

State Feedback Policies for Robust Receding Horizon Control: Uniqueness, Continuity, and Stability

Paul J. Goulart, Eric C. Kerrigan and Jan M. Maciejowski

Abstract—In this paper we consider the problem of controlling linear discrete-time systems subject to unknown disturbances and mixed constraints on the states and inputs, using a class of affine state-feedback control policies implemented in a receding horizon fashion. By defining a quadratic cost function in the disturbance-free sequence of states and controls, we demonstrate that this parameterization can be used in the synthesis of a nonlinear time-invariant receding horizon control law that is robustly invariant, unique and continuous in the initial state, and with guaranteed input-to-state (ISS) stability. Our method relies in part on the exploitation of an equivalent control policy parameterized as an affine function of the past disturbance sequence, and we show that this parameterization has the added benefit of enabling calculation of the control law at each stage using a single tractable quadratic program (QP) when the disturbance set is a polytope or affine map of a 1- or ∞ -norm bounded set.

I. INTRODUCTION

The problem of finding a nonlinear state feedback control law which guarantees that a set of state and input constraints are satisfied for all time, despite the presence of a persistent state disturbance, has been the subject of study for many authors [4], [5], [21], [22], [24]. However, the problem is that the solutions offered to date are exponentially complex or intractable for online implementation. As a consequence, many researchers have proposed compromise solutions, which, though not able to guarantee the same level of performance, are computationally tractable [1], [6], [17], [18], [25].

We propose a nonlinear control scheme that is calculated by optimizing over the set of admissible affine state feedback policies at each stage, and implemented in a receding horizon fashion. By defining a quadratic cost function in the predicted disturbance-free sequence of states and controls, we demonstrate that the resulting time-invariant control law is unique and continuous in the current state, and that the closed-loop system can be guaranteed to be input-to-state stable (ISS) given appropriate conditions.

For many of our results, we exploit a recently-proposed method for solving so-called *robust optimization* problems with hard constraints [3], [11]. The authors proposed that, instead of solving for a general, nonlinear function that guarantees that the constraints in the optimization problem are met for all values of the uncertainty, one could aim to parameterize the solution as an affine function of the *uncertainty*.

They proceeded to show that, if the uncertainty set is a polyhedron and the constraints in the robust optimization

problem are affine, then a robustly feasible affine function of the uncertainty can be found by solving a single, computationally tractable LP. An equivalent parameterization has also been proposed in the context of predictive control with bounded [19] and stochastic [26] disturbances.

We have previously shown that this affine uncertainty parameterization is *equivalent* to the class of affine state feedback policies [9], and thus transforms the *non-convex* optimization problem of finding a constraint-admissible affine state feedback policy into a *convex* one, solvable using standard techniques [9]. For problems with polytopic or 2-norm bounded disturbances, the proposed scheme requires the solution of either a single quadratic program (QP) or second order cone program (SOCP) in a tractable number of variables, depending on the disturbance characterization.

This paper is organized as follows. Section II introduces the control problem considered and some standing assumptions. Section III introduces the state-feedback control policy that we seek to employ in the solution of a finite horizon robust control problem. Section IV then introduces the alternative convex parameterization proposed in [3]. Section V demonstrates that, using a quadratic cost function, it is possible to formulate an optimal receding horizon control policy that is continuous in the current state, and is guaranteed to be input-to-state stable (ISS). Section VI ends the paper by drawing some conclusions and making recommendations for future research.

Notation and definitions: For matrices A and B , $A \otimes B$ is the Kronecker product of A and B , and $A \leq B$ denotes element-wise inequality. $\mathbf{1}$ is a column vector of ones. For vector x , $\|x\|_A^2 := x^T A x$. $\mathbb{Z}_{[k,l]}$ represents the set of integers $\{k, k+1, \dots, l\}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a \mathcal{K} -function if it is strictly increasing and $\gamma(0) = 0$; it is a \mathcal{K}_∞ -function if, in addition, $\gamma(s) \rightarrow \infty$ as $s \rightarrow \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a \mathcal{KL} -function if for all $k \geq 0$, the function $\beta(\cdot, k)$ is a \mathcal{K} -function and for each $s \geq 0$, $\beta(s, \cdot)$ is decreasing with $\beta(s, k) \rightarrow 0$ as $k \rightarrow \infty$.

II. STANDING ASSUMPTIONS

Consider the following discrete-time LTI system:

$$x^+ = Ax + Bu + w, \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state at the current time instant, x^+ is the state at the next time instant, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ is the disturbance. The current and future values of the disturbance are unknown and may change unpredictably from one time instant to the next, but

Department of Engineering, University of Cambridge, Cambridge CB2 1PZ, United Kingdom {pjpg42, eck21, jmm}@eng.cam.ac.uk

are contained in a convex and compact (closed and bounded) set W , which contains the origin. The actual values of the state, input and disturbance at time instant k are denoted by $x(k)$, $u(k)$ and $w(k)$, respectively; where it is clear from the context, x , u and w will be used to denote the current value of the state, input and disturbance (note that, since the system is time-invariant, the current time can always be taken as zero). It is assumed that (A, B) is stabilizable and that at each sample instant a measurement of the state is available. We also assume that a linear state feedback gain matrix $K \in \mathbb{R}^{m \times n}$ is given, such that $A+BK$ is strictly stable.

