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State-space μ-analysis for an experimental drive-by-wire vehicle

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Abstract— **This paper considers the application of the skewed structured singular value to the robust stability of systems subject to strictly real parametric uncertainty. Three state-space formulations that counteract the discontinuous nature of this problem are detailed. It is shown that the calculation of the supremum of the structured singular value over a frequency range using these formulations transforms into a single skewed structured singular value calculation. Like the structured singular value, the calculation of the exact value of the skewed structured singular value is a NP-hard problem, therefore alternative, less computationally demanding algorithms to determine upper and lower bounds are necessary. Two algorithms that determine upper and lower bounds on the skewed structured single value are presented. These algorithms are critically assessed by performing a robust stability analysis on a safety-critical experimental drive-by-wire vehicle.**

I. INTRODUCTION

For many physical systems, it is appropriate to consider the effect of parameter based uncertainty on system stability and performance. The structured singular value, μ, provides a rigorous means of analyzing the robustness of such systems [1]. Although μ analysis provides a general framework for robust analysis, in practice skew μ problems commonly occur where not all the uncertain parameters are allowed to vary freely. One example of skew μ is a stability analysis where frequency is considered as an uncertain variable. By reformulation of the problem incorporating frequency as a perturbation parameter, the gridded μ problem becomes a skew μ problem where no frequencies are missed in the search. Although these skew μ problems can be reformulated as iterative μ problems, direct methods for obtaining bounds on skew μ are desirable for computational reasons. Recent work on the development of direct methods to determine both upper and lower bounds on skew μ for suitably mixed uncertainty is detailed in [2], [3].

The focus in this work is on the special case where the parametric uncertainty is constrained to be real. It is known that this so-called "real μ" problem is a discontinuous function of the problem data [4]. The literature suggests that a "frequency interval" type solution using Linear Matrix Inequalities (LMIs) can address this type of problem [5]. While this type of LMI formulation generally provides a valid upper bound on μ, their practical implementation is limited. It is fair to say that the computational burden becomes impracticable for all but the most straightforward of problems. As an alternative approach, three state-space formulations are considered here that address the discontinuity issue associated with real μ. It is proposed that the calculation of the supremum of μ over a frequency range becomes a single skew μ computation. Due to the NP-hard nature of this problem, bounds on skew μ are sought. In order to obtain an upper bound on skew μ, the generalized eigenvalue formulation of [2] is implemented. Determining a lower bound on skew μ for strictly real parametric uncertainty is more difficult. It should be noted that the lower bound skew μ algorithm of [3] will not provide a solution to this problem due to convergence difficulties. In this work, a new skew μ optimization-based lower bound algorithm is presented. Coupled with the state-space formulations, promising results that provide useful worst-case information about the problem are returned. To demonstrate, a robust stability analysis for a safety-critical drive-by-wire application is performed where the parameter uncertainty is strictly real and repeated.

This paper is outlined as follows, section II introduces the nomenclature used and details the formal definitions of both the structured singular value and the skew structured singular value. Section III details the three state-space formulations that counteract the discontinuity issue with the real μ problem. Section IV presents an generalized eigenvalue formulation and an optimization-based algorithm used to determine upper and lower bounds on skew μ respectively. In section V, a robust stability analysis is performed on the drive-by-wire vehicle using standard μ-Tools techniques and the so-called "state-space skew μ" methods presented in this paper.

II. ROBUSTNESS ANALYSIS TECHNIQUES

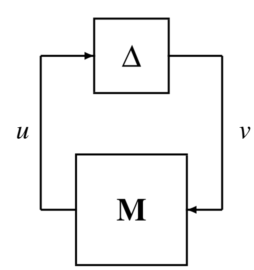

Fig. 1. Canonical μ analysis framework.

