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A Frequency-Domain Solution to the Sampled-Data H^2 Smoothing Problem

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***Abstract*— This paper studies the sampled-data H^2 smoothing problem. Using the compact representation of the lifted transfer functions introduced in the companion paper [12], a frequency-domain approach is used to solve the problem in the lifted domain. The resulting estimators have the structure of the cascade of a *linear discrete time-invariant* smoother and a *generalized hold* D/A converter.**

I. Introduction

Consider the sampled-data estimation setup in Fig. 1(a). Here $\mathcal{G}$ is a continuous-time system, which generates signals ν and y from an exogenous signal w. The output y is sampled by the ideal sampler $\mathcal{S}_h$ generating the measured discrete-time signal $\bar{y}$ as $\bar{y}_k = y(kh)$, where h is the sampling period. The problem is to construct a hybrid (having a discrete-time input and a continuous-time output) estimator $\mathcal{K}$ so that its output ν_e be "close" to the unmeasured signal ν. The closeness of ν and ν_e is conventionally measured by either H^2 or H^∞ norm of the system from the exogenous input w to the estimation error $e = \nu - \nu_e$.

When $\mathcal{K}$ is constrained to be causal, the problem is called the sampled-data *filtering* problem. This problem is currently fairly well-understood both in the H^2 [7] and in the H^∞ [15], [18] cases. Sampled-data estimation problems where the causality of $\mathcal{K}$ is relaxed are noticeably less studied. Such problems fall into the category of the sampled-data *smoothing* problems. In particular, when no causality constraints are imposed on $\mathcal{K}$, the problem is called the *fixed-interval smoothing*, whereas when $\mathcal{K}$ has only a finite-length preview, it is called the *fixed-lag smoothing* and the length of the preview is said to be the *smoothing lag*. The fixed-interval case may be motivated by off-line data processing problems or situations where the "time" variable is a spatial coordinate (so that no causality requirements exist). The fixed-lag case corresponds to situations in which some amount of delay or latency between the measurement and estimation generation can be tolerated.

The sampled-data smoothing proved to be a challenge. For example, the sampled signal reconstruction problem in the the H^∞ setting, which is a special case of the general smoothing setup in Fig. 1(a), is stated as Open Problem 51 in [1]. To the best of our knowledge, the only analysis of the achievable performance in the H^∞ fixed-lag smoothing problem was carried out in [13], yet only with a smoothing lag of *one sampling period*.

In this paper the sampled-data smoothing problem is studied in the H^2 setting. We derive the first solution to this problem and show that the frequency-domain solution procedure can be successfully carried out in the lifting domain by the use of the technique proposed in [12]. The resulting smoother is of the form of the cascade of a finite-dimensional LTI discrete part and a generalized hold function. Although the H^2 version of the problem is less challenging than its H^∞ counterpart, we believe that the solution procedure adopted here can be extended to the H^∞ case following the continuous-time solution of [17]. Moreover, in the fixed-interval case the H^2 and H^∞ solutions coincide [6], so that the proposed solution partially addresses the H^∞ smoothing as well.

The paper is organized as follows. In Section II the smoothing problem is formulated and its solution in the lifted domain is derived. As this solution is not readily implementable, the rest of the paper is devoted to its peeling-off back to the time domain. To this end, Section III addresses some underlying technical steps used in both the fixed-interval and the fixed-lag versions of the problem. The former is then completely solved in Section IV and the latter— in Section V. Concluding remarks are provided in Secttion VI. Appendix A contains some basic definitions related to STPBCs.

Notations: Some notations used throughout the paper are based on those introduced in the companion paper [12]. The other notations are fairly standard. $L^2(\mathbb{T})$, or just L^2, denotes the space of Hilbert-Schmidt functions, square integrable on the unit circle $\mathbb{T} \doteq \{z \in \mathbb{C} : |z| = 1\}$. $L^\infty(\mathbb{T})$, or simply L^∞, stands for the space of functions bounded on $\mathbb{T}$ (note that in the cases studied in this paper $L^\infty \subset L^2$). The Hardy space H^p, where p is either 2 or ∞, is the subspace of L^p comprised of functions analytic and bounded in $|z| > 1$. Given an $l \in \mathbb{Z}^+$, the space of functions $f(z)$ such that $z^{-l} f(z) \in H^p$ is referred to as $z^l H^p$. It is readily seen that $z^l H^p \subset z^{l+1} H^p \subset L^p$ for all l. Finally, by $\mathrm{proj}_{z^l H^2}(\cdot)$ we mean the orthogonal projection operator $L^2 \mapsto z^l H^2$.

II. Problem Formulation and Abstract Solution

The estimation problem addressed in this paper is formulated in the lifted domain. To this end, the hybrid (continuous/discrete) periodically time-varying system in Fig. 1(a) is converted to an equivalent purely discrete time-invariant system in Fig. 1(b) using the lifting transformation [2]. The lifting of the combination of $\mathcal{G}$ and $\mathcal{S}_h$ (the light gray box in Fig. 1(a)) is an LTI discrete signal generator of the form[1]

$$\breve{G}(z) = \begin{bmatrix} \breve{G}_\nu(z) \\ \acute{G}_y(z) \end{bmatrix}$$

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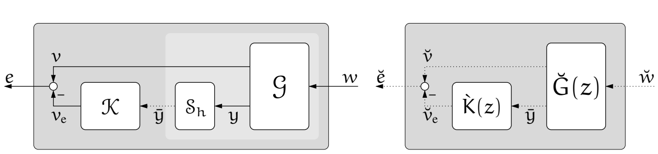

(a) In the time domain

(b) In the lifted domain

Fig. 1. General sampled-data estimation setup.

[1]Hereinafter a bar, like $\bar{O}$, indicates an operator whose input and output spaces are both finite dimensional; a grave accent, $\grave{O}$, indicates a finite-dimensional input space and a distributed output space, like $\mathcal{K}_h$; an acute accent, $\acute{O}$, indicates a finite-dimensional output space and a distributed input space; finally, a breve accent, $\breve{O}$, indicates that both the input and output take distributed values.

and the lifting of the estimator is $\grave{K}(z)$. The fixed-lag smoothing problem for this system is formulated as follows:

$\mathbf{SP}_{\breve{G},l}$: Given $\breve{G}(z)$ and $l \geqslant 0$, find $\grave{K}(z) \in z^l H^\infty$ such that

$$\breve{G}_e(z) \doteq \breve{G}_v(z) - \grave{K}(z)\acute{G}_y(z) \in z^l H^\infty$$

and $\|\breve{G}_e(z)\|_{L^2}$ is minimized.