The system is subject to mixed polyhedral constraints on the state and input:

$$Z := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\}, \quad (2)$$

where the matrices $C \in \mathbb{R}^{s \times n}$, $D \in \mathbb{R}^{s \times m}$ and the vector $b \in \mathbb{R}^s$; s is the number of affine inequality constraints that define Z . It is assumed that Z is bounded and contains the origin in its interior. A primary design goal is to guarantee that the state and input of the closed-loop system remain in Z for all time and for all allowable disturbance sequences.

In addition to Z , a target/terminal constraint set X_f is given by

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\}, \quad (3)$$

where the matrix $Y \in \mathbb{R}^{r \times n}$ and the vector $z \in \mathbb{R}^r$; r is the number of affine inequality constraints that define X_f . It is assumed that X_f is bounded and contains the origin in its interior. As will be seen in Section V, the set X_f can be used as terminal constraint in the design of a receding horizon controller with guaranteed invariance and stability properties.

Before proceeding, we define some additional notation. In the sequel, predictions of the system's evolution over a finite control/planning horizon will be used to define a number of suitable control policies. Let the length N of this planning horizon be a positive integer and define stacked versions of the predicted input, state and disturbance vectors $\mathbf{u} \in \mathbb{R}^{mN}$, $\mathbf{x} \in \mathbb{R}^{n(N+1)}$ and $\mathbf{w} \in \mathbb{R}^{nN}$, respectively, as

$$\mathbf{x} := [x_0^T, \dots, x_N^T]^T, \quad (4a)$$

$$\mathbf{u} := [u_0^T, \dots, u_{N-1}^T]^T, \quad (4b)$$

$$\mathbf{w} := [w_0^T, \dots, w_{N-1}^T]^T, \quad (4c)$$

where $x_0 = x$ denotes the current measured value of the state and $x_{i+1} := Ax_i + Bu_i + w_i$, $i = 0, \dots, N-1$ denote the prediction of the state after i time instants into the future. Finally, let the set $\mathcal{W} := W^N := W \times \dots \times W$, so that $\mathbf{w} \in \mathcal{W}$.

III. STATE FEEDBACK PARAMETERIZATION

One natural approach to controlling the system in (1), while ensuring the satisfaction of the constraints, is to search over the set of time-varying affine state feedback control policies with knowledge of prior states:

$$u_i = \sum_{j=0}^i L_{i,j} x_j + g_i, \quad \forall i \in \mathbb{Z}_{[0, N-1]}, \quad (5)$$

where each $L_{i,j} \in \mathbb{R}^{m \times n}$ and $g_i \in \mathbb{R}^m$. For notational convenience, we also define the block lower triangular matrix $\mathbf{L} \in \mathbb{R}^{mN \times n(N+1)}$ and stacked vector $\mathbf{g} \in \mathbb{R}^{mN}$ as

$$\mathbf{L} := \begin{bmatrix} L_{0,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} & 0 \end{bmatrix}, \quad \mathbf{g} := \begin{bmatrix} g_0 \\ \vdots \\ g_{N-1} \end{bmatrix}, \quad (6)$$

so that the control input sequence can be written as $\mathbf{u} = \mathbf{L}\mathbf{x} + \mathbf{g}$. For a given initial state x , we say that the pair (\mathbf{L}, \mathbf{g}) is admissible if the control policy (5) guarantees that, for all allowable disturbance sequences of length N , the constraints (2) are satisfied over the horizon $i = 0, \dots, N-1$ and that the state is in the target set (3) at the end of the horizon. More precisely, the set of admissible (\mathbf{L}, \mathbf{g}) is defined as

$$\Pi_N^{sf}(x) := \left\{ (\mathbf{L}, \mathbf{g}) \left| \begin{array}{l} (\mathbf{L}, \mathbf{g}) \text{ satisfies (6), } x = x_0 \\ x_{i+1} = Ax_i + Bu_i + w_i \\ u_i = \sum_{j=0}^i L_{i,j} x_j + g_i \\ (x_i, u_i) \in Z, x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0, N-1]}, \forall \mathbf{w} \in \mathcal{W} \end{array} \right. \right\} \quad (7)$$

The set of initial states x for which an admissible control policy of the form (5) exists is defined as

$$X_N^{sf} := \left\{ x \in \mathbb{R}^n \mid \Pi_N^{sf}(x) \neq \emptyset \right\}. \quad (8)$$

It is critical to note that it may not be possible to select a single (\mathbf{L}, \mathbf{g}) such that it is admissible for all $x \in X_N^{sf}$. Indeed, it is easy to find examples where there exists a pair $(x, \tilde{x}) \in X_N^{sf} \times X_N^{sf}$ such that $\Pi_N^{sf}(x) \cap \Pi_N^{sf}(\tilde{x}) = \emptyset$. Additionally, note that direct on-line computation of an admissible pair (\mathbf{L}, \mathbf{g}) is problematic, since the set $\Pi_N^{sf}(x)$ is non-convex in general.

