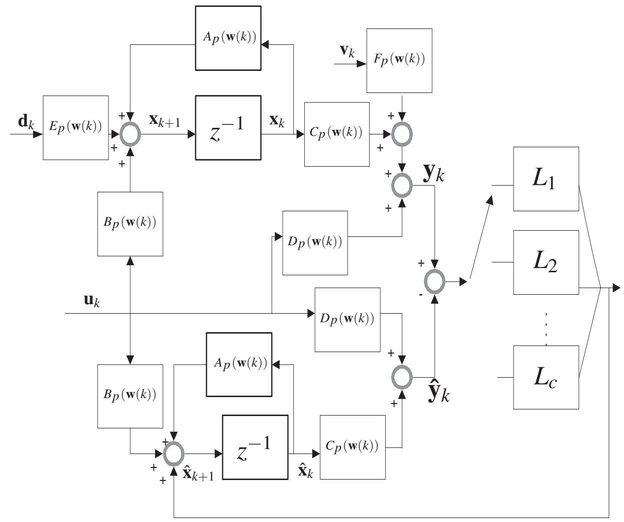

Fig. 1. The structure of the switching observer

where

$$\begin{aligned} \mathbf{w}(k) &= ((w_i(k)))_{s\times 1} \\ &\in \mathscr{W} = \{\mathbf{w}(k) : \ w_i(k) \ge 0, \ \sum_{i=1}^{s} w_i(k) = 1, \ k \ge 0\}. \end{aligned}$$

We aim at designing a switching observer of form (7) as depicted in Figure 1,

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} &= A_p(\mathbf{w}(k))\hat{\mathbf{x}}_k + B_p(\mathbf{w}(k))\mathbf{u}_k + L_p(\mathbf{y}_k - \hat{\mathbf{y}}_k) \\ \hat{\mathbf{y}}_k &= C_p(\mathbf{w}(k))\hat{\mathbf{x}}_k + D_p(\mathbf{w}(k))\mathbf{u}_k \end{aligned} \tag{7}$$

where $\hat{\mathbf{x}}_k \in \mathbf{R}^n$ is the estimated state vector, $\hat{\mathbf{y}}_k \in \mathbf{R}^l$ is the estimated output and L_p is the observer gain used when the current plant is the p_{th} plant. Consequently, the error dynamics can be expressed as

$$\mathbf{e}_{k+1} = \bar{A}\,\mathbf{e}_k + \bar{B}\,\bar{\mathbf{d}}_k, \tag{8}$$

where

$$\begin{aligned} \bar{A} &= A_p(\mathbf{w}(k)) - L_p C_p(\mathbf{w}(k)), \\ \bar{B} &= \begin{bmatrix} E_p(\mathbf{w}(k)) & L_p F_p(\mathbf{w}(k)) \end{bmatrix}, \\ \mathbf{e}_k &= \mathbf{x}_k - \hat{\mathbf{x}}_k, \\ \bar{\mathbf{d}}_k &= \begin{bmatrix} \mathbf{d}_k \\ \mathbf{v}_k \end{bmatrix}. \end{aligned}$$

Our design goal is to find L_p for each p such that the estimated error is minimized for any persistent-but-bounded disturbance $\bar{\mathbf{d}}_k$ and any $\mathbf{w}(k) \in \mathscr{W}$.

V. Superstable observer for switched systems

As mentioned in Section I, the disturbance rejection property of the superstable systems provides the motivation for the approach taken in this paper. More precisely, assume that for each subsystem p we can find an L_p such that

$$\|\bar{A}\|_1 = \|A_p(\mathbf{w}(k)) - L_p C_p(\mathbf{w}(k))\|_1 < 1 \tag{9}$$

for all admissible $\mathbf{w}(k) \in \mathscr{W}$. Then, one obtains superstable error dynamics and, hence, one would expect "good" disturbance attenuation.

In order to design a switching observer of form (7), we need to find a proper observer gain L_p for all possible time varying parameters $\mathbf{w}(k)$ in the admissible set $\mathscr{W}$ defined in Section IV. This can be easily addressed using the result below.

Theorem 2: For a fixed p, if one can find an L_p such that for all s vertices of subsystem p,

$$A_{p,i} - L_p C_{p,i}$$

is superstable, where $i = 1, \ldots, s$, then the system (8) is superstable.