The problem $\mathbf{SP}_{\breve{G},0}$, i.e., that for $l = 0$, corresponds to the filtering problem (so that $z^l H^\infty = H^\infty$). The other extreme case, $\mathbf{SP}_{\breve{G},\infty}$, corresponds to the fixed-interval smoothing in which the estimator has access to all future measurements (then $z^l H^\infty = L^\infty$).

Remark 2.1: The requirement on $\grave{K}$ and $\breve{G}_e$ to belong to $z^l H^\infty$ is the stability requirement relaxed to allow the estimator to have the l-step preview. When the preview is finite, the problem can be reformulated in a more conventional form of minimizing the H^2 norm of $z^{-l}\breve{G}_e$. This is the reason why we refer to $\mathbf{SP}_{\breve{G},l}$ as the H^2 problem.

To guarantee the solvability and well-posedness of the estimation problem we need to impose some constraints on the problem data. First, we assume that

$\mathcal{A}_1$: $\exists \grave{K} \in H^\infty$ such that $\breve{G}_v - \grave{K}\acute{G}_y \in H^\infty$.

This assumption might appear somewhat restrictive as we actually need the existence of a $\grave{K} \in z^l H^\infty$ such that $\breve{G}_v - \grave{K}\acute{G}_y \in z^l H^\infty$. This is clearly guaranteed by $\mathcal{A}_1$, yet, in general, not vice versa. $\mathcal{A}_1$, however, rules out admissible solutions only in the case when $\breve{G}_v(z)$ is not proper, which is not considered in this paper. Hence, this assumption does not impose any loss of generality for the problem in Fig. 1(a). We also assume that

$\mathcal{A}_2$: $\acute{G}_y(e^{j\theta})\acute{G}_y(e^{j\theta})^*$ is nonsingular $\forall\theta \in [0, 2\pi]$,

which guarantees the well-posedness of the optimization problem.

The solution of $\mathbf{SP}_{\breve{G},l}$ in the fixed-lag smoothing case is simplified when the following procedure from [11], [17] (it roots in [10]), which we call the *stabilification*, is applied to transform the original problem to that with stable normalized data. Toward this end, the following result plays a key role.

Proposition 2.1: Let $\breve{G}$ admit a left coprime factorization in H^∞. Then $\mathcal{A}_1$ holds iff there exists a left coprime factorization of $\breve{G}$ of the form

$$\breve{G}(z) = \begin{bmatrix} I & \grave{M}_v(z) \\ 0 & \bar{M}_y(z) \end{bmatrix}^{-1} \begin{bmatrix} \breve{N}_v(z) \\ \acute{N}_y(z) \end{bmatrix} \tag{1}$$

for some $\grave{M}_v, \bar{M}_y, \breve{N}_v, \acute{N}_y \in H^\infty$.

Proof: (If) Assume that the factorization (1) exists. It is then readily seen that $\grave{K} = -\grave{M}_v$ satisfies $\mathcal{A}_1$.

(Only if) Now assume that $\grave{K}$ satisfies $\mathcal{A}_1$. Let $\breve{M}^{-1}\breve{N}$ be a left coprime factorization of $\breve{G}$ in H^∞ and denote $\breve{K}_\alpha \doteq \begin{bmatrix} I & -\grave{K} \end{bmatrix} \breve{M}^{-1}$. Then $\mathcal{A}_1$ is equivalent to the condition

$$\breve{K}_\alpha \begin{bmatrix} \breve{M} & \breve{N} \end{bmatrix} \in H^\infty$$

which, together with the left coprimeness of $\breve{M}$ and $\breve{N}$, implies that $\breve{K}_\alpha \in H^\infty$ as well. Partition now $\breve{M}$ compatibly with the partitioning of $\begin{bmatrix} I & -\grave{K} \end{bmatrix}$, i.e., $\breve{M} = \begin{bmatrix} \breve{M}_1 & \grave{M}_2 \end{bmatrix}$. Then

$$\breve{K}_\alpha \begin{bmatrix} \breve{M}_1 & \grave{M}_2 \end{bmatrix} = \begin{bmatrix} I & -\grave{K} \end{bmatrix} \quad \Rightarrow \quad \breve{K}_\alpha \breve{M}_1 = I,$$

i.e., $\breve{M}_1$ is left invertible in H^∞. This implies that there exists a $\breve{U}, \breve{U}^{-1} \in H^\infty$ such that $\breve{U}\breve{M}_1 = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, so that $\breve{U}\breve{M}$ and $\breve{U}\breve{N}$ constitute the desired left coprime factorization of $\breve{G}$. $\square$

Having the factorization (1), the problem $\mathbf{SP}_{\breve{G},l}$ can be reformulated as an equivalent problem with stable data by replacing $\grave{K}$ with $\grave{K}_a \doteq (\grave{K} - \grave{M}_v)\bar{M}_y^{-1}$. It can be shown [11, Lemma 5] that the error transfer function is then equivalent to $\breve{N}_v - \grave{K}_a\acute{N}_y$ and, provided $\breve{G}_e \in z^l H^\infty$, $\grave{K} \in z^l H^\infty$ iff so does $\grave{K}_a$. Thus, the original smoothing problem can be equivalently formulated as $\mathbf{SP}_{\breve{N},l}$, in which the "stability" of the error transfer function should not be taken care of as it is redundant. The solution $\grave{K}_a$ of $\mathbf{SP}_{\breve{N},l}$ can then be used to produce the solution $\grave{K}$ of $\mathbf{SP}_{\breve{G},l}$ according to

$$\grave{K}(z) = \grave{K}_a(z)\bar{M}_y(z) - \grave{M}_v(z). \tag{2}$$

Now, $\mathcal{A}_2$ guarantees that the numerator in (1) can always be chosen to satisfy

$$\begin{bmatrix} \breve{N}_v(z) \\ \acute{N}_y(z) \end{bmatrix} \acute{N}_y^\sim(z) = \begin{bmatrix} \acute{V}^\sim(z) \\ I \end{bmatrix} \tag{3}$$

for some *strictly proper* $\acute{V} \in H^\infty$ (we show this by construction in Section III). Then, standard completion of squares arguments yield:

$$\breve{G}_e\breve{G}_e^\sim = (\acute{V}^\sim - \grave{K}_a)(\acute{V} - \grave{K}_a^\sim) + (\breve{N}_v\breve{N}_v^\sim - \acute{V}^\sim\acute{V}).$$