Remark 1. Note that the state feedback policy (5) subsumes the well-known class of ‘‘pre-stabilizing’’ control policies [6], [18], in which the control policy takes the form $u_i = Kx_i + c_i$, where K is computed off-line and on-line computation is limited to finding an admissible offset sequence $\{c_i\}_{i=0}^{N-1}$.

IV. DISTURBANCE FEEDBACK PARAMETERIZATION

An alternative to (5) is to parameterize the control policy as an affine function of the sequence of past disturbances, so that

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i, \quad \forall i \in \mathbb{Z}_{[0, N-1]}, \quad (9)$$

where each $M_{i,j} \in \mathbb{R}^{m \times n}$ and $v_i \in \mathbb{R}^m$. It should be noted that, since full state feedback is assumed, the past disturbance sequence is easily calculated as the difference between the predicted and actual states at each step, i.e.

$$w_i = x_{i+1} - Ax_i - Bu_i, \quad \forall i \in \mathbb{Z}_{[0, N-1]}. \quad (10)$$

The above parameterization appears to have originally been suggested some time ago within the context of stochastic programs with recourse [7]. More recently, it has been

revisited as a means for finding solutions to a class of robust optimization problems, called affinely adjustable robust counterpart (AARC) problems [3], [11], and robust model predictive control problems [19], [26].

For notational convenience, we define the vector $\mathbf{v} \in \mathbb{R}^{mN}$ and the strictly block lower triangular matrix $\mathbf{M} \in \mathbb{R}^{mN \times nN}$ such that

$$\mathbf{M} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \quad \mathbf{v} := \begin{bmatrix} v_0 \\ \vdots \\ \vdots \\ v_{N-1} \end{bmatrix}, \quad (11)$$

so that the control input sequence can be written as $\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}$. In a manner similar to (7), we define the set of admissible (\mathbf{M}, \mathbf{v}) as

$$\Pi_N^{df}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \left| \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfies (11), } x = x_0 \\ x_{i+1} = Ax_i + Bu_i + w_i \\ u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i \\ (x_i, u_i) \in Z, x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0, N-1]}, \forall \mathbf{w} \in \mathcal{W} \end{array} \right. \right\} \quad (12)$$

The set of initial states x for which an admissible control policy of the form (9) exists is defined as

$$X_N^{df} := \left\{ x \in \mathbb{R}^n \mid \Pi_N^{df}(x) \neq \emptyset \right\}. \quad (13)$$

Before proceeding, it is important to note that one can find matrices $F \in \mathbb{R}^{t \times mN}$, $G \in \mathbb{R}^{t \times nN}$, $H \in \mathbb{R}^{t \times n}$ and a vector $c \in \mathbb{R}^t$, where $t := sN + r$ (for completeness, the matrices and vectors are given in the Appendix), such that the expression for $\Pi_N^{df}(x)$ can be rewritten more compactly as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \left| \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfies (11)} \\ F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \leq c + Hx \\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right. \right\}. \quad (14)$$

which is easily shown to be convex and closed.

A. Convexity and Equivalence

The main advantage of the disturbance feedback parameterization in (9) over the state feedback parameterization in (5) is formalized in the following statements, proof of which may be found in [10]:

Theorem 1 (Convexity). *For a given state $x \in X_N^{df}$, the set of admissible affine disturbance feedback parameters $\Pi_N^{df}(x)$ is convex and closed. Furthermore, the set of states X_N^{df} , for which at least one admissible affine disturbance feedback parameter exists, is convex.*

Theorem 2 (Equivalence). *The set of admissible states $X_N^{df} = X_N^{sf}$. Additionally, for any admissible (\mathbf{L}, \mathbf{g}) an admissible (\mathbf{M}, \mathbf{v}) can be found that yields the same input and state sequence for all allowable disturbance sequences, and vice-versa.*

Remark 2. If \mathcal{W} is convex and compact, then it is conceptually possible to compute a pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ in a computationally tractable way, given the current state x . For

example, if the set W is a polytope (or the affine map of a 1- or ∞ -norm bounded ball), then an admissible pair (\mathbf{M}, \mathbf{v}) can be found by solving a single LP in a tractable number of variables [3], [11]. If W is 2-norm bounded (e.g. if W is the affine map of a Euclidean ball or an ellipsoid), an admissible pair may be found via the solution of a tractable second order cone program (SOCP) [10].

