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LMI based robust output feedback MPC

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Abstract—**This paper presents the synthesis of a robust model predictive controller (MPC) derived from an input-output representation with uncertainty. According to MPC approach, at each sampling time, the controller is calculated by minimizing an upper bound of a quadratic cost function over infinite time horizon. The optimization problem and the associated constraints on the input and output process are formulated in terms of linear matrix inequalities (LMI).**

From the input-output representation, an extended state space model is constructed, where the state is composed of present and past values of the system's inputs and outputs. The control is obtained as a function of known output and input signals, hence there is no need for any estimate of the unmeasured states, as it would be necessary in the "traditional" state space modelization. The proposed algorithm is illustrated on numerical examples.

I. Introduction

MPC solves an on-line optimization problem at each sampling time to compute the control law, therefore, it optimizes an open-loop control profile over the prediction horizon. Although more than one input move is computed, the controller only uses the first one. At the next sampling time, the optimization problem is solved again with new measurements, and the control input is updated [1], [2]. Since its introduction much research has been done regarding feasibility and stability [3]-[5].

The MPC performance depends on the accuracy of the process model. Since models are only approximations of real processes, it is important to look for MPC being robust to model uncertainty, see for instance: [6]-[8].

Most of the former results on robust MPC rely on state feedback control. One of the aims of this paper is to broaden these results to robust output feedback control in the MPC framework.

The predictive approach proposed in this paper, is based on the full state-feedback methodology described in [9], also developed by others autors [10]-[12], but only the measurable output is used.

The given approach for robust MPC is developed in the context of Semi Definite Programming (SDP) with LMI [13]. Due to the current status of LMI solvers and the continuing improvements under progress, such an approach seems reasonably suited to on line implementation and industrial applications, which is one of the main issues for MPC. Input/output constraints can be included using the concept of invariant ellipsoid in the form of additional LMIs. It is also possible to include other types of LMI constraints, i.e., those related to H_2 or H_∞ norm minimization or pole placement.

An equivalent extended state space model is derived from the input-output transfer representation. The model so obtained has the advantage that all states are available (measured) since they are constituted by past inputs and output signals. Thus, it is possible to use state feeedback robust MPC controller algorithms to derive the control.

In the traditional quadratic approach (the one used by [9]), the optimization problem is a guaranteed cost type which is solved by using a single "Lyapunov" matrix for all possible realizations of the system in its uncertainty domain. Recent work [14] makes it possible to reduce the conservatism of the quadratic approach by defining less restrictive conditions with parameter dependant Lyapunov functions. The conservatism reduction follows from the fact that in these new conditions, there is a decoupling between the parameter Lyapunov matrix and the matrices included in the controller determination.

II. Problem Statement

Consider the discrete time uncertain linear time invariant system represented by:

$$\alpha\left(z^{-1}\right)y(k) = \beta\left(z^{-1}\right)z^{-1}u(k) \tag{1}$$

where $u(k) \in \Re^m$ and $y(k) \in \Re^q$ are respectively, the input and output of the system, and

$$\begin{aligned} \alpha(z^{-1}) &= I_q + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + \alpha_{na} z^{-na} \\ \beta(z^{-1}) &= \beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots + \beta_{nb} z^{-nb} \end{aligned} \tag{2}$$

where α_i and β_j are matrices of dimension qxq and qxm, respectively. For each one of these matrices a polytopic uncertain domain Γ is defined.

$$\alpha_i \in \Gamma_{\alpha i}\,;\, \beta_j \in \Gamma_{\beta j} \tag{3}$$

with

$$\Gamma_{\alpha i} := Co\{\alpha_{i1}\ \ \alpha_{i2}\ \ \cdots\ \ \alpha_{iN_{\alpha i}}\},\ i = 1,2,\ldots,n_a \tag{4}$$

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and

$$\Gamma_{\beta j} := Co\{\beta_{j1} \;\; \beta_{j2} \;\; \cdots \;\; \beta_{jN_{\beta j}}\}, \; j = 1,2,\ldots,n_b \tag{5}$$

where $N_{\alpha i}$ and $N_{\beta j}$ are the number for the vertices of each α_i and β_j matrices respectively.