Since[2]

$$\|\breve{G}_e\|_{L^2}^2 = \frac{1}{2\pi}\,\mathrm{tr}\left(\int_0^{2\pi} \breve{G}_e(e^{j\theta})\breve{G}_e^\sim(e^{j\theta})d\theta\right)$$

and since $\mathrm{tr}(O_1 + O_2) = \mathrm{tr}(O_1) + \mathrm{tr}(O_2)$, $\mathbf{SP}_{\breve{N},l}$ is equivalent to the (one-block) problem of minimizing

$$\|\acute{V}^\sim(z) - \grave{K}_a(z)\|_{L^2}.$$

The latter problem then amounts to a straightforward application of the Projection Theorem and is thus solved by

$$\grave{K}_a = \mathrm{proj}_{z^l H^2}(\acute{V}^\sim) \doteq \acute{V}_+^\sim \tag{4}$$

with the optimal performance level of

$$\gamma_{\mathrm{opt}} = \sqrt{\|\breve{N}_v\|_{H^2}^2 - \|\acute{V}_+\|_{H^2}^2}$$

(here the orthogonality of $\acute{V}_+$ and $\acute{V} - \acute{V}_+$ was used). As $\|\breve{N}_v\|_{H^2}$ is the achievable performance in the filtering case, $\|\acute{V}_+\|_{H^2}$ actually quantifies the performance improvement due to smoothing.

When l is finite, $\acute{V}_+$ is a strictly proper FIR (finite impulse response) system, the (operator-valued) impulse response of which is the truncation of that of $\acute{V}(z)$ to the first $l+1$ steps (in the filtering case, i.e., $l = 0$, $\acute{V}_+ = 0$). In this case, the optimal estimator in (2) is the cascade of an IIR (infinite impulse response) system comprised of $\grave{M}_v$ and $\bar{M}_y$ and an FIR system.

When $l = \infty$ (fixed-interval smoothing), $\acute{V}_+ = \acute{V}$ is an IIR system. In this case, it might be more convenient to split the resulting $\grave{K} \in L^\infty$ into causal and anti-causal parts. The former corresponds to a part of $\grave{K}$, analytic outside the unit disc, whereas the latter—to its part, analytic in the unit disc.

III. Stabilification with Co-Inner Numerator

Bring in a *minimal* state-space realization of $\mathcal{G}$ in Fig. 1(a):

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C_v & 0 \\ C_y & 0 \end{array}\right].$$

Here, $G(\infty)$ is taken zero to guarantee the boundedness of the L^2 norm ($D_v = 0$) and to reflect the presence of an anti-aliasing filter in the measurement channel ($D_y = 0$), which is necessary for the boundedness of the ideal sampler $\mathcal{S}_h$, see [2]. We also impose the following assumptions on the parameters of this realization:

$\mathcal{A}_3$: the pair (C_y, e^{Ah}) is detectable;

$\mathcal{A}_4$: C_y has full row rank.

[2]By $\mathrm{tr}()$ we understand the trace of a Hilbert-Schmidt operator [3].

Assumption $\boldsymbol{\mathcal{A}_3}$ is a counterpart of $\boldsymbol{\mathcal{A}_1}$ and assumption $\boldsymbol{\mathcal{A}_4}$, which actually rules out trivially redundant measurement channels, is equivalent to $\boldsymbol{\mathcal{A}_2}$. Note that $\boldsymbol{\mathcal{A}_4}$ is milder than its counterpart in the continuous-time case, which requires the absence of invariant zeros on the $j\omega$-axis.

Making use of the developments and notations of [12], the factors in (1) are of the form

$$\begin{bmatrix} \check{N}_v(z) \\ \acute{N}_y(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{z}\mathcal{J}_h^* \end{bmatrix} \left(\begin{array}{c|c|c} A & zI \rightleftharpoons -(I+LC_y) & B \\ \hline C_v & & 0 \\ \tilde{C}_y & & 0 \end{array}\right) \tag{5a}$$

and

$$\begin{bmatrix} \grave{M}_v(z) \\ \bar{M}_y(z) \end{bmatrix} = \begin{bmatrix} 0 \\ J \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \frac{1}{z}\mathcal{J}_h^* \end{bmatrix} \left(\begin{array}{c|c|c} A & zI \rightleftharpoons -(I+LC_y) & L \\ \hline C_v & & 0 \\ \tilde{C}_y & & 0 \end{array}\right) \mathcal{J}_0, \tag{5b}$$

where L is any matrix such that the matrix

$$\bar{A}_L \doteq (I+LC_y)e^{Ah} \tag{6}$$

is Schur (such an L exists by $\boldsymbol{\mathcal{A}_3}$), $\tilde{C}_y \doteq JC_y$, and J is any nonsingular matrix.

The next step is to find L such that $\acute{N}_y$ is co-inner. To this end, note that

$$\acute{N}_y^{\sim}(z) = z\left(\begin{array}{c|c|c} -A' & z(I+LC_y)' \rightleftharpoons -I & \tilde{C}_y' \\ \hline -B' & & 0 \end{array}\right)\mathcal{J}_h,$$

so that

$$\begin{aligned} \bar{\Xi}(z) &\doteq J^{-1}\acute{N}_y(z)\acute{N}_y^{\sim}(z)J^{-\prime} \\ &= \mathcal{J}_h^* \left(\begin{array}{cc|c|c} A & -BB' & z\begin{bmatrix} I & 0 \\ 0 & (I+LC_y)' \end{bmatrix} \rightleftharpoons -\begin{bmatrix} I+LC_y & 0 \\ 0 & I \end{bmatrix} & 0 \\ 0 & -A' & & C_y' \\ \hline C_y & 0 & & 0 \end{array}\right)\mathcal{J}_h. \end{aligned}$$