Proof: According the definition of superstability, system (8) is superstable if and only if

$$\left\|\bar{A}\right\|_1 = \left\|A_p(\mathbf{w}(k)) - L_p C_p(\mathbf{w}(k))\right\|_1 < 1$$

It is known that for all $i = 1, \ldots, s$, $\left\|A_{p,i} - L_p C_{p,i}\right\|_1 < 1$, then

$$\begin{aligned}
\left\|A_p(\mathbf{w}(k)) - L_p C_p(\mathbf{w}(k))\right\|_1 &= \left\|\sum_{i=1}^{s} A_{p,i} w_i(k) - L_p \sum_{i=1}^{s} C_{p,i} w_i(k)\right\|_1 \\
&= \left\|\sum_{i=1}^{s} (A_{p,i} - L_p C_{p,i}) w_i(k)\right\|_1 \\
&\le \sum_{i=1}^{s} w_i(k) \left\|A_{p,i} - L_p C_{p,i}\right\|_1 \\
&< \sum_{i=1}^{s} w_i(k) = 1
\end{aligned}$$

■

Therefore, designing observer gain L_p for all $\mathbf{w}(k)$ of the p_{th} subsystem is equivalent to design L_p for the extreme points of the p_{th} subsystem.

The observer gain L_p is chosen according to which of the switched systems is currently "active". Nevertheless, the switching transient may cause unwanted oscillations of the estimation error. The approach taken in this paper leads to *a priori* bounds on the magnitude of these oscillations. The precise result is given below.

Theorem 3: Consider a superstable error system of the form (8), with any initial conditions $\mathbf{e}_{-1} \in \mathbf{R}^n$, and bounded disturbance $\left\|\bar{\mathbf{d}}_k\right\|_\infty \le 1$, $k \ge 0$. Then,

$$\|\mathbf{e}_k\|_\infty \le \mu + q^{k+1}(\|e_{-1}\|_\infty - \mu)_+ \tag{10}$$

where

$$q = \max_{p,i,j} \left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1,$$

$$\mu = \max_{p,i,j} \frac{\left\|\left[E_{p,i}(j,:) \quad L_p(j,:)F_{p,i}\right]\right\|_1}{1 - \left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1}.$$

Proof: From

$$\mu = \max_{p,i,j} \frac{\left\|\left[E_{p,i}(j,:) \quad L_p(j,:)F_{p,i}\right]\right\|_1}{1 - \left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1},$$

we can get for any p', i' and j',

$$\begin{gathered}
\mu \ge \frac{\left\|\left[E_{p',i'}(j',:) \quad L_p(j',:)F_{p',i'}\right]\right\|_1}{1 - \left\|A_{p',i'}(j',:) - L_p(j',:)C_{p',i'}\right\|_1} \\
\Updownarrow \\
\begin{aligned}
\mu \ge &\left\|\left[E_{p',i'}(j',:) \quad L_p(j',:)F_{p',i'}\right]\right\|_1 \\
&+ \mu \left\|A_{p',i'}(j',:) - L_p(j',:)C_{p',i'}\right\|_1
\end{aligned}
\end{gathered}$$

We now proceed by induction. First, consider the case $k = 0$. Then,

$$\begin{aligned}
\|\mathbf{e}_0\|_\infty &= \left\|\bar{A}\,\mathbf{e}_{-1} + \bar{B}\,\bar{\mathbf{d}}_{-1}\right\|_\infty \\
&\le \sum_{i=1}^{s} w_i(k) \left\|(A_{p,i} - L_p C_{p,i})\mathbf{e}_{-1} + \left[E_{p,i} \quad L_p F_{p,i}\right]\bar{\mathbf{d}}_{-1}\right\|_\infty \\
&\le \max_{p,i,j}\{\left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1 \|\mathbf{e}_{-1}\|_\infty \\
&\quad + \left\|\left[E_{p,i}(j,:) \quad L_p(j,:)F_{p,i}\right]\right\|_1 \left\|\bar{\mathbf{d}}_{-1}\right\|_\infty\} \\
&\le \max_{p,i,j}\{\left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1 (\|\mathbf{e}_{-1}\|_\infty - \mu + \mu) \\
&\quad + \left\|\left[E_{p,i}(j,:) \quad L_p(j,:)F_{p,i}\right]\right\|_1 \\
&\le \max_{p,i,j}\{\left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1 (\|\mathbf{e}_{-1}\|_\infty - \mu) + \mu\} \\
&\le \mu + q(\|\mathbf{e}_{-1}\|_\infty - \mu)_+
\end{aligned}$$