Remark 1. *$Co\{x_1 \;\; x_2 \;\; \cdots \;\; x_N\}$ stands for the convex hull of the vertices: $x_1, x_2, \cdots, x_N$.*

According to the MPC strategy, at each sampling time, the optimization block updates the controller parameters, by minimizing a quadratic cost function subject to constraints, uncertainty and partial information.

A. Extended state space model

The system (1), can also be expressed by the following difference equations system:

$$y(k)+\sum_{i=1}^{na}\alpha_i y(k-i)=\sum_{i=1}^{nb+1}\beta_{i-1}u(k-i) \tag{6}$$

An extended state space representation for the preceding model, can readily be obtained by taking as the state vector the present and past values of the system's inputs and outputs ([1], [2]), that is:

$$\begin{aligned}\tilde{x}(k) = \big[y(k)^T \;\; y(k-1)^T \;\; \cdots \;\; y(k-na+1)^T \;\; \ldots \\ \ldots \; u(k-1)^T \;\; u(k-2)^T \;\; \cdots \;\; u(k-nb)^T \big]^T\end{aligned} \tag{7}$$

An equivalent state space representation is then given by:

$$\begin{aligned}\tilde{x}(k+1) &= \tilde{A}\tilde{x}(k) + \tilde{B}u(k) \\ y(k) &= \tilde{C}\tilde{x}(k)\end{aligned} \tag{8}$$

where

$$\begin{aligned}\tilde{A} &= \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{na-1} & -\alpha_{na} & \beta_1 & \cdots & \beta_{nb-1} & \beta_{nb} \\ I_q & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_q & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_q & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I_m & 0 \end{bmatrix}, \\ \tilde{B} &= \left[\beta_0^T \;\; 0 \; 0 \cdots 0 \; I_m \;\; 0 \cdots 0\right]^T, \\ \tilde{C} &= \left[I_q \;\; 0 \cdots 0 \; 0 \; 0 \cdots 0 \; 0\right]\end{aligned} \tag{9}$$

I_q, I_m are respectively the q and m identity matrices.

Although the extended model is higher dimension than the original system, an obvious advantage arises from the fact that, at each sampling time, the extended state vector is known.

According to (3)-(5), $\tilde{A}$ and $\tilde{B}$ are such that they belong to an uncertain domain:

$$\left[\tilde{A} \mid \tilde{B}\right] \in \Omega \tag{10}$$

where

$$\Omega := Co\left\{\left[\tilde{A}_1 \mid \tilde{B}_1\right], \left[\tilde{A}_2 \mid \tilde{B}_2\right], \ldots, \left[\tilde{A}_N \mid \tilde{B}_N\right]\right\} \tag{11}$$

The vertices $\left[\tilde{A}_i \mid \tilde{B}_i\right]$ are obtained from the combination of the vertices of $\Gamma_{\alpha i}$ and $\Gamma_{\beta j}$. Let N be the number of these vertices. Thus, any couple $\left[\tilde{A}, \tilde{B}\right]$ is a convex combination of the vertices:

$$\lambda_j \geq 0, \sum_{j=1}^{N}\lambda_j = 1, \; \left[\tilde{A} \mid \tilde{B}\right] = \sum_{j=1}^{N}\lambda_j\left[\tilde{A}_j \mid \tilde{B}_j\right] \tag{12}$$

Note that unmeasured output disturbance can be considered in the MPC strategy as an uncertain domain such as above mentioned.