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The μ approach for systems analysis is based on the observation that problems involving interconnections of linear time invariant (LTI) systems with uncertain parameters and unmodelled dynamics can be reduced to considering the constant matrix feedback interconnection in Fig. 1. The uncertainty block Δ is structured where three non-negative integers m_r, m_c and m_C specify the number of uncertainty blocks of each type. The block structure $\mathcal{K}(m_r, m_c, m_C)$ is an m-tuple of positive integers.

$$\mathcal{K} = (k_1, \ldots, k_{m_r}, k_{m_r+1}, \ldots, k_{m_r+m_c}, k_{m_r+m_c+1}, \ldots, k_m)$$

with $m = m_r + m_c + m_C$. This m-tuple specifies the dimensions of the perturbation blocks, which determines the set of allowable perturbations, namely define:

$$X_{\mathcal{K}} = \Big\{ \Delta = \text{block diag}\big(\delta_1^r I_{k_1}, \ldots, \delta_{m_r}^r I_{k_{m_r}}, \ldots \\ \delta_1^c I_{k_{m_r+1}}, \ldots, \delta_{m_c}^c I_{k_{m_r+m_c}}, \Delta_1^C, \ldots, \Delta_{m_C}^C\big) \Big\}$$

with:

$$\delta_i^r \in \mathbf{R},\ \delta_i^c \in \mathbf{C},\ \Delta_i^C \in \mathbf{C}^{k_{m_r+m_c+i} \times k_{m_r+m_c+i}}$$

Note that $X_{\mathcal{K}} \subset \mathbf{C}^{n \times n}$ (where $n = \sum_{i=1}^m k_i$) and that this block structure allows for repeated real scalars ($\delta_i^r I$), repeated complex scalars ($\delta_i^c I$), and full complex blocks (Δ_i^C). Noting this block structure, the following definition, taken from [1] is introduced.

Definition 1: The structured singular value, $\mu_{\mathcal{K}}(M)$, of a matrix $M \in \mathbf{C}^{n \times n}$ with respect to a block structure $\mathcal{K}(m_r, m_c, m_C)$ is defined as:

$$\mu_{\mathcal{K}}(M) = \frac{1}{\min\limits_{\Delta \in X_{\mathcal{K}}} \{\overline{\sigma}(\Delta) : \det(I_n - \Delta M) = 0\}} \tag{1}$$

with $\mu_{\mathcal{K}}(M) = 0$ if no $\Delta \in X_{\mathcal{K}}$ solves $\det(I_n - \Delta M) = 0$. Linear Fractional Transformations (LFTs) are used to reorganize a perturbed problem with uncertainty into the feedback interconnection in Fig. 1. In particular, if $M \in \mathbf{C}^{n \times n}$ is partitioned as:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{2}$$

with $M_{11} \in \mathbf{C}^{n_1 \times n_1}$, $M_{11} \in \mathbf{C}^{n_1 \times n_1}$ and $n = n_1 + n_2$, then an upper LFT will be described as:

$$\Delta * M = M_{22} + M_{21}\Delta(I_{n_1} - M_{11}\Delta)^{-1} M_{12} \tag{3}$$

If two block structures are defined as $\mathcal{X}_{\mathcal{K}_1} \subset \mathbf{C}^{n_1 \times n_1}$, $\mathcal{X}_{\mathcal{K}_2} \subset \mathbf{C}^{n_2 \times n_2}$, then the augmented block structure $\mathcal{X}_{\hat{\mathcal{K}}} \in \mathbf{C}^{n \times n}$ is defined as:

$$\mathcal{X}_{\hat{\mathcal{K}}} = \{\Delta = \text{block diag}(\Delta_f, \Delta_v) : \\ \Delta_f \in \mathbf{B}\mathcal{X}_{\mathcal{K}_1},\ \Delta_v \in \mathcal{X}_{\mathcal{K}_2}\}$$

where $\mathbf{B}\mathcal{X}_{\mathcal{K}_1} = \big\{\Delta_f \in \mathcal{X}_{\mathcal{K}_1} : \overline{\sigma}(\Delta_f) \le 1\big\}$.