Before proceeding, we introduce the following standard assumption (cf. [21]):

Assumption 1 (Terminal constraint). *The state feedback gain matrix K and terminal constraint X_f have been chosen such that:*

- X_f is contained inside the set of states for which the constraints (2) are satisfied under the control $u = Kx$, i.e. $X_f \subseteq \{x \mid (x, Kx) \in Z\} = \{x \mid (C + DK)x \leq b\}$.
- X_f is robust positively invariant for the closed-loop system $x^+ = (A + BK)x + w$, i.e. $(A + BK)x + w \in X_f$, for all $x \in X_f$ and all $w \in W$.

Under some additional, mild technical assumptions, it is easy to compute a K and a polytopic X_f that satisfies Assumption 1 if W is a polytope, an ellipsoid or the affine map of a p -norm ball. The reader is referred to [5], [16], [18], [23] and the references therein for details.

Proposition 1 (Size of X_N^{sf}). *If Assumption 1 holds, then the following set inclusion holds:*

$$X_f \subseteq X_1^{sf} \subseteq \cdots \subseteq X_{N-1}^{sf} \subseteq X_N^{sf} \subseteq X_{N+1}^{sf} \subseteq \cdots, \quad (15)$$

where each X_i^{sf} is defined as in (8) with $N = i$.

Proof. Proof of this is straightforward, and can be found in [10]. Recall that since $X_N^{df}(x) = X_N^{sf}(x)$, an identical set inclusion result also holds for the sets $X_i^{df}(x)$. \square

V. UNIQUENESS, CONTINUITY AND STABILITY OF RHC LAWS

We consider the important problem of how to synthesize an RHC law such that the closed-loop system is robustly stable. We choose to minimize the value of a cost function that is quadratic in the disturbance-free states and control inputs and demonstrate that this allows for the synthesis of a continuous RHC law, which guarantees that the closed-loop system is input-to-state stable (ISS). We rely heavily on Theorem 2 in order to derive these results, moving freely between the two parameterizations and using whichever is most natural in each context.

We first note that alternative cost functions are certainly possible; the reader is referred to [13] where a worst-case quadratic cost function, in which the disturbance is negatively weighted as in \mathcal{H}_∞ control, is used.

A. Cost Function

We define an optimal policy to be one that minimizes the value of a cost function that is quadratic in the disturbance-free state and input sequences. We thus define:

$$V_N(x, \mathbf{L}, \mathbf{g}, \mathbf{w}) := \sum_{i=0}^{N-1} \frac{1}{2} (\|\tilde{x}_i\|_Q^2 + \|\tilde{u}_i\|_R^2) + \frac{1}{2} \|\tilde{x}_N\|_P^2 \quad (16)$$

where $\tilde{x}_0 = x$, $\tilde{x}_{i+1} = A\tilde{x}_i + B\tilde{u}_i + w_i$ and $\tilde{u}_i = \sum_{j=0}^i L_{i,j}\tilde{x}_j + g_i$ for $i = 0, \dots, N-1$, and P, Q and R are positive definite; and define an optimal policy pair as

$$(\mathbf{L}^*(x), \mathbf{g}^*(x)) := \operatorname{argmin}_{(\mathbf{L}, \mathbf{g}) \in \Pi_N^{sf}(x)} V_N(x, \mathbf{L}, \mathbf{g}, 0). \quad (17)$$

The time-invariant receding horizon control law $\mu_N : X_N^{sf} \rightarrow \mathbb{R}^m$ is defined by the first part of the optimal affine state feedback control policy, i.e.

$$\mu_N(x) := L_{0,0}^*(x)x + g_0^*(x) \quad (18)$$

which is *nonlinear*, in general. The closed-loop system becomes

$$x^+ = Ax + B\mu_N(x) + w. \quad (19)$$

We also define the value function $V_N^* : X_N^{sf} \rightarrow \mathbb{R}_{\geq 0}$ to be

$$V_N^*(x) := \min_{(\mathbf{L}, \mathbf{g}) \in \Pi_N^{sf}(x)} V_N(x, \mathbf{L}, \mathbf{g}, 0). \quad (20)$$

The problem with the control law in (18) is that an optimal pair $(\mathbf{L}^*(x), \mathbf{g}^*(x))$ is difficult to find due to non-convexity of the admissible set $\Pi_N^{sf}(x)$ and of the function $(\mathbf{L}, \mathbf{g}) \mapsto V_N(x, \mathbf{L}, \mathbf{g}, 0)$. However, by exploiting the equivalent affine disturbance feedback parameterization (9), we will show that the resulting control law can be calculated using convex optimization techniques.