The following shows that indeed the robust output MPC can be solved through the extended state feedback problem. Let:

$$u(k) = K_C(k)\tilde{x}(k) \tag{13}$$

B. Objective function

Consider an infinite horizon objective function:

$$\begin{aligned}J_\infty(k) = \sum_{i=0}^{\infty}\begin{bmatrix}\tilde{x}(k+i/k) \\ u(k+i/k)\end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R\end{bmatrix}\begin{bmatrix}\tilde{x}(k+i/k) \\ u(k+i/k)\end{bmatrix}, \\ Q \geq 0, \; R > 0\end{aligned} \tag{14}$$

A classical approach in robust control is to look for a guaranteed cost control which, in the MPC strategy, can be defined at each sampling time. This asks for the definition of an upper bound for the objective cost over the uncertainty domain Ω. Let:

$$V(i,k) = \tilde{x}(k+i/k)^T S(k)\tilde{x}(k+i/k), \; S(k) > 0 \tag{15}$$

Assuming a stable closed loop system, an upper bound for the performance index $J_\infty(k)$, is obtained whenever the following inequality is satisfied:

$$\begin{aligned}& V(i+1,k) - V(i,k) \leq \\ & -\left(\tilde{x}(k+i/k)^T Q\tilde{x}(k+i/k) + u(k+i/k)^T R u(k+i/k)\right), \\ & \forall \left[\tilde{A} \mid \tilde{B}\right] \in \Omega\end{aligned} \tag{16}$$

Adding inequalities (16) from $i = 0$ up to $i = \infty$ with the assumption of asymptotic stability ($\hat{x}(\infty/k) = 0$), yields:

$$V(0,k) \geq J_\infty(k), \; \forall \tilde{A}, \tilde{B} \in \Omega \tag{17}$$

The approach is to determine a state feedback control minimizing the upper bound $V(0,k)$ in (17). The algorithm proposed, as will be shown later, will be decreasing in time, i.e., $V(0,k+1) \le V(0,k)$.

The proposed algorithm is roughly as follow:

Step 1.- Solve:

$$\min_{G,\tilde{S},Y} V(0,k)$$

subject to (8), (16). (18)

Step 2.- Apply $u^*(k) = K_C^*(k)\tilde{x}(k)$ where K_C^* is the solution of the optimization problem.

Step 3.- Make $k = k+1$ and go back to *Step 1*.

C. Constraints definitions

Real processes have constraints in their variables. Input constraints normally represent physical limits (such as valve saturation, power limitations, etc). We will consider these constraints through the Euclidean norm:

$$\|u(k+i/k)\|_2 \le u_{\max},\ \ i \ge 0 \tag{19}$$

Output constraints represent performance or safety requirements. They will by considered similarly, through the Euclidean norm:

$$\|y(k+i/k)\|_2 \le y_{\max},\ i \ge 1 \tag{20}$$

Vector $y(k+i/k)$, represents system's predicted output at time $k+i$, based on the output at time k, $y(k/k)$. The output constraints are imposed on future values ($i>0$), because it does not make any sense to apply it to the actual value ($i=0$).

III. Main Result

The uncertain discrete time system is stabilized by output MPC if, at each sampling time, the following theorem is satisfied.

Theorem 1. *The uncertain discrete system (1) is output MPC robustly stabilized by the algorithm proposed in section B, if there exists matrices $Y \in \Re^{m\,x\,n}$, $G \in \Re^{n\,x\,n}$ and positive definite symmetric matrices $\tilde{S}_j \in \Re^{n\,x\,n}$ solutions of the following problem:*

$$\min_{G,\tilde{S},Y} \gamma \tag{21}$$

subject to

$$\begin{bmatrix} \tilde{S}_j & \tilde{x}(k/k) \\ \tilde{x}(k/k)^T & I \end{bmatrix} > 0,\ \forall j = 1,\ldots,N \tag{22}$$