The skewed structured singular value is the smallest structured singular value of a *subset* of perturbations that destabilizes the system M with the remainder of the perturbations contained within a fixed range. Formally stating this:

Definition 2: The skewed structure singular value, $\mu^s_{\hat{\mathcal{K}}}(M)$, of a matrix $M \in \mathbf{C}^{n \times n}$ with respect to a block structure $\hat{\mathcal{K}}(m_{r_f}, m_{c_f}, m_{C_f}, m_{r_v}, m_{c_v}, m_{C_v})$ is defined as:

$$\mu^s_{\hat{\mathcal{K}}}(M) = \frac{1}{\min\limits_{\Delta \in \mathcal{X}_{\hat{\mathcal{K}}}} \{\overline{\sigma}(\Delta_v) : \det(I_n - \Delta M) = 0\}} \tag{4}$$

with $\mu^s_{\hat{\mathcal{K}}}(M) = 0$ if no $\Delta \in \mathcal{X}_{\hat{\mathcal{K}}}$ solves $\det(I_n - \Delta M) = 0$.

III. STATE-SPACE APPROACHES USING SKEW μ

In general, robustness analysis problems correspond to a question of checking the value for:

$$\mu_{\mathcal{K}}(M(s)) \tag{5}$$

over the closed right-half-plane (where $M(s)$ is a stable system). This approach can be computationally intensive and an appropriate frequency range and the fineness of the grid must be decided. Even with such an approach, there still remains the possibility of missing important points especially as real μ may be discontinuous. Instead three grid-free state-space formulations are presented that counteract all the issues associated with a grid search.

A. State-space μ (Bilinear Transform)

The development of the first two tests is based on the fact that a transfer function can be expressed as a LFT of a constant matrix on the frequency variable. Given a transfer function $M(s)$ its differential equation representation is considered and expanded using the usual state-space formula:

$$M(s) = C(sI_p - A)^{-1}B + D = \frac{1}{s}I_p * \hat{M} \tag{6}$$

where $\hat{M}$ is the constant matrix:

$$\hat{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{7}$$

and p is the dimension of the state-space. The idea is to replace $M(s)$ with this expression and include $\frac{1}{s}I_p$ as one of the uncertainties. Instead of testing $\frac{1}{s}I_p$ over the right-half-plane, a better solution is to test within a unit circle. This is achieved by employing a bilinear transform, where the transformation:

$$T = \begin{bmatrix} I_p & \sqrt{2}I_p \\ \sqrt{2}I_p & I_p \end{bmatrix}$$

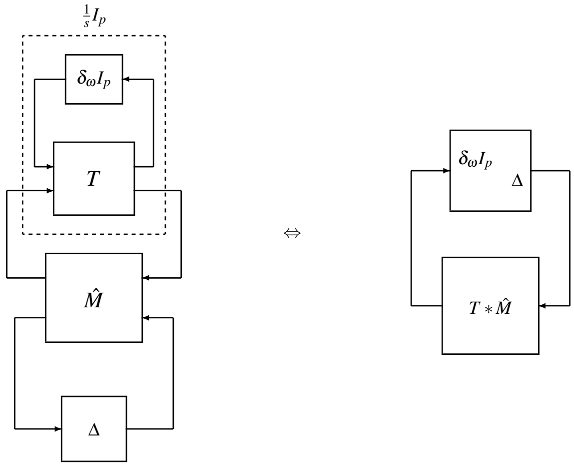
is used to generate $\frac{1}{s}I_p$ in the right-half-plane from $\delta_\omega I_p * T$ where $\delta_\omega I_p$ lies within the unit disk. The test now follows by applying the main loop theorem [6] and evaluating $T * \hat{M}$ (Fig. 2).

Theorem 1 ([7]): (Bilinear transform)
Suppose that $M(s)$ has all of its poles in the open left-half-plane (i.e. nominal stability) and let $\beta > 0$. Let a minimal state-space representation for $M(s)$ be given as:

$$M(s) = C(sI_p - A)^{-1}B + D \tag{8}$$

Given $X_{\mathcal{K}}$ compatible with $M(s)$, define a new uncertainty structure $\mathcal{X}_{\hat{\mathcal{K}}}$ as:

$$\mathcal{X}_{\hat{\mathcal{K}}} = \big\{\text{block diag}(\delta_\omega I_p, \Delta) : \delta_\omega \in \mathbf{C},\ \Delta \in X_{\mathcal{K}}\big\} \tag{9}$$



(a) Transformed LFT (b) Augmented LFT

Fig. 2. State-space μ test.