Now, to complete the proof, assume that equation (10) is valid for $0, \ldots, k-1$. Then,

$$\begin{aligned}
\|\mathbf{e}_k\|_\infty &= \left\|\bar{A}\,\mathbf{e}_{k-1} + \bar{B}\,\bar{\mathbf{d}}_{-1}\right\|_\infty \\
&\le \sum_{i=1}^{s} w_i(k) \left\|(A_{p,i} - L_p C_{p,i})\mathbf{e}_{k-1} + \left[E_{p,i} \quad L_p F_{p,i}\right]\bar{\mathbf{d}}_{-1}\right\|_\infty \\
&\le \max_{p,i,j}\{\left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1 \|\mathbf{e}_{k-1}\|_\infty \\
&\quad + \left\|\left[E_{p,i}(j,:) \quad L_p(j,:)F_{p,i}\right]\right\|_1 \left\|\bar{\mathbf{d}}_{-1}\right\|_\infty\} \\
&\le \max_{p,i,j}\{\left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1 [\mu + q^k(\|\mathbf{e}_{-1}\|_\infty - \mu)_+] \\
&\quad + \left\|\left[E_{p,i}(j,:) \quad L_p(j,:)F_{p,i}\right]\right\|_1\} \\
&\le \max_{p,i,j}\{\mu + \left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1 [q^k(\|\mathbf{e}_{-1}\|_\infty - \mu)_+]\} \\
&\le \mu + q^{k+1}(\|\mathbf{e}_{-1}\|_\infty - \mu)_+
\end{aligned}$$

■

The results above show that although in practice the initial conditions of the error states are unknown, the ∞-norm of the estimated error will converge to a value below μ. Moreover, if ∞-norm of the initial condition of the error states is less than or equal to the bound μ, i.e. $\|\mathbf{e}_{-1}\|_\infty \le \mu$, then from equation (10), we obtain the following result (can be also derived directly from Theorem 1, see [8]):

Lemma 1: Consider a superstable error system of the form (8), with bounded initial conditions $\|\mathbf{e}_{-1}\|_\infty \le \mu$, and bounded disturbance $\left\|\bar{\mathbf{d}}_k\right\|_\infty \le 1$, $k \ge 0$. Then,

$$\|\mathbf{e}_k\|_\infty \le \mu,$$

where

$$\mu = \max_{p,i,j} \frac{\left\|\left[E_{p,i}(j,:) \quad L_p(j,:)F_{p,i}\right]\right\|_1}{1 - \left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1}. \tag{11}$$

for all $i = 1, \ldots, s$, $p = 1, \ldots, c$ and $j = 1, \ldots, n$.

From Lemma 1, it can be seen that the invariant set of the error dynamics (8) for all $\left\|\bar{\mathbf{d}}_k\right\|_\infty \leq 1$ is

$$Q_e = \{\mathbf{e} \in \mathbf{R} : \|\mathbf{e}\|_\infty \leq \mu\},$$

i.e., with the bounded disturbance $\left\|\bar{\mathbf{d}}_k\right\|_\infty \leq 1$, if at time instance k_1, $\left\|\mathbf{e}_{k_1}\right\|_\infty \leq \mu$, then $\left\|\mathbf{e}_{k_2}\right\|_\infty \leq \mu$, for all $k_2 \geq k_1$, independently of the switching order and value of $\mathbf{w}(k)$. Therefore, switching among the observer gains L_p according to which subsystem is activated will not lower the performance of the switching observer.

A. Remarks

The equalized performance μ in equation (11) is the overall equalized performance of the switched system. This is valid for any switching sequence and any value of the time varying parameters $\mathbf{w}(k)$. Hence, it is inherently a conservative estimate of the performance of the observer. More precisely, given a specific switching sequence, we have:

(a) Each subsystem $p = p_1$ exhibits equalized performance

$$\mu_{p_1} = \max_{i,j} \frac{\left\| \begin{bmatrix} E_{p_1,i}(j,:) & L_{p_1}(j,:)F_{p_1,i} \end{bmatrix} \right\|_1}{1 - \left\|A_{p_1,i}(j,:) - L_{p_1}(j,:)C_{p_1,i}\right\|_1} \leq \max_{p,i,j} \frac{\left\| \begin{bmatrix} E_{p,i}(j,:) & L_p(j,:)F_{p,i} \end{bmatrix} \right\|_1}{1 - \left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1} = \mu.$$