$$\begin{bmatrix} G+G^T-\tilde{S}_j & G^T\tilde{A}_j^T + Y^T\tilde{B}_j^T & G^TQ^{1/2} & Y^TR^{1/2} \\ \tilde{A}_jG+\tilde{B}_jY & \tilde{S}_j & 0 & 0 \\ Q^{1/2}G & 0 & \gamma I & 0 \\ R^{1/2}Y & 0 & 0 & \gamma I \end{bmatrix} > 0, \tag{23}$$
$\forall j = 1,\ldots,N$

$$\begin{bmatrix} G+G^T-\tilde{S}_j & Y^T \\ Y & u_{\max}^2 I \end{bmatrix} > 0,\ \forall j = 1,\ldots,N \tag{24}$$

$$\begin{bmatrix} G+G^T-\tilde{S}_j & (\tilde{A}_jG+\tilde{B}_jY)^T\tilde{C}^T \\ \tilde{C}(\tilde{A}_jG+\tilde{B}_jY) & y_{\max}^2 I \end{bmatrix} > 0,\ \forall j = 1,\ldots,N \tag{25}$$

The state feedback matrix is given by $K_C(k) = YG^{-1}$.

Remark 2. *The variables $\gamma, Y, G, \tilde{S}_j$ in the optimization problem are calculated at each iteration k.*

Proof. Let $\tilde{S} = \gamma S^{-1}$, then

$$\begin{aligned} &\tilde{x}(k/k)^T S(k)\tilde{x}(k/k) < \gamma \Leftrightarrow \\ &\qquad \tilde{x}(k/k)^T \tilde{S}^{-1}(k)\tilde{x}(k/k) < 1 \end{aligned} \tag{26}$$

Let $\tilde{S} = \sum_{j=1}^N \lambda_j \tilde{S}_j, \tilde{S}_j > 0$. $\tilde{S}^{-1}$ being convex with respect to the λ_j, (26) is equivalent to

$$\tilde{x}(k/k)^T \tilde{S}_j^{-1}(k)\tilde{x}(k/k) < 1,\ j = 1,2,\ldots N \tag{27}$$

giving (22) by Schur complement.

Assume that (16) is satisfied for any realization $[\tilde{A} \mid \tilde{B}] \in \Omega$. Replacing, $u(k+i/k) = K_C(k)\tilde{x}(k+i/k), i \ge 0$ in (8), the inequality (16) can be written:

$$\begin{aligned} &(\tilde{A}+\tilde{B}K_C(k))^T S(\tilde{A}+\tilde{B}K_C(k)) - S\ldots \\ &\qquad \ldots + Q + K_C(k)^T RK_C(k) < 0 \end{aligned} \tag{28}$$

Replacing $S = \gamma\tilde{S}^{-1}$ and applying the Schur complement, (28) is equivalent to

$$\begin{bmatrix} \gamma\tilde{S}^{-1} & (\tilde{A}^T + K_C(k)^T\tilde{B}^T)\gamma\tilde{S}^{-1} & Q^{1/2} & K_C(k)^TR^{1/2} \\ \gamma\tilde{S}^{-1}(\tilde{A}+\tilde{B}K_C(k)) & \gamma\tilde{S}^{-1} & 0 & 0 \\ Q^{1/2} & 0 & I & 0 \\ R^{1/2}K_C(k) & 0 & 0 & I \end{bmatrix} > 0 \tag{29}$$

Pre multiplying the last inequality by, $blockdiag(\gamma^{-1/2}G^T, \gamma^{-1/2}\tilde{S}_j, I, I)$ and post multiplying by $blockdiag(\gamma^{-1/2}G, \gamma^{-1/2}\tilde{S}_j, I, I)$, yields:

$$\begin{bmatrix} G^T\tilde{S}^{-1}G & G^T\left(\tilde{A}+\tilde{B}K_C(k)\right)^T & G^TQ^{1/2}\gamma^{-1/2} & G^TK_C(k)^TR^{1/2}\gamma^{-1/2} \\ \left(\tilde{A}+\tilde{B}K_C(k)\right)G & \tilde{S} & 0 & 0 \\ \gamma^{-1/2}Q^{1/2}G & 0 & I & 0 \\ \gamma^{-1/2}R^{1/2}K_C(k)G & 0 & 0 & I \end{bmatrix} > 0, \quad (30)$$