Then for all $\Delta \in \mathscr{M}(X_{\mathscr{K}})$ with $\|\Delta\|_\infty \leqslant \beta$, the perturbed closed-loop system in Fig. 1 is (well-posed and) uniformly stable if and only if:

$$\mu_{\hat{\mathscr{K}}}\left(T * \hat{M}\right) < 1$$

where:

$$T * \hat{M} = \begin{bmatrix} \mathscr{A}_1^{-1}(I_p + A) & \sqrt{\frac{2}{\beta}}\mathscr{A}_1^{-1}B \\ \sqrt{\frac{2}{\beta}}C\mathscr{A}_1^{-1} & \frac{1}{\beta}(C\mathscr{A}_1^{-1}B + D) \end{bmatrix} \tag{10}$$

and:

$$\mathscr{A}_1^{-1} = (I_p - A)^{-1}$$

B. State-space μ (Real Interval Transform)

The benefit of the first state-space approach is that the uncertainty block is both real and complex so reasonable solutions for the bounds are expected. However it may be desirable to exploit the fact that the frequency uncertainty parameter is real-valued to obtain possibly better results. The transformation:

$$T = \begin{bmatrix} -I_p & \sqrt{2}I_p \\ j\sqrt{2}I_p & -jI_p \end{bmatrix}$$

is used to generate $\frac{1}{s}I_p$ in the right-half-plane from $\delta_\omega I_p * T$ where $\delta_\omega I_p$ lies within the interval $[-1, 1]$ along the real axis. The test now follows by once again applying the main loop theorem and evaluating $T * \hat{M}$.

Theorem 2: (Real interval transform)
Suppose that $M(s)$ has all of its poles in the open left-half-plane and let $\beta > 0$. Given a minimal state-space representation of $M(s)$ defined in theorem 1 and given $X_{\mathscr{K}}$ compatible with $M(s)$, define a new uncertainty structure $\mathscr{X}_{\hat{\mathscr{K}}}$ as:

$$\mathscr{X}_{\hat{\mathscr{K}}} = \left\{\text{block diag}(\delta_\omega I_p, \Delta) : \delta_\omega \in \mathbf{R}, \Delta \in X_{\mathscr{K}}\right\} \tag{11}$$

Then for all $\Delta \in \mathscr{M}(X_{\mathscr{K}})$ with $\|\Delta\|_\infty \leqslant \beta$, the perturbed closed-loop system in Fig. 2 is uniformly stable if and only if:

$$\mu_{\hat{\mathscr{K}}}\left(T * \hat{M}\right) < 1$$

where:

$$T * \hat{M} = \begin{bmatrix} \mathscr{A}_2^{-1}(-I_p + jA) & \sqrt{\frac{2}{\beta}}\mathscr{A}_2^{-1}B \\ j\sqrt{\frac{2}{\beta}}C\mathscr{A}_2^{-1} & -\frac{1}{\beta}(jC\mathscr{A}_2^{-1}B - D) \end{bmatrix} \tag{12}$$

and:

$$\mathscr{A}_2^{-1} = (I_p + jA)^{-1}$$

C. State-space μ (Bounded Frequency Test)

The basis for the third approach differs from the first two but remains a frequency independent approach with frequency represented as an uncertain (real) parameter. Unlike the previous two approaches, instead of checking for frequency points over the right-half-plane corresponding to $\omega \in [0, \infty]$, this approach allows a frequency interval to be selected *a priori* where $\omega \in [\underline{\omega}, \overline{\omega}]$. Using the transformation:

$$T = \begin{bmatrix} 0 & I_p \\ \frac{1}{2}I_p & \frac{1}{2}I_p \end{bmatrix}$$

and introducing the parameters:

$$\begin{aligned} \omega_0 &= (\overline{\omega} + \underline{\omega})/2 \\ \alpha_\omega &= (\overline{\omega} - \underline{\omega})/2 \end{aligned}$$

The following theorem extends results first presented in [8] where the test again follows by applying the main loop theorem and evaluating $T * \hat{M}$.