Following the procedure in [20, §13.6], we split $\bar{\Xi}(z)$ to the sum of stable and anti-stable transfer functions. This is accomplished by a time-varying state transformation with the matrix $T(t) = \left[\begin{smallmatrix} I & Q(t) \\ 0 & I \end{smallmatrix}\right]$, where $Q(t)$ satisfies the differential Lyapunov equation

$$\dot{Q}(t) = AQ(t) + Q(t)A' + BB', \quad Q(0) = Q_0 \tag{7a}$$

(we also denote $Q_h \doteq Q(h)$). This brings the "A" matrix of the STPBC above to the diagonal form, $\operatorname{diag}\{A, -A'\}$, thus decoupling the intersample dynamics, and the boundary conditions to the form

$$z\begin{bmatrix} I & -Q_0 \\ 0 & (I+LC_y)' \end{bmatrix} \rightleftharpoons -\begin{bmatrix} I+LC_y & -(I+LC_y)Q_h \\ 0 & I \end{bmatrix}.$$

To decouple the boundary conditions, premultiply both sides by $\left[\begin{smallmatrix} I & (I+LC_y)Q_h \\ 0 & I \end{smallmatrix}\right]$. We obtain equivalent boundary conditions

$$z\begin{bmatrix} I & (I+LC_y)Q_h(I+LC_y)' - Q_0 \\ 0 & (I+LC_y)' \end{bmatrix} \rightleftharpoons -\begin{bmatrix} I+LC_y & 0 \\ 0 & I \end{bmatrix}.$$

Thus, the stipulation

$$Q_0 = (I+LC_y)Q_h(I+LC_y)' \tag{7b}$$

would guarantee the decoupling of the boundary conditions and result in the following representation:

$$\bar{\Xi}(z) = \mathcal{J}_h^* \left(\begin{array}{cc|c|c} A & 0 & z\begin{bmatrix} I & 0 \\ 0 & (I+LC_y)' \end{bmatrix} \rightleftharpoons -\begin{bmatrix} I+LC_y & 0 \\ 0 & I \end{bmatrix} & QC_y' \\ 0 & -A' & & C_y' \\ \hline C_y & -C_yQ & & 0 \end{array}\right)\mathcal{J}_h.$$

To express the boundary solutions of (7) in a more conventional form, introduce the matrix function

$$\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ 0 & \Sigma_{22}(t) \end{bmatrix} \doteq \exp\left(\begin{bmatrix} A & -BB' \\ 0 & -A' \end{bmatrix} t\right) \tag{8}$$

(when $t = h$ we omit the time argument and write simply Σ). It can be shown [2, Lemma 10.5.1] that $\Sigma_{11}(t) = e^{At}$, $\Sigma_{22}(t) = e^{-A't}$, and

$$\Sigma_{12}(t) = -\int_0^t e^{A\tau}BB'e^{A'\tau}d\tau\, e^{-A't}.$$

It is also known [5] that the solution of (7a) is

$$Q(t) = \big(\Sigma_{11}(t)Q_0 - \Sigma_{12}(t)\big)\Sigma_{22}^{-1}(t). \tag{9}$$

This equation for $t = h$ combined with (7b) leads to the following equation for Q_h:

$$Q_h = \Sigma_{11}(I+LC_y)Q_h(I+C_y'L')\Sigma_{11}' - \Sigma_{12}\Sigma_{11}'. \tag{10}$$

This is a discrete Lyapunov equation and since $\Sigma_{12}\Sigma_{11}' < 0$ and $\Sigma_{11}(I+LC_y) = \Sigma_{11}\bar{A}_L\Sigma_{11}^{-1}$ is Schur, it is always solvable with $Q_h > 0$. It can also be easily shown that Q_0 satisfies a Lyapunov equation too and is actually the controllability Gramian of the "natural" (the one having $\bar{A}_L$ as its "A" matrix) state-space realization of $\check{N}(z)$.

Remark 3.1: As already mentioned, the technique for obtaining the coprime factorization with a co-inner numerator presented above follows the steps described in [20, §13.6]. The key difference from the "standard" procedure applied to continuous-time transfer matrices is that the similarity transformation splitting $\bar{\Xi}(z)$ into two parts is time varying. The reason is that in order to split STPBCs we need to decouple *both* the continuous-time dynamics *and* the boundary conditions. A time-invariant transformation would not offer enough freedom to affect both these components separately.

Now, denoting

$$\Xi_s(z) \doteq C_y e^{Ah}(zI - \bar{A}_L)^{-1}(I+LC_y)Q_hC_y'$$

and applying [12, Proposition A.1] to $\bar{\Xi}(z)$ above, we obtain:

$$\begin{aligned} \bar{\Xi}(z) &= \Xi_s(z) - C_yQ_he^{-A'h}\big(z(I+LC_y)' - e^{-A'h}\big)^{-1}C_y' \\ &= \Xi_s(z) + \Xi_s^{\sim}(z) + C_yQ_hC_y'. \end{aligned}$$

It is readily seen that if there exists L satisfying

$$(I+LC_y)Q_hC_y' = 0, \tag{11}$$

then $\Xi_s(z) = 0$ and therefore $\bar{\Xi}(z) = C_yQ_hC_y'$ is static. Since $Q_h > 0$ and $\boldsymbol{\mathcal{A}_4}$ holds, the matrix $C_yQ_hC_y'$ is nonsingular, so that

$$L = -Q_hC_y'(C_yQ_hC_y')^{-1} \tag{12}$$

and the choice $J = (C_yQ_hC_y')^{-1/2}$ leads to the required equality $\acute{N}_y\acute{N}_y^{\sim} = I$. The substitution of L from (12) to (10) yields the following equation for Q_h:

$$Q_h = \Sigma_{11}\big(Q_h - \Sigma_{11}^{-1}\Sigma_{12} - Q_hC_y'(C_yQ_hC_y')^{-1}C_yQ_h\big)\Sigma_{11}'. \tag{13}$$

This is actually the discrete H^2 ARE associated with $\acute{G}_y$ [19] and the existence of its stabilizing solution (i.e., such that $\Sigma_{11}(I+LC_y)$ is Schur) is guaranteed by $\boldsymbol{\mathcal{A}_1}$ and $\boldsymbol{\mathcal{A}_2}$.