Inequality (30) is satisfied for any realization $\left[\tilde{A} \mid \tilde{B}\right] \in \Omega$, if (convexity with respect to λ_j).

$$\begin{bmatrix} G^T\tilde{S}_j^{-1}G & G^T\left(\tilde{A}_j+\tilde{B}_jK_C(k)\right)^T & G^TQ^{1/2}\gamma^{-1/2} & G^TK_C(k)^TR^{1/2}\gamma^{-1/2} \\ \left(\tilde{A}_j+\tilde{B}_jK_C(k)\right)G & \tilde{S}_j & 0 & 0 \\ \gamma^{-1/2}Q^{1/2}G & 0 & I & 0 \\ \gamma^{-1/2}R^{1/2}K_C(k)G & 0 & 0 & I \end{bmatrix} > 0, \quad (31)$$

$\forall j = 1,\ldots,N$

then, since $\tilde{S}_j > 0$, one has

$$G^T\tilde{S}_j^{-1}G \geq G^T + G - \tilde{S}_j > 0 \quad (32)$$

which follows from:

$$\left(G-\tilde{S}_j\right)^T\tilde{S}_j^{-1}\left(G-\tilde{S}_j\right) \geq 0 \quad (33)$$

with $Y = K_C(k)G$, (31) is written as (23) after pre and post multiplication by $blockdiag\left[I, I, \gamma^{1/2}I, \gamma^{1/2}I\right]$.

To derive (24) and (25), one needs the following lemma :

Lemma 1. ***(Invariant ellipsoid)*** **[9]**. *Consider the uncertain system (8), at sampling time* k*, suppose there exist* $\gamma, Y, G, \tilde{S}_j > 0$ *such that (22) and (23) holds. Also* $u(k+i/k) = K_C(k)\tilde{x}(k+i/k), i \geq 0$.

For the quadratic optimization problem under constraints, $\tilde{S}^{-1}$ *is a Lyapunov matrix so that:*

$$\tilde{x}(k+i/k)^T\tilde{S}^{-1}\tilde{x}(k+i/k) \leq 1,\ i \geq 0\ \ \forall\left[\tilde{A} \mid \tilde{B}\right] \in \Omega \quad (34)$$

or equivalently,

$$\tilde{x}(k+i/k)^TS\tilde{x}(k+i/k) \leq \gamma,\ i \geq 0,\ \forall\left[\tilde{A} \mid \tilde{B}\right] \in \Omega \quad (35)$$

and therefore $\varepsilon = \left\{z \middle| z^TSz \leq \gamma\right\}$ *is an invariant ellipsoid for the predicted states, for any realization* $\left[\tilde{A} \mid \tilde{B}\right] \in \Omega$ *of the uncertain system (1).*

The proof of lemma 1 is in [9]. The proof of (24) and (25), can be derived from lemma 1 by following the reasoning proposed in [9] and [14], is omitted for the sake of brevity.

It only rests to prove that the optimal solution of (22)-(25) is indeed a Lyapunov function and hence robust stability is assured.

Lemma 2. (Feasibility). *Provided that the LMIs (22)-(25) are feasible at time* k*, then the optimization problem is feasible for all subsequent times.*