Theorem 3: (Bounded frequency test)
Suppose that $M(s)$ has all of its poles in the open left-half-plane and let $\beta > 0$. Given a minimal state-space representation of $M(s)$ and the uncertainty structure $\mathscr{X}_{\hat{\mathscr{K}}}$ defined in theorem 1 and theorem 2 respectively, then for all $\Delta \in \mathscr{M}(X_{\mathscr{K}})$ with $\|\Delta\|_\infty \leqslant \beta$, the perturbed closed-loop system is uniformly stable if and only if:

$$\mu_{\hat{\mathscr{K}}}(T * \hat{M}) < 1$$

where:

$$T * \hat{M} = \begin{bmatrix} j\alpha_\omega \mathscr{A}_3^{-1} & \sqrt{\frac{1}{\beta}}\mathscr{A}_3^{-1}B \\ -j\sqrt{\frac{1}{\beta}}\alpha_\omega C\mathscr{A}_3^{-1} & -\frac{1}{\beta}\left(C\mathscr{A}_3^{-1}B - D\right) \end{bmatrix} \tag{13}$$

with:

$$\mathscr{A}_3^{-1} = (A - j\omega_0 I_p)^{-1}$$

It is with abuse of notation that $T * \hat{M}$ is used for all three cases, the reasons for which will become apparent. A full description of the third approach is presented in [9]. As they exist, all three state-space formulations from theorems 1, 2 and 3 provide a constant matrix μ test for the general robust stability problem with a yes/no answer. This can be improved upon by reformulating as a skew μ problem. This is quantified in the following proposition.

Proposition 1: (state-space skew μ)
Formulation of frequency as an uncertainty parameter is a skew μ problem. Since the frequency parameter is skewed (fixed in range), (10), (12) and (13) can be recast as a skew μ formulation.

$$\mu^s_{\hat{\mathscr{K}}}(T * \hat{M}) < 1 \tag{14}$$

with $\delta_\omega I = \Delta_f$ and $\Delta = \Delta_v$.

It is now possible to obtain the worst-case perturbation from each test and the value of δ_ω containing the worst-case frequency information. Furthermore, this information may be determined or unwrapped using the following expressions for each of the three approaches:

$$
\begin{aligned}
s &= \frac{1-\delta_\omega}{1+\delta_\omega} \\
s &= -j\frac{1+\delta_\omega}{1-\delta_\omega} \\
s &= j(\omega_0 + \alpha_\omega \delta_\omega)
\end{aligned}
$$

IV. BOUNDS ON SKEW μ

It is possible to use the transformations to determine the supremum of μ through a single skew μ analysis. It is therefore necessary to consider algorithms to determine bounds on skew μ that adequately address strictly real parametric uncertainty. An upper bound (ub) on skew μ may be obtained using the generalized eigenvalue formulation of [2]. This algorithm is quantified in the following theorem.

Theorem 4: (Mixed skew μ upper bound)
For $M \in \mathscr{C}^{n\times n}$ and any compatible block structure $\hat{\mathscr{K}}$, a skew μ upper bound, ν_u, can be calculated from:

$$\left(\mathscr{A} + \frac{1}{\nu_u}\mathscr{B}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad (15)$$

where:

$$\mathscr{A} = \left[\begin{array}{cc|c} M_{11}M_{11}^* - I_f & M_{11}M_{21}^* & 0 \\ M_{21}M_{11}^* & M_{21}M_{21}^* & I \\ \hline 0 & I & 0 \end{array}\right] \qquad (16)$$

and:

$$\mathscr{B} = \left[\begin{array}{cc|c} M_{12}M_{12}^* & M_{12}M_{22}^* & 0 \\ M_{22}M_{12}^* & M_{22}M_{22}^* & 0 \\ \hline 0 & 0 & I \end{array}\right] \qquad (17)$$

In its current format, the formulation in (15) does not detail or consider standard D and G scaling matrices associated with calculation of the standard μ-Tools upper bound, reflecting the structure of the associated perturbation block. It is detailed in [10] how this scaling is considered and implemented. An optimization-based approach may be used to determine a lower bound (lb) on skew μ where the parametric uncertainty may be real or complex valued.