Now, consider the first part of (3). Following the arguments above, it can be shown that

$$\begin{aligned} \check{N}_v\acute{N}_y^{\sim} &= z\left(\begin{array}{cc|c|c} A & 0 & z\begin{bmatrix} I & 0 \\ 0 & (I+LC_y)' \end{bmatrix} \rightleftharpoons -\begin{bmatrix} I+LC_y & 0 \\ 0 & I \end{bmatrix} & Q\tilde{C}_y' \\ 0 & -A' & & \tilde{C}_y' \\ \hline C_v & -C_vQ & & 0 \end{array}\right)\mathcal{J}_h \\ &= z\left(\begin{array}{c|c|c} -A' & z(I+LC_y)' \rightleftharpoons -I & \tilde{C}_y' \\ \hline -C_vQ & & 0 \end{array}\right)\mathcal{J}_h, \end{aligned}$$

so that

$$\acute{V}(z) = z^{-1}\mathcal{J}_h^*\left(\begin{array}{c|c|c} A & zI \rightleftharpoons -(I+LC_y) & QC_v' \\ \hline \tilde{C}_y & & 0 \end{array}\right). \tag{14}$$

Summarizing the discussion above, the following result can be formulated:

Lemma 3.1: Let assumptions $\boldsymbol{\mathcal{A}_3}$ and $\boldsymbol{\mathcal{A}_4}$ hold. Then the Riccati equation (13) admits the stabilizing solution $Q_h > 0$ and the left coprime factorization (1) of $\check{G}(z)$ satisfying (3) exists and is given by (5), where L is chosen according to (12) and $\tilde{C}_y = JC_y$ for any J satisfying $J'J = (C_yQ_hC_y')^{-1}$.

Remark 3.2: It can be shown that Q_0 is actually the stabilizing solution of the discrete ARE associated with the extended symplectic pencil

$$\lambda \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \Sigma_{11} & \Sigma_{12} \\ C_y' & 0 & \Sigma_{22} \\ 0 & C_y\Sigma_{11} & C_y\Sigma_{12} \end{bmatrix}$$

and it satisfies $C_yQ_0 = 0$. This is exactly the ARE obtained in [16] (for the case $\gamma \to \infty$).

IV. Fixed-Interval Smoothing Formulae

We first consider the fixed-interval smoothing problem $\mathbf{SP}_{\check{G},\infty}$, i.e., the problem where the estimator has all future measurements of $\bar{y}$ available. In §IV-A we derive the optimal estimator by decomposing it into causal and anti-causal parts. The optimal performance level is then calculated in §IV-B.

A. Optimal estimator

In this case the optimal estimator is

$$\grave{K}(z) = \grave{K}_{\mathrm{FIS}}(z) \doteq \acute{V}^{\sim}(z)\bar{M}_y(z) - \grave{M}_v(z).$$

As discussed at the end of Section II, we decompose this estimator into two parts, analytical outside and inside the unit disc. The former yields the causal part, while the latter—the anti-causal part of $\grave{K}_{\mathrm{FIS}}(z)$.

We start with the decomposition of

$$\begin{aligned}
\grave{\Psi} &\doteq \acute{V}^{\sim}(\bar{M}_y - J) \\
&= \left(\begin{array}{c|c|c} -A' & \boxed{z(I+LC_y)' \rightleftharpoons -I} & \tilde{C}_y' \\ \hline -C_vQ & & 0 \end{array}\right)\mathcal{J}_h\mathcal{J}_h^{*}\left(\begin{array}{c|c|c} A & \boxed{zI \rightleftharpoons -(I+LC_y)} & L \\ \hline \tilde{C}_y & & 0 \end{array}\right)\mathcal{J}_0 \\
&= \left(\begin{array}{cc|c|c} A & 0 & \boxed{z\left[\begin{smallmatrix} I & 0 \\ 0 & (I+LC_y)' \end{smallmatrix}\right] \rightleftharpoons -\left[\begin{smallmatrix} I+LC_y & 0 \\ \tilde{C}_y'\tilde{C}_y & I \end{smallmatrix}\right]} & L \\ 0 & -A' & & 0 \\ \hline 0 & -C_vQ & & 0 \end{array}\right)\mathcal{J}_0,
\end{aligned}$$

where the last equality is obtained by Proposition A.1. Apply now a time-varying state transformation with the matrix $\left[\begin{smallmatrix} I & 0 \\ P(t) & I \end{smallmatrix}\right]$, where $P(t)$ satisfies the differential Lyapunov equation

$$\dot{P}(t) = -P(t)A - A'P(t), \quad P(0) = P_0 \tag{15a}$$

(we also denote $P_h \doteq P(h)$). This transformation does not alter the "A" matrix, yet transforms the boundary conditions to

$$z\begin{bmatrix} I & 0 \\ -(I+LC_y)'P_0 & (I+LC_y)' \end{bmatrix} \rightleftharpoons -\begin{bmatrix} I+LC_y & 0 \\ \tilde{C}_y'\tilde{C}_y - P_h & I \end{bmatrix}.$$

To decouple the boundary conditions, premultiply both sides by $\left[\begin{smallmatrix} I & 0 \\ (I+LC_y)'P_0 & I \end{smallmatrix}\right]$, which results in

$$z\begin{bmatrix} I & 0 \\ 0 & I + C_y'L' \end{bmatrix} \rightleftharpoons -\begin{bmatrix} I+LC_y & 0 \\ \tilde{C}_y'\tilde{C}_y - P_h + (I+LC_y)'P_0(I+LC_y) & I \end{bmatrix}.$$

Taking into account the definition of $\tilde{C}_y$, the boundary conditions are then decoupled if

$$P_h = (I+LC_y)'P_0(I+LC_y) + C_y'(C_yQ_hC_y')^{-1}C_y. \tag{15b}$$

It is readily verifiable that the solution of (15a) is

$$P(t) = \Sigma_{22}(t)P_0\Sigma_{11}^{-1}(t), \tag{16}$$

which yields the relation $P_h = \Sigma_{11}^{-\prime}P_0\Sigma_{11}^{-1}$. Thus, equations (15) result in the following discrete Lyapunov equation for P_0:

$$P_0 = \bar{A}_L'P_0\bar{A}_L + \Sigma_{11}'C_y'(C_yQ_hC_y')^{-1}C_y\Sigma_{11}, \tag{17}$$

which is the observability Gramian of the "natural" realizations of $\acute{N}_y(z)$ and $\acute{V}(z)$ (those having $\bar{A}_L$ as their "A" matrix).