B. Using Equivalence to Compute the Value of the RHC Law

For the equivalent affine disturbance feedback parameterization (9), we define a cost function $J_N(\cdot)$ analogous to the one defined in (16), i.e.

$$J_N(x, \mathbf{M}, \mathbf{v}, \mathbf{w}) := \frac{1}{2} \sum_{i=0}^{N-1} (\|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2) + \frac{1}{2} \|\bar{x}_N\|_P^2$$

where $\bar{x}_0 = x$, $\bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i + w_i$ and $\bar{u}_i = \sum_{j=0}^{i-1} M_{i,j}w_j + v_i$ for $i = 0, \dots, N-1$. If we define

$$(\mathbf{M}^*(x), \mathbf{v}^*(x)) := \operatorname{argmin}_{(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)} J_N(x, \mathbf{M}, \mathbf{v}, 0) \quad (21)$$

then the proof of the following result follows by a straightforward application of Theorem 2.

Proposition 2 (Equivalence for computation of RHC law). *The RHC law $\mu_N(\cdot)$, defined in (18), is given by the first part of the optimal control sequence $\mathbf{v}^*(\cdot)$, i.e.*

$$\mu_N(x) = v_0^*(x) = L_{0,0}^*(x)x + g_0^*(x), \quad \forall x \in X_N^{sf}. \quad (22)$$

The minimum value of $J_N(x, \cdot, \cdot, 0)$ taken over the set of admissible affine disturbance feedback parameters is equal to $V_N^(x)$, defined in (20), i.e.*

$$V_N^*(x) = \min_{(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)} J_N(x, \mathbf{M}, \mathbf{v}, 0). \quad (23)$$

Together with Theorem 1 and the remarks immediately thereafter, the above result implies that, for a given $x \in X_N^{sf}$, the value of the RHC law $u = \mu_N(x)$ can be computed via the minimization of a convex function over a convex set. In particular, we remark that if W is a polytope, then the

optimization problem in (21) can be written as a convex quadratic program (QP) in a tractable number of variables and constraints [8]. If W is an ellipsoid or the affine map of a Euclidean ball, then the optimization problem in (21) becomes a tractable SOCP. In all these cases, the number of decision variables and constraints in the convex optimization problem is $\mathcal{O}(N^2)$.

C. Continuity of the RHC Law and Value Function

Proposition 3 (Continuity of μ_N and V_N^*). *If W is a polytope, then the receding horizon control law $\mu_N(\cdot)$ in (18) is unique and Lipschitz continuous on X_N^{sf} . Furthermore, the value function $V_N^*(\cdot)$ in (20) is strictly convex and Lipschitz continuous on X_N^{sf} .*

Proof. Note that $J_N(x, \mathbf{M}, \mathbf{v}, 0) = J_N(x, 0, \mathbf{v}, 0)$ for all \mathbf{M} . Hence, if we define the set

$$\mathcal{V}_N(x) := \left\{ \mathbf{v} \mid \exists \mathbf{M} \text{ s.t. } (\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x) \right\},$$

then, from (21) and Proposition 2 respectively,

$$\mathbf{v}^*(x) = \operatorname{argmin}_{\mathbf{v} \in \mathcal{V}_N(x)} J_N(x, 0, \mathbf{v}, 0) \quad (24)$$

$$V_N^*(x) = \min_{\mathbf{v} \in \mathcal{V}_N(x)} J_N(x, 0, \mathbf{v}, 0). \quad (25)$$

Recalling the discussion in Section IV-A, it can be shown that if \mathcal{W} is a polytope, then $\Pi_N^{df}(x)$ in (14) is a polyhedron, and $\mathcal{V}_N(x)$ is also a polyhedron since it is the projection of $\Pi_N^{df}(x)$ onto a subspace. It is also easy to verify that $(x, \mathbf{v}) \mapsto J_N(x, 0, \mathbf{v}, 0)$ is a strictly convex quadratic function, and thus that (25) is a strictly convex QP. By applying the results in [2], it follows that $\mathbf{v}^*(\cdot)$ and hence $\mu_N(\cdot)$ are continuous, piecewise affine functions on X_N^{sf} , and that $V_N^*(\cdot)$ is a strictly convex, piecewise quadratic function on X_N^{sf} . Lipschitz continuity follows from the assumption that Z is compact, hence X_N^{sf} is also compact. \square

Finally, we present the following result, which will be useful in proving stability in the next section:

Lemma 1 (Values at the origin). *If Assumption 1 holds, then $V_N^*(0) = 0$ and $\mu_N(0) = 0$.*

Proof. Proposition 1 implies that the origin is in the interior of X_N^{sf} . Note that if $x \in X_f$, then (\mathbf{L}, \mathbf{g}) is admissible if $\mathbf{g} = 0$, $L_{i,i} = K$ for $i = 0, \dots, N-1$ and $L_{i,j} = 0$ for all $i \neq j$. Hence, $V_N^*(0) \leq V_N(0, \mathbf{L}, 0, 0) = 0$. Since $V_N^*(x) \geq 0$ for all $x \in X_N^{sf}$, it follows that $V_N^*(0) = 0$, hence $\mu_N(0) = 0$. \square

D. Input-to-State Stability (ISS) for RHC

Since the disturbance is non-zero, it is not possible to guarantee that the origin is asymptotically stable, as in conventional RHC without disturbances [21]. As an alternative, we use the notion of input-to-state stability (ISS) [12], [14], which has proven to be effective in the study of RHC laws with input constraints only [15] and in the analysis and synthesis of RHC laws with robust constraint satisfaction guarantees [20].