(b) When switching from subsystem p_1 to p_2 at time instant k_1, if $\mu_{p_1} \leq \mu_{p_2}$, and assuming that subsystem p_1 has already achieved equalized performance μ_{p_1}, then subsystem p_2 can obtain equalized performance μ_{p2} immediately, i.e., $\left\|\mathbf{e}_{k_2}\right\|_\infty \leq \mu_{p2}$ for $k_2 \geq k_1$; if $\mu_{p_1} > \mu_{p_2}$, and assume that subsystem p_1 has already achieved equalized performance μ_{p_1},then subsystem p_2 can achieve

$$\left\|\mathbf{e}_{k_2}\right\|_\infty \leq \mu_{p_2} + q_{p_2}^{k_2-k_1+1}(\mu_{p_1} - \mu_{p_2})_+ \leq \mu_{p_1}$$

where

$$q_{p_2} = \max_{i,j} \left\|A_{p_2,i}(j,:) - L_{p_2}(j,:)C_{p_2,i}\right\|_1.$$

for all $i = 1,\ldots,s$ and $j = 1,\ldots,n$.

(c) Switching between subsystems p, where $p = 1,\ldots,c$, can achieve equalized performance

$$\mu = \max_p(\mu_p) = \max_p \left\{ \max_{i,j} \frac{\left\| \begin{bmatrix} E_{p,i}(j,:) & L_p(j,:)F_{p,i} \end{bmatrix} \right\|_1}{1 - \left\|A_{p,i}(j,:) - L_p(j,:)C_{p,i}\right\|_1} \right\}$$

This μ is the upper bound on the performance and it can only be reached if the worst input and output disturbance, and the worst parameters occur at the same time. Usually, the performance of the estimated error system is much better than this upper bound. This can be seen in the results of the example presented in Section VII.

Given the comments above, it is desirable to design the matrices L_p in order to obtain the best possible performance for each of the subsystems. A linear programming approach to solve this problem is presented in the next section.

VI. LP Formulation of Optimal Observer Design

In this section, a switching observer with gains L_p, where $p = 1,\ldots,c$, is designed using an LP approach.

Lemma 2: Consider the system (6) and the observer (7). If there exist L_p, for $p = 1,\ldots,c$, satisfying the set of linear inequalities (12), then the error dynamics of the switching observer have equalized performance $\mu = \max_p(\mu_p)$.

$$\begin{aligned}
&\Theta_{p,i,ef} \geq 0 \\
&\Theta_{p,i,ef} - \Phi_{p,i,ef} \geq 0 \\
&\Theta_{p,i,ef} + \Phi_{p,i,ef} \geq 0 \\
&\Psi_{p,i,eg} \geq 0 \\
&\Psi_{p,i,eg} + \Lambda_{p,i,eg} \geq 0 \\
&\Psi_{p,i,eg} - \Lambda_{p,i,eg} \geq 0 \\
&1 - \sum_{n'=1}^{n} \Theta_{p,i,en'} \geq 0 \\
&\mu_p - \mu_p \sum_{n'=1}^{n} \Theta_{p,i,en'} - \sum_{m'=1}^{m} \Psi_{p,i,em'} \geq 0
\end{aligned} \tag{12}$$

for all $p = 1,\ldots,c$, $i = 1,\ldots,s$, $e = 1,\ldots,n$, $f = 1,\ldots,n$, $g = 1,\ldots,m$, $\Theta_{p,i} \in \mathbf{R}^{n\times n}$, $\Phi_{p,i} = A_{p,i} - L_pC_{p,i} \in \mathbf{R}^{n\times n}$, $\Psi_{p,i} \in \mathbf{R}^{n\times m}$, $\Lambda_{p,i} = \begin{bmatrix} E_{p,i} & L_pF_{p,i} \end{bmatrix} \in \mathbf{R}^{n\times m}$ and $\Theta_{p,i,ef}$, $\Psi_{p,i,ef}$, $\Phi_{p,i,ef}$, $\Lambda_{p,i,ef}$ denote the element (e, f) in the matrices $\Theta_{p,i}$, $\Psi_{p,i}$, $\Phi_{p,i}$ and $\Lambda_{p,i}$.