Theorem 5: (Optimized skew μ lower bound)
Let $\emptyset_d \in \mathbf{R}$ where $0 \leq \emptyset_d \leq 10^{-8}$. For $M \in \mathscr{C}^{n\times n}$ and any compatible block structure $\hat{\mathscr{K}}(m_{r_f}, m_{c_f}, 0, m_{r_v}, m_{c_v}, 0)$, a lower bound on $\mu^s_{\hat{\mathscr{K}}}(M)$ can be determined from:

$$\mu^s_{\hat{\mathscr{K}}}(M) = \frac{1}{\min\limits_{\Delta \in X_{\hat{\mathscr{K}}}} \{\|\Delta_{v_l}\| : |\det(I_n - \Delta M)| \leq \emptyset_d\}} \qquad (18)$$

where the solution is (the inverse of) a constrained minimization problem.

In order to obtain quality solutions, it is necessary to relax the nonlinear equality constraint in (4) to an inequality with the introduction of $\emptyset_d$ to counteract the non-convex nature of the problem search. Note that "$\emptyset_d$" is the digital implementation of zero and is generally of magnitude 10^{-8} or less. The uncertainty block structure is implemented where one optimization variable represents a real-valued uncertainty parameter and two optimization variables are used to represent the real and imaginary parts of a complex uncertainty parameter. This lower bound algorithm has been developed in Matlab using the Optimization Toolbox [11]. The authors have found that improved and reliable lower bound solutions are returned with reasonable associated computation times over a wide variety of problems where the perturbation sets are in excess of 20+ parameters using this algorithm [10]. Combined with the state-space formulations, the computational burden in obtaining a lower bound is further reduced. Indeed, since no form of regularization is required, valid problem perturbations are always provided [12]. Both upper and lower bound algorithms are available in beta form and will be implemented as part of the next release of the freely downloadable "MuExplorer" software [13]. The performance of each approach is now illustrated on a safety-critical experimental drive-by-wire example.

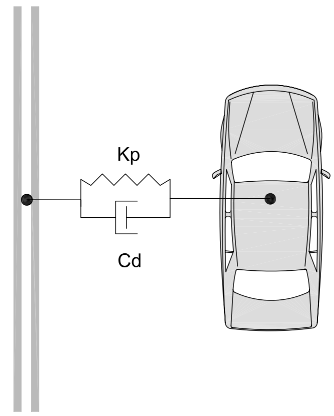


Fig. 3. Lanekeeping analogy.

V. DRIVE-BY-WIRE APPLICATION

In this section an experimental drive-by-wire application is considered. Drive-by-wire systems have many advantages over conventional automotive steering, braking, and throttle mechanisms. By electronically actuating the throttle, brakes, and steering, driving inputs can be fully integrated with vehicle safety systems. The absence of conventional mechanical associations between the driver and the vehicle raises some new questions regarding the nature of interactions that occur between driver, vehicle, and environment in a drive-by-wire vehicle. The question of stability and performance requirements are paramount with the introduction of such systems in order to lead to certifiably safe robust control law design.

For a constant longitudinal speed, V, the linearized model is represented by four states where the state vector is given

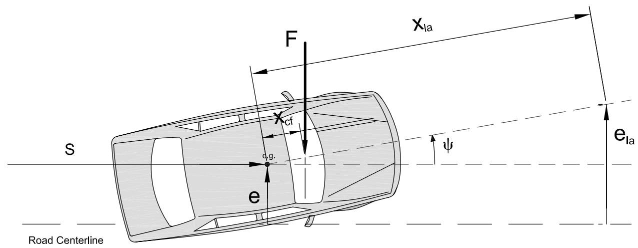


Fig. 4. Vehicle with lookahead.

as:

$$x = \begin{bmatrix} e & \dot{e} & \psi & \dot{\psi} \end{bmatrix}^T \quad (19)$$

where e is the lateral offset from the lane centre and ψ is the heading angle. The goal of the robustness analysis is to study the vehicle's driver assistance response to small deviations from the lane centre (Fig. 3). The linear state space matrices and controller are given with:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{C_f+C_r}{mV} & \frac{C_f+C_r-kx_{la}}{m} & \frac{-aC_f+bC_r}{mV} \\ 0 & 0 & 0 & 1 \\ -\frac{kx_{cf}}{I_z} & \frac{-aC_f+bC_r}{I_zV} & \frac{aC_f-bC_r-kx_{la}x_{cf}}{I_z} & \frac{-(a^2C_f+b^2C_r)}{I_zV} \end{bmatrix}$$