Proof. Feasibility of the system of LMIs (22)-(25) at time k implies that an initial feedback gain $K_C(k) = YG^{-1}$ can be found which is a stabilizing one satisfying the constraints (input-output) with:

$$\tilde{S}^{-1}(k) = \left(\sum_{j=1}^{N}\lambda_j\tilde{S}_j\right)^{-1} \quad (36)$$

as a Lyapunov matrix for the controlled system. Then

$$\tilde{x}(k+i/k)^T\tilde{S}^{-1}(k)\tilde{x}(k+i/k) \leq 1,\ i \geq 0 \quad (37)$$

for any realization of $\left[\tilde{A} \mid \tilde{B}\right] \in \Omega$ and hence:

$$\tilde{x}(k+1/k+1)^T\tilde{S}(k)^{-1}\tilde{x}(k+1/k+1) \leq 1 \quad (38)$$

or

$$\begin{bmatrix} \tilde{S}(k) & \tilde{x}(k+1/k+1) \\ \tilde{x}(k+1/k+1)^T & I \end{bmatrix} > 0 \quad (39)$$

equivalently

$$\begin{bmatrix} \tilde{S}_j(k) & \tilde{x}(k+1/k+1) \\ \tilde{x}(k+1/k+1)^T & I \end{bmatrix} > 0, \quad (40)$$

$j = 1,2,\ldots N$

Thus, the solution of the optimization ($\tilde{S}_j(k),\ j = 1,2,\ldots N$) problem at instant k is also feasible at instant $k+1$. Hence, the optimization is feasible at instant $k+1$. By repeating this argument for the successive instant, the proof is complete. □

Theorem 2. (Robust stability). *The feasible receding state feedback law obtained from the solution of Theorem 1 robustly asymptotically stabilizes the closed loop system.*

Proof. It is done by establishing that the MPC type procedure defines a sequence of decreasing positive definite quadratic functions $V(0,k)$: $\tilde{S}(k) > 0$ and $\lim_{k\to\infty}\tilde{x}(k/k)^T\tilde{S}(k)^{-1}\tilde{x}(k/k) \to 0$ implies $\lim_{k\to\infty}\left\|\tilde{x}(k/k)\right\| \to 0$.

It is assumed that the problem (18) is feasible at instant k . That ensure feasibility of the optimization problem at all times $k > 0$ (Lemma 2). The problem being convex, therefore, has a unique minimum and a corresponding optimal solution ($\gamma, Y, G, \widetilde{S}_j > 0$) at each time $k \geq 0$.

The optimal solution at instant k, is feasible (not necessarily optimal) at instant $k+1$ (Lemma 2). Denoting $\widetilde{S}(k)$ and $\widetilde{S}(k+1)$ the optimal solutions at instant k and $k+1$ respectively, one has:

$$\begin{aligned} &\widetilde{x}(k+1/k+1)^T \widetilde{S}(k+1)^{-1} \widetilde{x}(k+1/k+1) \leq \ldots \\ &\quad \ldots \widetilde{x}(k+1/k+1)^T \widetilde{S}(k)^{-1} \widetilde{x}(k+1/k+1) \end{aligned} \tag{41}$$

since at time $k+1$ $\widetilde{S}(k+1)^{-1}$ is optimal while $\widetilde{S}(k)^{-1}$ is only feasible.

By applying the control $u(k) = K_C(k)\widetilde{x}(k)$, at instant k, the one step prediction $\widetilde{x}(k+1/k)$ is obtained for some realization of the uncertain pair $\left[\widetilde{A} \mid \widetilde{B}\right] \in \Omega$, that is, $\widetilde{x}(k+1/k+1) = \widetilde{x}(k+1/k)$. Then:

$$\begin{aligned} &\widetilde{x}(k+1/k)^T \widetilde{S}^{-1}(k+1)^{-1} \widetilde{x}(k+1/k) \leq \ldots \\ &\quad \ldots \widetilde{x}(k+1/k+1)^T \widetilde{S}^{-1}(k)^{-1} \widetilde{x}(k+1/k+1) \\ &= \widetilde{x}(k+1/k)^T \widetilde{S}(k)^{-1} \widetilde{x}(k+1/k) \leq \widetilde{x}(k/k)^T \widetilde{S}(k)^{-1} \widetilde{x}(k/k) \end{aligned} \tag{42}$$

The last inequality in (42) is a consequence of $\widetilde{S}(k)$ being a Lyapunov matrix for the closed loop system determined at time k . This proves the decreasing property of $V(0,k)$ as k increase. □

IV. NUMERICAL EXAMPLES

In this section, two examples illustrate the feasibility of the proposed algorithm. For these examples, an LMI control toolbox has been used.