Consider a nonlinear, time-invariant, discrete-time system of the form

$$x^+ = f(x, w), \quad (26)$$

where $x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^l$ is a disturbance that takes on values in a compact set $W \subset \mathbb{R}^l$ containing the origin. It is assumed that the state is measured at each time instant, that $f : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}^n$ is continuous and that $f(0, 0) = 0$. Given a disturbance sequence $w(\cdot)$, let the solution to (26) at time k be denoted by $\phi(k, x, w(\cdot))$, where $w(\cdot)$ is taken from \mathcal{M}_W , the set of infinite disturbance sequences with values in W . For systems of this type, a useful definition of stability is *input-to-state stability* (ISS):

Definition 1 (ISS). For system (26), the origin is *input-to-state stable* (ISS) with region of attraction $\mathcal{X} \subseteq \mathbb{R}^n$, which contains the origin in its interior, if there exists a \mathcal{KL} -function $\beta(\cdot)$ and a \mathcal{K} -function $\gamma(\cdot)$ such that for all initial states $x \in \mathcal{X}$ and disturbance sequences $w(\cdot) \in \mathcal{M}_W$, the solution of the system satisfies $\phi(k, x, w(\cdot)) \in \mathcal{X}$ and for all $k \in \mathbb{N}$,

$$\|\phi(k, x, w(\cdot))\| \leq \beta(\|x\|, k) + \gamma \sup \{ \|w(\tau)\| \mid \tau \in \mathbb{Z}_{[0, k-1]} \}. \quad (27)$$

Note that ISS implies that the origin is an asymptotically stable point for the undisturbed system $x^+ = f(x, 0)$ with region of attraction \mathcal{X} , and also that all state trajectories are bounded for all bounded disturbance sequences. Furthermore, every trajectory $\phi(x, k, w(\cdot)) \rightarrow 0$ if $w(k) \rightarrow 0$ as $k \rightarrow \infty$.

In order to be self-contained, we introduce the following useful result from [12, Lem 3.5]:

Lemma 2 (ISS-Lyapunov function). For the system (26), the origin is input-to-state stable (ISS) with region of attraction $\mathcal{X} \subseteq \mathbb{R}^n$ if the following conditions are satisfied:

- \mathcal{X} contains the origin in its interior and X_f is robust positively invariant for (26), i.e. $f(x, w) \in \mathcal{X}$ for all $x \in \mathcal{X}$ and all $w \in W$.
- There exist \mathcal{K}_∞ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$, a \mathcal{K} -function $\sigma(\cdot)$, and a continuous function $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ such that for all $x \in \mathcal{X}$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad (28a)$$

$$V(f(x, w)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|) \quad (28b)$$

Remark 3. A function $V(\cdot)$ that satisfies the conditions in Lemma 2 is called an *ISS-Lyapunov function*.

The above result leads immediately to the following:

Lemma 3 (Lipschitz Lyapunov function for undisturbed system). Let $\mathcal{X} \subseteq \mathbb{R}^n$ contain the origin in its interior and be a robust positively invariant set for (26). Furthermore, let there exist \mathcal{K}_∞ -functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ and a function $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ that is Lipschitz continuous on \mathcal{X} such that for all $x \in \mathcal{X}$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad (29a)$$

$$V(f(x, 0)) - V(x) \leq -\alpha_3(\|x\|) \quad (29b)$$

The function $V(\cdot)$ is an ISS-Lyapunov function and the origin is ISS for the system (26) with region of attraction \mathcal{X} if, in addition, either of the following conditions are satisfied:

- $f : \mathcal{X} \times W \rightarrow \mathbb{R}^n$ is Lipschitz continuous on $\mathcal{X} \times W$.
- $f(x, w) := g(x) + w$, where $g : \mathcal{X} \rightarrow \mathbb{R}^n$ is continuous on \mathcal{X} .

Proof. Let L_V be the Lipschitz constant of $V(\cdot)$.

(i) Since $\|V(f(x, w)) - V(f(x, 0))\| \leq L_V \|f(x, w) - f(x, 0)\| \leq L_V L_f \|w\|$, where L_f is the Lipschitz constant of $f(\cdot)$, it follows that $V(f(x, w)) - V(x) = V(f(x, 0)) - V(x) + V(f(x, w)) - V(f(x, 0)) \leq -\alpha_3(\|x\|) + L_V L_f \|w\|$. The proof is completed by letting $\sigma(s) := L_V L_f s$ in Lemma 2.