Thus, the transfer matrix $\grave{\Psi}(z)$ is decoupled as

$$\begin{aligned}
\grave{\Psi}(z) = &\left(\begin{array}{c|c|c} A & \boxed{zI \rightleftharpoons -(I+LC_y)} & L \\ \hline C_vQP & & 0 \end{array}\right)\mathcal{J}_0 \\
&+ \left(\begin{array}{c|c|c} -A' & \boxed{z(I+LC_y)' \rightleftharpoons -I} & PL \\ \hline -C_vQ & & 0 \end{array}\right)\mathcal{J}_0.
\end{aligned}$$

Therefore, the causal part of the optimal estimator is

$$\begin{aligned}
\grave{K}_c(z) &= \left(\begin{array}{c|c|c} A & \boxed{zI \rightleftharpoons -(I+LC_y)} & L \\ \hline C_vQP & & 0 \end{array}\right)\mathcal{J}_0 - \grave{M}_v(z) \\
&= \left(\begin{array}{c|c|c} A & \boxed{zI \rightleftharpoons -(I+LC_y)} & L \\ \hline C_v(QP - I) & & 0 \end{array}\right)\mathcal{J}_0
\end{aligned}$$

(note that it contains the feedthrough term $C_y(I + QP)e^{At}L$) and, taking into account the equalities

$$\begin{aligned}
&\left(\begin{array}{c|c|c} -A' & \boxed{z(I+LC_y)' \rightleftharpoons -I} & PL \\ \hline -C_vQ & & 0 \end{array}\right)\mathcal{J}_0 \\
&\qquad = z\left(\begin{array}{c|c|c} -A' & \boxed{z(I+LC_y)' \rightleftharpoons -I} & (I+LC_y)'P_0L \\ \hline -C_vQ & & 0 \end{array}\right)\mathcal{J}_h
\end{aligned}$$

and $C_y'JJ' = -P_hL$, the strictly anti-causal part of the estimator is

$$\begin{aligned}
\grave{K}_{ac}(z) &= \acute{V}^{\sim}(z)\,J + z\left(\begin{array}{c|c|c} -A' & \boxed{z(I+LC_y)' \rightleftharpoons -I} & PL \\ \hline -C_vQ & & 0 \end{array}\right)\mathcal{J}_0 \\
&= z\left(\begin{array}{c|c|c} -A' & \boxed{z(I+LC_y)' \rightleftharpoons -I} & (P_h - (I+LC_y)'P_0)L \\ \hline C_vQ & & 0 \end{array}\right)\mathcal{J}_h.
\end{aligned}$$

It is readily seen that both $\grave{K}_c$ and $\grave{K}_{ac}$ can be implemented as a combination of finite-dimensional discrete systems and zero-order generalized hold functions. Indeed, it follows from the solution of the corresponding STPBCs that $\grave{K}_c$ is the cascade of

$$\bar{K}_c(z) = -z\left[\begin{array}{c|c} \bar{A}_L & L \\ \hline I & 0 \end{array}\right] \tag{18a}$$

(with $\bar{A}_L$ defined by (6)) and the generalized hold having the following hold function, defined for $t \in [0, h]$:

$$\begin{aligned}
\phi_c(t) &= C_v\bigl(I - Q(t)P(t)\bigr)e^{At} \\
&= \begin{bmatrix} C_v & 0 \end{bmatrix}\Sigma(t)\left[\begin{smallmatrix} I - Q_0P_0 \\ P_0 \end{smallmatrix}\right],
\end{aligned} \tag{19a}$$

where (9) and (16) are used to obtain the last expression. Similarly, $\grave{K}_{ac}$ is the cascade of

$$\bar{K}_{ac}(z^{-1}) = -\left[\begin{array}{c|c} \bar{A}_L' & e^{A'h}(P_h - (I+LC_y)'P_0)L \\ \hline I & 0 \end{array}\right] \tag{18b}$$

(mind the inverse of z) and the generalized hold having the following hold function, defined for $t \in [0, h]$:

$$\begin{aligned}
\phi_{ac}(t) &= C_vQ(t)e^{-A't} \\
&= \begin{bmatrix} C_v & 0 \end{bmatrix}\Sigma(t)\left[\begin{smallmatrix} Q_0 \\ -I \end{smallmatrix}\right]
\end{aligned} \tag{19b}$$

(in fact, $\phi_c(t) = C_ve^{At} - \phi_{ac}(t)P_0$ and C_ve^{At} is the hold function in the filtering case).

Remark 4.1: It is known [6] that the solution of the $L^2(\mathbb{T})$ estimation (fixed-interval smoothing) problem coincides with that of the $L^{\infty}(\mathbb{T})$ problem. Hence, the optimal estimator derived above also solves the sampled-data H^{∞} fixed-interval smoothing problem.

B. Optimal performance

To complete the solution, we only need to calculate the (optimal) performance level γ_{opt} achievable with $\grave{K}_c+\grave{K}_{ac}$. It follows from the analysis in Section II that the optimal performance for $\mathbf{SP}_{\check{G},\infty}$ is

$$\gamma_{\text{opt}} = \sqrt{\|\check{N}_v\|_{H^2}^2 - \|\acute{V}\|_{H^2}^2}.$$

We calculate each of the terms in the right-hand side above separately. To this end, introduce the matrix exponential

$$\begin{bmatrix} \Sigma & \Delta \\ 0 & \Sigma \end{bmatrix} \doteq \exp\left(\left[\begin{array}{cc:cc} A & -BB' & 0 & 0 \\ 0 & -A' & -C_v'C_v & 0 \\ \hdashline 0 & 0 & A & -BB' \\ 0 & 0 & 0 & -A' \end{array}\right] h\right) \tag{20}$$

The result for $\|\check{N}_v\|$ is then as follows.

