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On the stabilisation of a class of SISO switched linear systems

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***Abstract*— We consider stabilisation of SISO switched linear systems of arbitrary order. Our approach is motivated by applications in which the major design objective is to achieve similar behaviour of the closed-loop system in each mode. We exploit the algebraic properties of a class of matrices to develop design guidelines to achieve this goal for a class of SISO switched linear systems. It is shown that closed-loop stability and transient-free switching can be achieved simultaneously.**

I. INTRODUCTION

In this paper we consider stabilisation of classes of linear time-varying single-input single-output systems of the form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + b(t)u(t),\\ y(t) &= c^T(t)x(t)\end{aligned}$$

where $A(\cdot), b(\cdot), c^T(\cdot)$ are piecewise constant functions such that $\{A(t), b(t), c^T(t)\} \in \{\{A_1, b_1, c_1^T\}, \ldots, \{A_N, b_N, c_N^T\}\}$. We refer to the LTI system defined by the triple $\{A_k, b_k, c_k^T\}$ as mode k of the switched system, $k \in \mathcal{I} = \{1, \ldots, N\}$. At any time-instant the dynamics of the system are described by exactly one of those LTI systems.

Our approach in this paper is to find a set of stabilising controllers for a given switched linear process with N modes. The mode-switches of the process may occur arbitrarily but are detected by the switching unit. We apply a controller structure as depicted in Figure 1 where the controller for each mode is realised as a single LTI system. At any switching instant of the plant, the appropriate controller is deployed in the closed loop by switching the plant input to the respective controller output. To aid analysis we make the assumption that there is no time-delay between the switching of the plant and switching of the controller output. Such a controller structure is referred to as local-state controller [1].

Systems of this generic structure appear naturally in many applications; see [2], [3], [4], [5], [1]. Despite considerable progress on the analysis of switched systems (see [6], [7] and references therein) many problems in this general area remain unsolved. In particular, given the frequency with which one finds switched linear control systems in practice, one of the most pressing needs is for the development of analytic tools for the design of such systems.

Our contribution in this paper is to develop tools for the design of a class of such systems.

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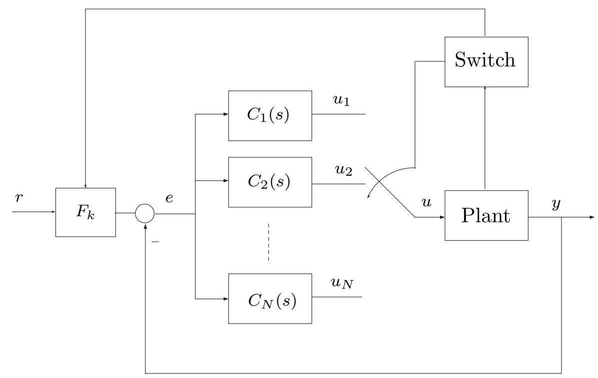

Fig. 1. Structure of the considered switched linear system

II. PROBLEM STATEMENT

The plant dynamics are given by the linear time-varying differential equation of the form

$$y^{(n_p)} = \sum_{l=0}^{n_p-1} q_l(t) y^{(l)} + p_0(t)u \tag{1}$$

where $y^{(n_p)}$ denotes the n_p'th derivative of $y(t)$ and $p_0(t)$, $q_l(t)$ are piecewise constant functions taking on values in the finite sets $p_0(t) \in \{p_{01}, \ldots, p_{0N}\}$, and $q_l(t) \in \{q_{l1}, \ldots, q_{lN}\}\ \forall\, l = 0, \ldots, n_p - 1$. The discontinuities occur simultaneously such that $p_0(t) = p_{0k}$ whenever $q_l(t) = q_{lk}$ for all $l = 0, \ldots, n_p - 1$ where k denotes the plant-mode $k \in \mathcal{I} = \{1, \ldots, N\}$.

Thus at any time instant the plant dynamics in Figure 1 correspond to exactly one of the N linear systems

$$\dot{x}(t) = A_k x(t) + b_k u(t) \tag{2a}$$
$$y(t) = c^T x(t) \tag{2b}$$

where

$$A_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -q_{0k} & -q_{1k} & \cdots & \cdots & -q_{n-1k} \end{pmatrix}$$

$$c^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \quad b_k = \begin{pmatrix} 0 & \cdots & 0 & p_{0k} \end{pmatrix}^T.$$

With each mode $k \in \mathcal{I}$ we associate the proper transfer function

$$P_k(s) = c^T(sI - A_k)^{-1} b_k.$$

We shall assume that the mode-switches of the plant are immediately detectable such that the switching instances can be assumed to be known for the controller. Given these assumptions, our objective is to design a controller such that the closed-loop system

- is asymptotically stable for arbitrary switching signals,
- has the poles $\Lambda_t \subset \mathbb{C}_-$, specified independently of the plant mode $k \in \mathcal{I}$,
- and has little or no transients induced by the switching action.