A. Example 1 (SISO system)

The following fourth order monovariable system corresponds to the example 2 reported in [9]. The system consists of a two-mass-spring model whose discrete-time input-output representation is:

$$\frac{y(k)}{u(k)} = \frac{(0.0001\kappa)z^{-4}}{1-4z^{-1}+(6+0.2\kappa)z^{-2}-(4+0.4\kappa)z^{-3}+(1+0.2\kappa)z^{-4}}$$

It is assumed that the parameter κ is uncertain, the actual value being in the interval $[0.75, 1.25]$. The initial values for the seven order ($n_a = 4, n_b = 3$) extended state are: $\widetilde{x}(0) = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 & 0 \end{bmatrix}^T$.

The input and output constraints are defined by: $u_{\max} = 1$, $y_{\max} = 1$ and $Q = I$, $R = 1$ have been chosen as the weighting matrices.

Fig 1 shows the behavior of the closed loop output for the uncertain parameter κ ranging from 0.75 and 1.25 .

Of course a stable behavior for the realizations chosen in the uncertainty domain is observed.

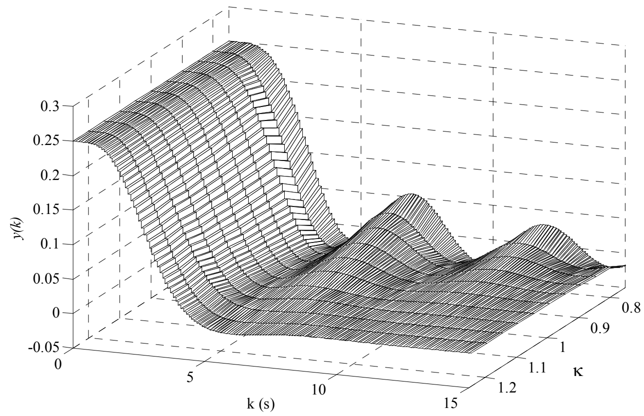

Fig. 1. Performance profile of the output for $0.75 \leq \kappa \leq 1.25$

B. Example 2 (MIMO system)

The following example corresponds to a system reported in section 6.2.1 in [1] to which a uncertainty has been added. The system consists of a chemical reactor with a cooling jacket. The decomposition of a product *A* in another product *B* occurs in the tank. The reaction is exothermic and consequently the temperature must be controlled by water circulation through the cooling jacket which surrounds the walls of the tank. The objective is to regulate the temperature in the tank and the concentration of the product at exit. The uncertain system is represented by:

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} \dfrac{0.0420\kappa z^{-1}}{1-0.958z^{-1}} & \dfrac{0.4758\kappa z^{-1}}{1-0.9048z^{-1}} \\ \dfrac{0.0582\kappa z^{-1}}{1-0.9418z^{-1}} & \dfrac{0.1445\kappa z^{-1}}{1-0.9277z^{-1}} \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

where the control variables u_1 and u_2 are food flow and the flow of cooling in the jacket respectively. The controlled variables y_1 and y_2 are the concentration of the output product at and the temperature in the reactor respectively.

In order to get the formulation (6) the transfer matrix is factorized as $A(z^{-1})B(z^{-1})z^{-1}$, with $A(z^{-1})$ diagonal [1]. This is done by using least common multiple of the denominators resulting in:

$$A(z^{-1}) = \begin{bmatrix} 1-1.8629z^{-1}+0.8669z^{-2} & 0 \\ 0 & 1-1.8695z^{-1}+0.8737z^{-2} \end{bmatrix}$$

$$B(z^{-1}) = \begin{bmatrix} 0.0420\kappa - 0.0380\kappa z^{-1} & 0.04758\kappa - 0.4559\kappa z^{-1} \\ 0.0582\kappa - 0.0540\kappa z^{-1} & 0.1445\kappa - 0.1361\kappa z^{-1} \end{bmatrix}$$