Proof:

$$\left.\begin{aligned}
\Theta_{p,i,ef} &\geq 0 \\
\Theta_{p,i,ef} - \Phi_{p,i,ef} &\geq 0 \\
\Theta_{p,i,ef} + \Phi_{p,i,ef} &\geq 0
\end{aligned}\right\} \Longleftrightarrow -\Theta_{p,i,ef} \leq \Phi_{p,i,ef} \leq \Theta_{p,i,ef}$$

$$\left.\begin{aligned}
\Psi_{p,i,ef} &\geq 0 \\
\Psi_{p,i,ef} - \Lambda_{p,i,ef} &\geq 0 \\
\Psi_{p,i,ef} + \Lambda_{p,i,ef} &\geq 0
\end{aligned}\right\} \Longleftrightarrow -\Psi_{p,i,eg} \leq \Lambda_{p,i,ef} \leq \Psi_{p,i,ef}$$

Moreover,

$$1 - \sum_{n'=1}^{n} \Theta_{p,i,en'} \geq 0$$
$$\Updownarrow$$
$$\max_{i,e} \left\|A_{p,i}(e,:) - L_p(e,:)C_{p,i}\right\|_1 \leq 1$$

which guarantees the superstability of each error subsystem p. And

$$\mu_p - \mu_p \sum_{n'=1}^{n} \Theta_{p,i,en'} - \sum_{m'=1}^{m} \Psi_{p,i,em'} \geq 0$$
$$\Updownarrow$$
$$\mu_p \geq \max_{i,e} \frac{\left\| \begin{bmatrix} E_{p,i}(e,:) & L_p(e,:)F_{p,i} \end{bmatrix} \right\|_1}{1 - \left\|A_{p,i}(e,:) - L_p(e,:)C_{p,i}\right\|_1}$$

■

TABLE I

TIME TABLE OF SYSTEM NUMBER

Time period (second)	0-5	5-5.1	5.1-9	9-9.1	9.1-15
System number	2	1	2	1	2

Hence, given the fact that we can address the performance of each switched system independently, a way to optimize the performance of the observer is to design optimal observers for each one of the switched systems. More precisely, optimal observer design can be done by solving c generalized eigenvalue problems; i.e., for $p = 1, 2, \ldots, c$ determine

$$\begin{array}{cc} \min & \{\mu_p\}, \\ \text{subject to} & (12). \end{array}$$

VII. NUMERICAL EXAMPLE

Consider a system given by:

$$\begin{aligned} \mathbf{x}_{k+1} &= A_p(w(k))\mathbf{x}_k + B_p(w(k))\mathbf{u}_k + E_p(w(k))\mathbf{d}_k \\ \mathbf{y}_k &= C_p(w(k))\mathbf{x}_k + D_p(w(k))\mathbf{u}_k + F_p(w(k))\mathbf{v}_k \end{aligned}$$

where $\mathbf{x}_k \in \mathbf{R}^2$, $\mathbf{u}_k \in \mathbf{R}^2$, $\mathbf{d}_k \in \mathbf{R}^2$, $\mathbf{y}_k \in \mathbf{R}$, $\mathbf{v}_k \in \mathbf{R}$, $p \in \{1, 2\}$, and $w(k) \in [0, 1]$.