Lemma 4.1: Let $\check{N}_v(z)$ be as defined in Lemma 3.1. Then,

$$\|\check{N}_v\|_{H^2} = \sqrt{\frac{1}{h}\,\mathrm{tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} \Sigma^{-1}\Delta \begin{bmatrix} -Q_0 \\ I \end{bmatrix}\right)}.$$

Proof: Since $\|\check{N}_v\|_{H^2}$ is the achievable performance of the sampled-data H^2 filtering problem (see Section II), the formula follows from [16, Lemma 5.5]. □

Lemma 4.2: Let $\acute{V}(z)$ be as in (14). Then,

$$\|\acute{V}\|_{H^2} = \sqrt{\frac{1}{h}\,\mathrm{tr}\left(P_0 \begin{bmatrix} I & Q_0 \end{bmatrix} \Sigma^{-1}\Delta \begin{bmatrix} -Q_0 \\ I \end{bmatrix}\right)}.$$

Proof: Omitted because of space limitations. □

V. Fixed-Lag Smoothing Formulae

The solution in this case requires the construction of the orthogonal projection transfer function $\acute{V}_+$ from $\acute{V}$ as defined by (4). This is an FIR system, the impulse response of which has support in $[h,(l+1)h]$ (so that the impulse response of $\acute{V}_+^{\sim}$ has support in $[-lh,0]$). It can be verified that this projection is of the form $\acute{V}_+ = \acute{V} - z^{-l}\acute{V}_l$, where

$$\acute{V}_l(z) = z^{-1}\mathfrak{J}_h^*\left(\begin{array}{c|c|c} A & \boxed{zI \rightleftharpoons -(I+LC_y)} & QC_v' \\ \hline \tilde{C}_y\Phi^l & & 0 \end{array}\right) \tag{21}$$

is the truncated "tail" and $\Phi \doteq e^{Ah}(I+LC_y) = e^{Ah}\bar{A}_L e^{-Ah}$ is Schur. Hence,

$$\acute{V}_+^{\sim}(z) = z\left(\begin{array}{c|c|c} -A' & \boxed{z(I+LC_y)' \rightleftharpoons -I} & (I-z^l(\Phi')^l)\tilde{C}_y' \\ \hline -C_vQ & & 0 \end{array}\right)\mathfrak{J}_h \tag{22}$$

is the required projection.

A. Optimal estimator

The optimal estimator in the fixed-lag case is

$$\grave{K}(z) = \grave{K}_{\text{FLS}}(z) \doteq \acute{V}_+^{\sim}(z)\bar{M}_y(z) - \grave{M}_v(z).$$

Our aim is to convert this lifted transfer function back to the time domain (peeling-off procedure).

To this end, define the following two discrete-time systems:

$$\begin{aligned} \bar{V}_{\text{FIR}}(z) &= z\mathfrak{J}_0^*\left(\begin{array}{c|c|c} -A' & \boxed{z(I+LC_y)' \rightleftharpoons -I} & (I-z^l(\Phi')^l)\tilde{C}_y'J \\ \hline I & & 0 \end{array}\right)\mathfrak{J}_h \\ &= \left(\sum_{i=1}^{l} z^i(\bar{A}_L')^{i-1}\right)e^{A'h}C_y'(C_yQ_hC_y')^{-1}, \end{aligned}$$

which belongs to z^lH^∞, and

$$\bar{M}_x(z) = \tfrac{1}{z}\mathfrak{J}_h^*\left(\begin{array}{c|c|c} A & \boxed{zI \rightleftharpoons -(I+LC_y)} & -L \\ \hline I & & 0 \end{array}\right)\mathfrak{J}_0 \in H^\infty,$$

which, in fact, is the discrete-time Kalman filter for the sampled state vector of $G(s)$, so that $\bar{M}_y = J(I - C_y\bar{M}_x)$ generates a scaled estimation error, $J(\bar{y}_k - C_y\hat{x}(kh))$. It can now be shown that

$$\grave{K}_{\text{FLS}}(z) = \grave{H}_h \begin{bmatrix} \bar{M}_x(z) \\ \bar{V}_{\text{FIR}}(z)(I - C_y\bar{M}_x(z)) \end{bmatrix},$$

where $\grave{H}_h : \mathbb{R} \mapsto L^2[0,h]$ is a static gain in the lifted domain corresponding to a generalized hold with the following hold function:

$$\phi_{\text{FIS}}(t) = \begin{bmatrix} C_v & 0 \end{bmatrix}\Sigma(t)\begin{bmatrix} I & -Q_0 \\ 0 & I \end{bmatrix}. \tag{23}$$

Thus, the optimal fixed-lag smoother is a cascade of a discrete-time system comprised of a Kalman filter and an FIR part and a generalized hold with the hold function (23). This cascade can be described as the following system, generating the continuous-time estimated signal $v_e(t)$ (the details are omitted because of space limitations):

$$\begin{cases} \hat{x}_{k+1} = e^{Ah}\hat{x}_k - L\bar{\epsilon}_k \\ v_e(kh+\tau) = C_ve^{A\tau}\hat{x}_k + C_v\Sigma(\tau)\left[\begin{smallmatrix} -Q_0 \\ I \end{smallmatrix}\right] \\ \qquad\qquad \times \displaystyle\sum_{i=1}^{l}(\bar{A}_L')^{i-1}e^{A'h}C_y'(C_yQ_hC_y')^{-1}\bar{\epsilon}_{k+i} \end{cases} \tag{24}$$

where $\bar{\epsilon}_k = \bar{y}_k - C_ye^{Ah}\hat{x}_k$.

Remark 5.1: It is worth emphasizing that the causality constraint arises in our treatment as part of the optimization problem. This is a clear advantage over the conventional approach in digital signal processing, where causality constraints are imposed *after* the fixed-interval solution is calculated by truncating the impulse response of the latter. The performance of the resulting solution depends then on the decay properties of the fixed-interval solution. In this respect, it is of interest to compare the optimal fixed-lag smoother above with its fixed-interval counterpart in Section IV. It can be shown that the former solution is the sum of the truncated fixed-interval estimator and the IIR system

$$z^{l+1}\left(\begin{array}{c|c|c} A & \boxed{zI \rightleftharpoons -(I+LC_y)} & L \\ \hline C_vQP_l & & 0 \end{array}\right)\mathfrak{J}_0 \in z^lH^\infty,$$

where $P_l(t)$ is the solution of the Lyapunov equation (15a) with the initial condition $(\bar{A}_L^l)'P_0$. This system is thus the correction term with respect to the truncated non-causal solution.