and:

$$B = \begin{bmatrix} 0 & \frac{1}{m} & 0 & \frac{x_{cf}}{I_z} \end{bmatrix}^T$$

The state feedback controller K is given by:

$$K = \begin{bmatrix} k & 0 & kx_{la} & 0 \end{bmatrix}$$

For a constant speed of V = 30m/s and the application point of the environmental 'virtual' force, F, is applied 0.5m in front of the neutral steer point of the vehicle (Fig. 4). It is shown in [14] that the nominal system remains stable with all the poles remaining strictly in the left-half-plane when the lookahead distance x_{la} is varied from 10m to 50m. This technique may work for the nominal case but does not guarantee robust stability against parameter uncertainties. Instead a better approach is to incorporate uncertainty into the system (Table I) and use the analysis tools outlined in section II to assess the robust stability of the system.

A. Frequency Sweeps

For each frequency sweep outlined below, the lookahead is included as an uncertain parameter with the structured perturbation block now given as:

$$\Delta = \mathrm{diag}(\delta^r_m, \delta^r_{I_z}, \delta^r_{C_f}I_4, \delta^r_{C_r}I_4, \delta^r_a I_4, \delta^r_b I_4, \delta^r_{x_{cf}}I_3, \delta^r_V I_3, \delta^r_{x_{la}}I_2)$$

Running the `mu` command from the Matlab μ-Tools toolbox for a frequency sweep of 300 grid points, the upper bound plot included in Fig. 5 is produced. This upper bound indicates that this system is not robustly stable for this level of uncertainty. The lower bound algorithm failed to return any solution for all grid points. Very little information

TABLE I

PARAMETER VALUES FOR VEHICLE.

Parameter	Units	Nominal Value	Variation
m	kg	1640	± 10%
I_z	N/m^2	3500	± 10%
C_f	N/rad	100000	± 5%
C_r	N/rad	160000	± 5%
a	m	1.3	± 10%
b	m	1.5	± 10%
x_{cf}	m	0.0772	[-0.1314, 0.2858]
V	m/s	30	± 10%
x_{la}	m	30	[10, 50]
k	N/m	5000	± 0%

TABLE II

SUMMARY OF RESULTS.

Description	μ	Max μ Value	Frequency
Mu-Tools UB 'U'	μ_U	1.2251	5.6560
State-space skew μ 1 UB	μ^s_{U1}	1.6899	-
State-space skew μ 2 UB	μ^s_{U2}	1.6532	-
State-space skew μ 3 UB	μ^s_{U3}	1.7470	$[4,5]$
State-space skew μ 1 LB	μ^s_{opt1}	0.9898	5.4721
State-space skew μ 2 LB	μ^s_{opt2}	0.9929	4.5270
State-space skew μ 3 LB	μ^s_{opt3}	0.9965	4.5364

is obtained from this analysis using the μ-Tools toolbox, considering that the purpose of the lower bound to be twofold - to obtain a tight gap on the bounds, and to return a valid destabilizing problem perturbation Δ_d. This information is essential in determining the vector set of parameters that gives the worst-case response of the system from a time domain perspective.

B. Frequency Independent Methods

This subsection considers the frequency independent state-space formulations (10), (12) and (13) outlined in section III. The augmented perturbation block includes an uncertain repeated frequency variable. Upper bound results on μ were obtained using the generalized eigenvalue formulation outlined in theorem 4. The results are given in Table II. The peak upper bound value of μ is returned at 1.7470 from the frequency-bounded state-space approach for the frequency interval $[4,5]$ rad/s. The clear advantage of this approach is the ability to select or subdivide frequency intervals *a priori*. The first and second state-space formulations also return good results in this instance. The recursive "finite frequency" method of [15] may be employed to find a relatively narrow frequency interval where these peaks occur. The LMI method of [5] was also used to determine an upper bound on real μ used since it too allows frequency intervals to be selected *a priori*. Implemented using the `mubnd` function from the LMI Toolbox, this algorithm failed to return any solution for any of the 14 frequency intervals selected due to the computational effort required. Other similar mixed approaches based on the result in [5], specifically algorithms outlined in [16], also failed to return a solution for any frequency interval for this case study.