$$\begin{array}{ll}
A_p(w(k)) = w(k)A_{p,1} + (1 - w(k))A_{p,2} & \text{, with} \\
A_{1,1} = \begin{bmatrix} 0.9 & 0.5 \\ -0.3 & 1 \end{bmatrix}, & A_{1,2} = \begin{bmatrix} 0.85 & 0.55 \\ -0.35 & 1 \end{bmatrix}, \\
A_{2,1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.8 & 0 \end{bmatrix}, & A_{2,2} = \begin{bmatrix} 0.35 & -0.25 \\ -0.8 & 0.05 \end{bmatrix}; \\
B_p(w(k)) = w(k)B_{p,1} + (1 - w(k))B_{p,2} & \text{, with} \\
B_{1,1} = \begin{bmatrix} -0.2 & 0.4 \\ 0.1 & -0.5 \end{bmatrix}, & B_{1,2} = \begin{bmatrix} -0.25 & 0.45 \\ 0.15 & -0.55 \end{bmatrix}, \\
B_{2,1} = \begin{bmatrix} 0.5 & 0 \\ -0.1 & 0.6 \end{bmatrix}, & B_{2,2} = \begin{bmatrix} 0.53 & -0.05 \\ -0.05 & 0.56 \end{bmatrix}; \\
E_p(w(k)) = w(k)E_{p,1} + (1 - w(k))E_{p,2} & \text{, with} \\
E_{1,1} = \begin{bmatrix} -0.02 & 0.04 \\ 0.01 & -0.05 \end{bmatrix}, & E_{1,2} = \begin{bmatrix} -0.025 & 0.045 \\ 0.015 & -0.055 \end{bmatrix}, \\
E_{2,1} = \begin{bmatrix} 0.05 & 0 \\ -0.01 & 0.06 \end{bmatrix}, & E_{2,2} = \begin{bmatrix} 0.053 & -0.005 \\ -0.005 & 0.056 \end{bmatrix}; \\
C_p(w(k)) = w(k)C_{p,1} + (1 - w(k))C_{p,2} & \text{, with} \\
C_{1,1} = \begin{bmatrix} 1 & 2 \end{bmatrix}, & C_{1,2} = \begin{bmatrix} 1.1 & 1.9 \end{bmatrix}, \\
C_{2,1} = \begin{bmatrix} 1 & 2 \end{bmatrix}, & C_{2,2} = \begin{bmatrix} 1.1 & 1.9 \end{bmatrix}; \\
D_p(w(k)) = w(k)D_{p,1} + (1 - w(k))D_{p,2} & \text{, with} \\
D_{1,1} = \begin{bmatrix} 2 & 3 \end{bmatrix}, & D_{1,2} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \\
D_{2,1} = \begin{bmatrix} 1.4 & 2.5 \end{bmatrix}, & D_{2,2} = \begin{bmatrix} -2 & 2.5 \end{bmatrix}. \\
F_p(w(k)) = w(k)F_{p,1} + (1 - w(k))F_{p,2} & \text{, with} \\
F_{1,1} = \begin{bmatrix} 0.03 \end{bmatrix}, & F_{1,2} = \begin{bmatrix} 0.07 \end{bmatrix}, \\
F_{2,1} = \begin{bmatrix} 0.05 \end{bmatrix}, & F_{2,2} = \begin{bmatrix} 0.04 \end{bmatrix}.
\end{array}$$

Using the Linear Matrix Inequality (LMI) toolbox in Matlab, we optimize L_1 and L_2 for subsystem 1 and subsystem 2. The solutions are

$$L_1 = \begin{bmatrix} 0.3045 \\ 0.5263 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.0738 \\ 0.0070 \end{bmatrix}$$