Remark 5.2: One of the motivations of the sampled-data smoothing problem is the problem of reconstructing a continuous-time signal from its sampled measurements [8]. This problem corresponds to $C_v = C_y$. It can be shown that in this case the impulse response of the optimal estimator (interpolator) in (24) is a continuous function for every $l > 0$ (this is not true when $l = 0$). Moreover, the impulse response is zero at almost all sampling points, namely, at $t = kh$ for $k \neq 0$ (it is the identity at $k = 0$). This means that $v_e(kh) = \bar{y}_k$, $\forall k \in \mathbb{Z}$, i.e., the reconstruction is *consistent*.

B. Optimal performance

Since $\acute{V}_+$ and $z^{-l-1}\acute{V}_l$ are orthogonal in H^2, the optimal performance level is

$$\gamma_{\text{opt}} = \sqrt{\|\check{N}_v\|_{H^2}^2 - \|\acute{V}\|_{H^2}^2 + \|\acute{V}_l\|_{H^2}^2}.$$

The first two terms in the right-hand side above are already calculated in Lemmas 4.1 and 4.2. The third term is given by the following result.

Lemma 5.1: Let $\acute{V}_l(z)$ be as in (21). Then

$$\|\acute{V}_l\|_{H^2} = \sqrt{\frac{1}{h}\,\mathrm{tr}\left((\bar{A}_L^l)'P_0\bar{A}_L^l \begin{bmatrix} I & Q_0 \end{bmatrix}\Sigma^{-1}\Delta\begin{bmatrix} -Q_0 \\ I \end{bmatrix}\right)},$$

which vanishes as $l \to \infty$.

Proof: Follows the proof of Lemma 4.2 and the fact that the observability Gramian of $\acute{V}_l$ is $(\bar{A}_L^l)'P_0\bar{A}_L^l$. $\square$

In other words, the quantity

$$\frac{1}{h}\,\mathrm{tr}\left(-\big(P_0-(\bar{A}_L^l)'P_0\bar{A}_L^l\big)\begin{bmatrix} I & Q_0\end{bmatrix}\Sigma^{-1}\Delta\begin{bmatrix}-Q_0\\ I\end{bmatrix}\right)$$

determines the performance improvement offered by the smoothing.

VI. Concluding Remarks

In this paper the sampled-data H^2 smoothing problem has been studied. Complete solutions to both the fixed-interval and fixed-lag versions of the problem have been derived. In both cases the solution has the form of the cascade of a finite-dimensional discrete estimator and a generalized hold. While the discrete parts of the optimal sampled-data smoothers are quite similar to their filtering counterparts, the optimal hold functions are more complicated than those appearing in sampled-data filtering.

Appendix A
Systems with Two-Point Boundary Conditions

Systems with two-point boundary conditions (STPBCs) utilized throughout this paper are systems operating over the interval $[0,h]$ and driven by the following dynamics [4], [9]:

$$\begin{cases}\dot{x}(t)=Ax(t)+B(t)u(t), & \Omega x(0)+\Upsilon x(h)=0,\\ y(t)=C(t)x(t),\end{cases}$$

where the square matrices Ω and Υ shape the boundary conditions of the state vector x and the matrix functions $B(t)$ and $C(t)$ are assumed to be continuous in $[0,h]$. The boundary conditions are said to be well-posed if $\det(\Omega+\Upsilon e^{Ah})\neq 0$. If this condition holds, the mapping $y=\breve{O}u$ is well defined $\forall u\in L^2[0,h]$ with

$$\begin{aligned} y(t) = {}& -C(t)\int_t^h e^{A(t-\theta)}B(\theta)u(\theta)d\theta \\ &+ C(t)e^{At}(\Omega+\Upsilon e^{Ah})^{-1}\Omega\int_0^h e^{-A\theta}B(\theta)u(\theta)d\theta \end{aligned} \tag{25}$$

STPBC's are denoted by the following compact block notation:

$$\left(\begin{array}{c|c|c} A & \Omega\rightleftharpoons\Upsilon & B\\ \hline C & & 0\end{array}\right),$$

where the time dependence of $B(t)$ and $C(t)$ is omitted for brevity. Much like in the case of systems operating over an infinite horizon, similarity transformation does not change STPBCs, i.e.,

$$\left(\begin{array}{c|c|c} A & \Omega\rightleftharpoons\Upsilon & B\\ \hline C & & 0\end{array}\right)=\left(\begin{array}{c|c|c} TAT^{-1}+\dot{T}T^{-1} & S\Omega T(0)^{-1}\rightleftharpoons S\Upsilon T(h)^{-1} & TB\\ \hline CT^{-1} & & 0\end{array}\right)$$

for any non-singular $T(t)$ and S.

To make the STPBC formalism applicable to the representation of sampled-data systems in the lifted domain, the following two operators are also required:

- *The impulse operator* $\mathcal{I}_\theta$, which transforms a vector $\eta\in\mathbb{R}^n$ into a modulated δ-impulse as follows:

$$\big(\mathcal{I}_\theta\eta\big)(t)=\delta(t-\theta)\eta.$$

- *The sampling operator* $\mathcal{I}_\theta^*$, which transforms a continuous function $\zeta\in\mathcal{K}_h$ into a vector from $\mathbb{R}^n$:

$$\mathcal{I}_\theta^*\zeta=\zeta(\theta).$$

The manipulations over STPBC can be performed in the state space, much like the manipulations over standard finite-dimensional state-space systems [4]. Moreover, as shown in [14], the sampling and impulse operators fit well into the STPBC formalism. Especially important for the developments in this paper is the following proposition:

Proposition A.1: Let $\det(\Omega+\Upsilon e^{Ah})\neq 0$. Then the systems

$$\begin{cases}\dot{x}_1=Ax_1+B_u u+B_\eta\mathcal{I}_h\bar{\eta}, & \Omega x_1(0)+\Upsilon x_1(h)=0\\ y_1=Cx_1\end{cases}$$

and

$$\begin{cases}\dot{x}_2=Ax_2+B_u u, & \Omega x_2(0)+\Upsilon\big(x_2(h)+B_\eta\bar{\eta}\big)=0\\ y_2=Cx_2\end{cases}$$

are equivalent as mappings from $L^2[0,h]\times\mathbb{R}$ to $L^2[0,h]$.

Proof: Similar to the proof of [12, Proposition A.2]. $\square$

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