Finally:

$$\alpha_1 = \begin{bmatrix} 1.8629 & 0 \\ 0 & 1.8695 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} -0.8669 & 0 \\ 0 & -0.8737 \end{bmatrix},$$

$$\beta_0 = \begin{bmatrix} 0.0420\kappa & 0.04758\kappa \\ 0.0582\kappa & 0.14450\kappa \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} -0.0380\kappa & -0.4559\kappa \\ -0.0540\kappa & -0.1361\kappa \end{bmatrix}$$

It is supposed that the uncertainty on κ has a value contained in the interval: $0.5 \le \kappa \le 2$, $Q = I$, $R = I$, $u_{\max} = 1$, $y_{\max} = 1$ and it is considered that the initial condition is $\tilde{x}(0) = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 & 0 & 0 \end{bmatrix}^T$.

Fig 2 and Fig 3 shows the behavior of the closed loop output for the uncertain parameter κ ranging from 0.5 and 2.

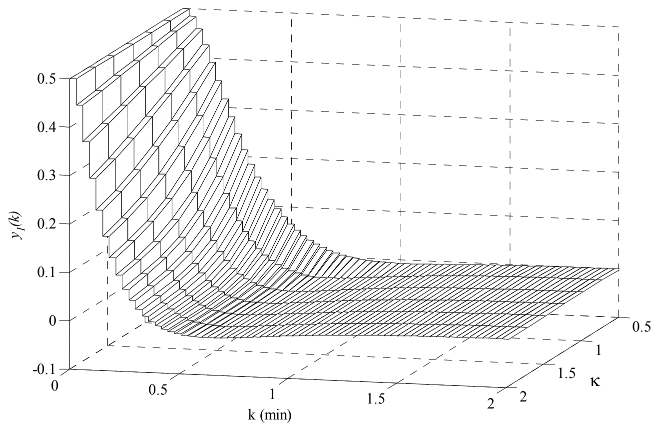

Fig. 2. Performance profile of the output y_1 for $0.5 \le \kappa \le 2$

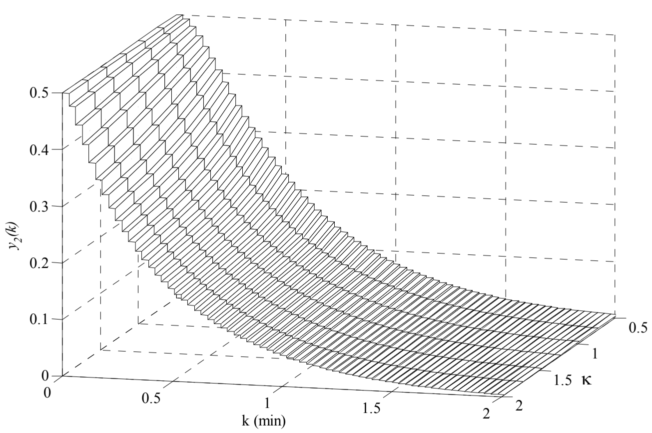

Fig. 3. Performance profile of the output y_2 for $0.5 \le \kappa \le 2$

It can be verified a stable behavior for the realizations chosen in the uncertainty domain.

V. Conclusion

Using an input-output representation, the paper shows how to derive an MPC controller which only uses the actual output measurements together with past values of the applied control and observed outputs. This is done in presence of uncertainties which affect the matrices of the input-output transfer representation. Constraints on the control and states are taken into account.

The design is made in the so called *G* shaping paradigm context [15] whose big benefit comes from its ability of using different Lyapunov matrices instead of the single one of use in classical quadratic approach, also termed as Lyapunov shaping algorithm ([16]). Min-max input/output constraints may be considered much as [9] deals with peak bounds.

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