with performance values $\mu_1 = 1.504$, $\mu_2 = 0.394$, and

$$\hat{\mu} = \max_p \{\mu_p\} = 1.504.$$

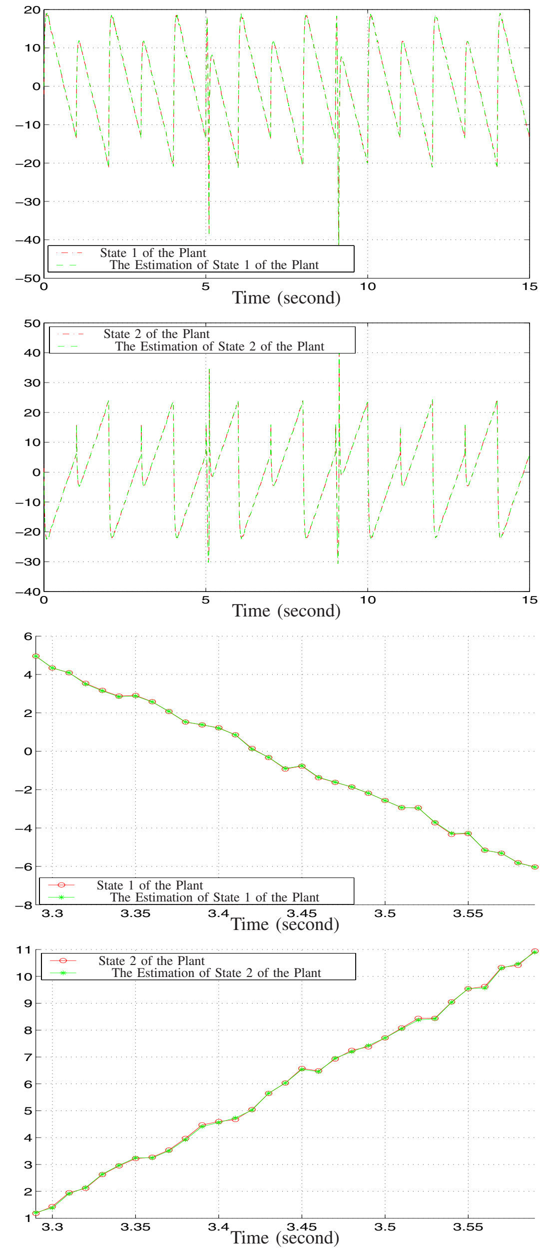

Fig. 2. The trajectories of the states of the plant and the observer

As mentioned before, we use the observer structure depicted in Figure 1. The command inputs are one sawtooth wave with amplitude of 15 and frequency of 1 Hz and one square wave with amplitude of 10 and frequency of 0.5 Hz. The disturbance inputs are two unit-amplitude uniformly-distributed random signals. The disturbance in the output is a unit-amplitude uniformly-distributed random signal. The initial conditions of the system states and the states of the estimator are $[-3, 2.5]$ and $[0, 0]$ respectively. Moreover the subsystems are chosen according to the table (I), and parameter $w(k)$ is a random number uniformly distributed in $[0, 1]$. The trajectories of the states and their estimates are plotted in Figure (2).

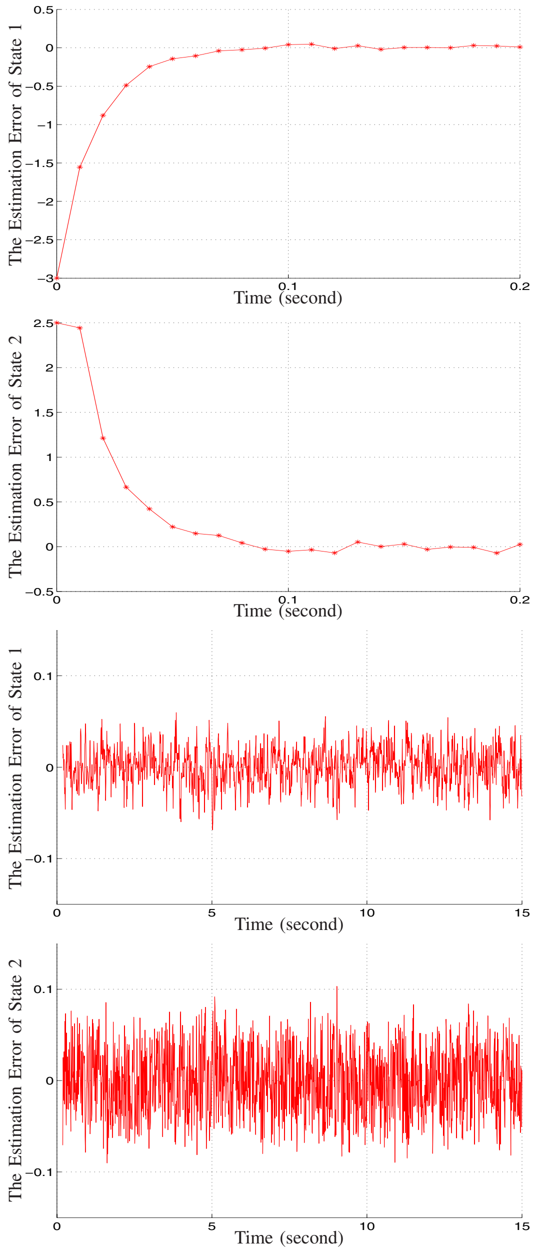

Fig. 3. The trajectories of the estimation error

As predicted by the results in this paper, one has first a transient behavior where the estimation error starts at $[-3, 2.5]$ and converges to the invariant set. After that, the estimation error remains in this invariant set where the ∞-norm of the error is below $\mu = 1.504$. The evolution of the estimation error is depicted in Figure (3).

VIII. Conclusions

The main topic of this paper is observer design for a class of of systems which we refer to as superdetectable systems. The procedure presented is applicable to switched LPV systems and it mitigates the effect of unmeasured input/output disturbances. Moreover, observer design can be done using an LP approach.

Effort is now being put in the extension of the proposed approach to the design of observers for linear and nonlinear uncertain systems.

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