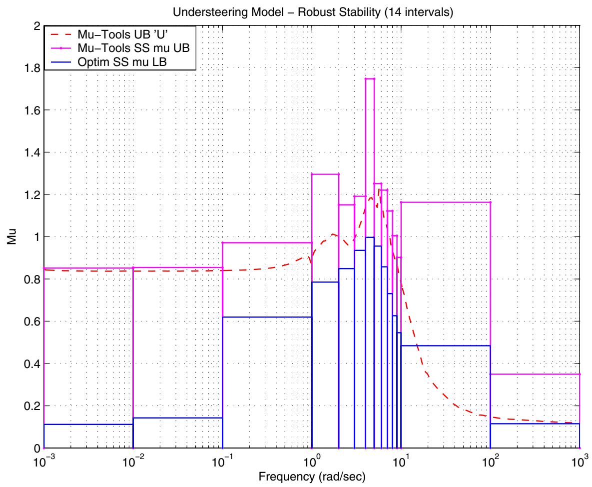


Fig. 5. Robust stability μ-Tools bounds.

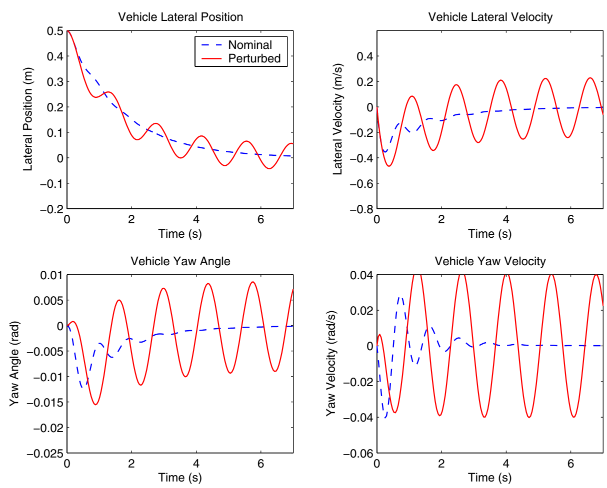


Fig. 6. Nominal and worst case time responses for $x_{la} = 50$m.

Lower bounds on skew μ are obtained using the optimization-based skew μ algorithm. The results returned from each of the three state-space approaches are again included in Table II. The maximum value of μ found is 0.9965 for a critical frequency of $\omega = 4.5364$ rad/s using the frequency bounded state-space formulation. This critical frequency complements the result from the upper bound approach lying within the frequency interval of $[4,5]$ rad/s. A further observation stresses the importance of tight bounds on μ, since the peak upper bound on μ found is greater than unity while the peak lower bound remains marginally less than unity. If considered independently, these are contradicting results. To interpret, the system is not robustly stable for the levels of uncertainty specified but no destabilizing problem perturbation that lies within these allocated uncertainty ranges has been found. For illustrative purposes, plots of both the upper and lower bounds for each frequency interval are shown in Fig. 5.

C. Worst-case Time Responses

The accuracy of $\det(I - M_{11}\Delta)$ for each interval for the optimization algorithm was found to be of the magnitude 10^{-9} or less, indicating a destabilizing problem perturbation was found in each case. Fig. 6 shows the time-domain responses for each of the system states (outputs) to an initial offset of 0.5m from the road centreline. The lookahead distance, x_{la}, is 50m for both the nominal and perturbed time responses. Clearly the optimization-based lower bound returns a candidate worst-case problem perturbation corresponding to very oscillatory responses in Fig. 6. This perturbation set corresponds to the peak lower bound μ value of 0.9965.

VI. Conclusions

In this paper, algorithms that determine reasonable bounds on skew μ for the case of strictly real and repeated parametric uncertainty are presented. These algorithms are used with three different state-space formulations to counteract the discontinuity issue associated with "real μ". The methodologies outlined have been tested on an experimental drive-by-wire vehicle, where the physical parameter uncertainty is real-valued. The frequency bounded state-space approach where the frequency interval can be selected *a priori* returned the best results for this analysis.

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