(ii) Note that $\|V(f(x, w)) - V(f(x, 0))\| \leq L_V \|w\|$. The proof is completed as for (i), but by letting $\sigma(s) := L_V s$ in Lemma 2. \square

Remark 4. If \mathcal{X} in Lemmas 2 and 3 is compact, then the condition that $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ be of class \mathcal{K}_∞ can be relaxed to the condition that they only be of class \mathcal{K} .

Finally, we add the following assumption, which will allow the value function defined in (20) to be used as an ISS-Lyapunov function:

Assumption 2 (Terminal cost). The terminal cost $F(x) := x^T P x$ is a Lyapunov function in the terminal set X_f for the undisturbed closed-loop system $x^+ = (A + BK)x$, in the sense that, for all $x \in X_f$,

$$F((A + BK)x) - F(x) \leq -x^T (Q + K^T R K) x. \quad (30)$$

We can now state our final result:

Theorem 3 (ISS for RHC). Let W be a polytope and the RHC law $\mu_N(\cdot)$ be defined as in (18). If Assumptions 1 and 2 hold, then the origin is ISS for the closed-loop system (19) with region of attraction X_N^{sf} . Furthermore, the input and state constraints (2) are satisfied for all time and for all allowable disturbance sequences if and only if the initial state $x(0) \in X_N^{sf}$.

Proof. For the system of interest, we of course let $f(x, w) := Ax + B\mu_N(x) + w$. Lemma 1 implies that $f(0, 0) = 0$.

Combining Proposition 3 with Lemma 1, it follows that $V_N^*(\cdot)$ is a continuous, positive definite function. Hence, there exist \mathcal{K}_∞ -functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that (29a) holds with $V(\cdot) := V_N^*(\cdot)$ [14, Lem. 4.3].

Using standard techniques [21], it is easy to show that $V(\cdot) := V_N^*(\cdot)$ is a Lyapunov function for the *undisturbed* system $x^+ = Ax + B\mu_N(x)$. More precisely, the methods in [21] can be employed to show that (29b) holds with $\alpha_3(r) := (1/2)\lambda_{\min}(Q)r^2$.

It can be shown [10] from Assumption 1 that X_N^{sf} is robust positively invariant for the closed-loop system (19). Proposition 1 implies that the origin is in the interior of X_N^{sf} . Finally, recall from Proposition 3 that $\mu_N(\cdot)$ and $V_N^*(\cdot)$ are Lipschitz continuous on X_N^{sf} . By combining all of the above, it follows from Lemma 3 that $V_N^*(\cdot)$ is an ISS Lyapunov function for the closed-loop system (19). \square

Remark 5. On examination of the proof of Theorem 3, it is easy to show that the origin is an exponentially stable equilibrium for the undisturbed system $x^+ = Ax + B\mu_N(x)$ with region of attraction X_N^{sf} .

VI. CONCLUSIONS

We have demonstrated that the the disturbance feedback policy defined in Section IV, which is a convex reparameterization of an affine state feedback policy, is useful for synthesizing robust control laws with guaranteed closed-loop properties such as robust invariance, continuity and input-to-state stability. However, there are still a number of issues that need to be addressed. In particular, the paper only considers polyhedral uncertainty representations when dealing with stability, although most of the other results presented here are true for general compact disturbance sets.

The results in Section V on computational tractability may be extended to exploit any additional structure inherent in the robust finite horizon control problem for different classes of disturbance; some results along these lines are already available for a class of problems with ∞ -norm bounded disturbances [8]. It would also be interesting to extend these results to other cost functions, for example worst-case quadratic cost functions where the disturbance is negatively weighted, as in \mathcal{H}_∞ control.

ACKNOWLEDGEMENTS

Eric Kerrigan would like to thank the Royal Academy of Engineering for supporting this research. Paul Goulart would like to thank the Gates Cambridge Trust for their support.

REFERENCES

- [1] A. Bemporad. Reducing conservativeness in predictive control of constrained systems with disturbances. In *Proc. 37th IEEE Conf. on Decision and Control*, pages 1384–1391, Tampa, FL, USA, December 1998.
- [2] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38(1):3–20, January 2002.
- [3] A. Ben-Tal, A. Goryashko, E. Guslitser, and A Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, March 2004.
- [4] D. P. Bertsekas and I. B. Rhodes. On the minimax reachability of target sets and target tubes. *Automatica*, 7(2):233–247, 1971.
- [5] F. Blanchini. Set invariance in control. *Automatica*, 35(1):1747–1767, November 1999.
- [6] L. Chisci, J. A. Rossiter, and G. Zappa. Systems with persistent state disturbances: predictive control with restricted constraints. *Automatica*, 37(7):1019–1028, July 2001.
- [7] S. J. Gartska and R. J-B. Wets. On decisions rules in stochastic programming. *Mathematical Programming*, 7:117–143, 1974.
- [8] P. J. Goulart and E. C. Kerrigan. An efficient decomposition-based formulation for robust control with constraints. In *Proc. 16th IFAC World Congress on Automatic Control*, Prague, Czech Republic, July 2005.
- [9] P. J. Goulart and E. C. Kerrigan. Relationships between affine feedback policies for robust control with constraints. In *16th IFAC World Congress on Automatic Control*, Prague, Czech Republic, July 2005.

- [10] P. J. Goulart, E. C. Kerrigan, and J. M. Maciejowski. Optimization over state feedback policies for robust control with constraints. Technical Report CUED/F-INFENG/TR.494, Cambridge University Engineering Department, August 2005. Accepted to *Automatica*.
- [11] E. Guslitser. Uncertainty-immunized solutions in linear programming. Master’s thesis, Technion, Israeli Institute of Technology, June 2002.
- [12] Z. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37(6):857–869, June 2001.
- [13] E. C. Kerrigan and T. Alamo. A convex parameterization for solving constrained min-max problems with a quadratic cost. In *Proc. 2004 American Control Conference*, Boston, MA, USA, June 2004.
- [14] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, USA, 2002.
- [15] J. Kim, T. Yoon, A. Jadbabaie, and C. De Persis. Input-to-state stabilizing MPC for naturally stable linear systems subject to input constraints. In *Proc. 43rd IEEE Conference on Decision and Control*, pages 5041–5046, Paradise Island, Bahamas, December 2004.
- [16] I. Kolmanovsky and E. G. Gilbert. Theory and computations of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4(4):317–363, 1998.
- [17] W. I. Langson, I. Chrysochoos, S. V. Raković, and D. Q. Mayne. Robust model predictive control using tubes. *Automatica*, 40:125–133, 2004.
- [18] Y. I. Lee and B. Kouvaritakis. Constrained receding horizon predictive control for systems with disturbances. *International Journal of Control*, 72(11):1027–1032, August 1999.
- [19] J. Löfberg. Approximations of closed-loop MPC. In *Proc. 42nd IEEE Conference on Decision and Control*, pages 1438–1442, Maui, Hawaii, USA, December 2003.
- [20] D. Limón Marruedo, T. Álamo, and E. F. Camacho. Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive disturbances. In *Proc. 41st IEEE Conference on Decision and Control*, pages 4619–2624, Las Vegas, Nevada USA, December 2002.
- [21] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, June 2000. Survey paper.
- [22] D. Q. Mayne and W. R. Schroeder. Robust time-optimal control of constrained linear systems. *Automatica*, 33:2103–2118, 1997.
- [23] S. V. Rakovic, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne. Invariant approximations of the minimal robust positively invariant set. *IEEE Trans. Automatic Control*, 50(3), March 2005.
- [24] P. O. M. Scokaert and D. Q. Mayne. Min-max feedback model predictive control for constrained linear systems. *IEEE Transactions on Automatic Control*, 43(8):1136–1142, August 1998.
- [25] R. S. Smith. Robust model predictive control of constrained linear systems. In *Proc. 2004 American Control Conference*, Boston, MA, USA, June 2004.
- [26] D. H. van Hessem and O. H. Bosgra. A conic reformulation of model predictive control including bounded and stochastic disturbances under state and input constraints. In *Proc. 41st IEEE Conference on Decision and Control*, pages 4643–4648, December 2002.

APPENDIX

Define $\mathbf{A} \in \mathbb{R}^{n(N+1) \times n}$ and $\mathbf{E} \in \mathbb{R}^{n(N+1) \times nN}$ as

$$\mathbf{A} := \begin{bmatrix} I_n \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \mathbf{E} := \begin{bmatrix} 0 & 0 & \dots & 0 \\ I_n & 0 & \dots & 0 \\ A & I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1} & A^{N-2} & \dots & I_n \end{bmatrix}.$$

The matrices $\mathbf{B} \in \mathbb{R}^{n(N+1) \times mN}$, $\mathbf{C} \in \mathbb{R}^{t \times n(N+1)}$ and $\mathbf{D} \in \mathbb{R}^{t \times mN}$ are defined as $\mathbf{B} := \mathbf{E}(I_N \otimes B)$, $\mathbf{C} := \begin{bmatrix} I_N \otimes C & 0 \\ 0 & Y \end{bmatrix}$, and $\mathbf{D} := \begin{bmatrix} I_N \otimes D \\ 0 \end{bmatrix}$. It is easy to verify that (12) is equivalent to (14) with $F := \mathbf{CB} + \mathbf{D}$, $G := \mathbf{CE}$, $H := -\mathbf{CA}$, and $c := \begin{bmatrix} \mathbf{1}_N \otimes b \\ z \end{bmatrix}$.