III. Preliminary discussion: Basic ideas

In order to achieve the design objectives we associate an individual controller for each plant mode $k \in \mathcal{I}$. We choose a controller architecture where each controller is realised as an LTI system as depicted in Figure 1.

The dynamics of the individual controllers are given by

$$\dot{x}_k(t) = K_k x_k(t) + l_k e(t) \tag{3a}$$
$$u_k(t) = m_k^T x_k(t) + j_k e(t) \tag{3b}$$

where $x_k(t) \in \mathbb{R}^{n_c}$ is the state-vector of the controller associated with mode $k \in \mathcal{I}$; the input $e(t) \in \mathbb{R}$ is shared by all controllers and each controller has an individual control signal $u_k(t) \in \mathbb{R}$. For the realisation of the controllers we choose the control canonical form as above with $K_k \in \mathbb{R}^{n_c \times n_c}$, $l_k, m_k^T \in \mathbb{R}^{n_c}$ and $j_k \in \mathbb{R}$. The respective transfer functions are given by

$$C_k(s) = m_k^T (sI - K_k)^{-1} l_k.$$

As design-law for the controllers we choose a set of stable target-poles Λ_t and design the controllers using pole-placement such that the closed-loop system in each mode has the specified target-poles Λ_t.

Assumption 3.1 (Pole-placement): For each plant-mode $k \in \mathcal{I}$ the controller $C_k(s)$ is designed such that the poles of the closed loop transfer function

$$\frac{C_k(s) P_k(s)}{1 + C_k(s) P_k(s)}$$

lie in the open left half-plane and are constant for all $k \in \mathcal{I}$. We denote those target poles by $\Lambda_t = \{\lambda_1, \ldots, \lambda_{n_p + n_c}\}$ accounting for multiplicity. The resulting controllers have poles in the open left half-plain.

The state x of the switched closed-loop system consists of the plant-states x_p and the controller-states x_k

$$x = \begin{pmatrix} x_p^T & x_1^T & \ldots & x_N^T \end{pmatrix}^T$$

where $x \in \mathbb{R}^n$, $n = n_p + N n_c$. For the switched closed-loop system we then obtain

$$\dot{x}(t) = H(t) x(t), \tag{4}$$

where $H(\cdot)$ is an arbitrary piecewise constant function $H : \mathbb{R} \to \mathcal{H} = \{H_1, \ldots, H_N\} \subset \mathbb{R}^{n \times n}$. The constituent system matrices in each mode $k \in \mathcal{I}$ are given by

$$H_k = \begin{pmatrix} A_k - b_k j_k c_k & b_k m_1^T \delta_{k1} & \cdots & b_k m_N^T \delta_{kN} \\ -l_1 c^T & K_1 & & 0 \\ \vdots & & \ddots & \\ -l_N c^T & 0 & & K_N \end{pmatrix} \tag{5}$$

where δ_{kj} is the Kronecker symbol.

Before we present the main results we note some preliminary observations. Given the plants (2) and controllers (3) in control-canonical form, all closed-loop system matrices H_k are identical except for the n_p-th row, Furthermore as all but one of the sets of Kronecker symbols are equal to 0, we have that $\sigma(H_k) \supset \sigma(K_l)$ for $l \neq k$. By design (Assumption 3.1) the remaining eigenvalues are given by Λ_t for all $k \in \mathcal{I}$. Thus the spectrum of H_k is given by

$$\sigma(H_k) = \Lambda_t \cup \bigcup_{l \neq k} \sigma(K_l),$$

accounting for multiplicities. Therefore the matrices H_k have pairwise $n_p + (N-1)n_c$ common eigenvalues.

A useful consequence of this approach is that the subspace corresponding to the target poles do not depend on k given some mild conditions. This fact shall be useful in the following discussion and we state it formally as the following lemma.

Lemma 3.1: Let $\lambda \in \Lambda_t$ be a simple eigenvalue of each H_k, then there exists a vector $v \neq 0$ such that for all $k \in \mathcal{I}$

$$H_k v = \lambda v. \tag{6}$$

Proof: As $\lambda \in \sigma(H_k)$ we have that the rows $\tilde{h}_{jk}$ of $\lambda I - H_k$ are linearly dependent for each k. On the other hand, all the rows, but the n_p'th are independent of k. By inspection the set of $n-1$ rows obtained by omitting the n_p'th row is linearly independent, since by assumption λ is not an eigenvalue of one of the controllers K_k, $k \in \mathcal{I}$. Thus for each k there are constants γ_{jk} such that

$$\tilde{h}_{nk} = \sum_{j \neq n} \gamma_{jk} \tilde{h}_{jk}. \tag{7}$$

Now by definition an eigenvector v of H_1 corresponding to the eigenvalue λ satisfies $\tilde{h}_{j1} v = 0, j = 1, \ldots, n$. This implies that $\tilde{h}_{jk} v = 0, j = 1, \ldots, n, j \neq n_p$ for each $k \in \mathcal{I}$. This, however, implies by (7) that also $\tilde{h}_{nk} v = 0$, so that we have $(\lambda I - H_k) v = 0$. ■

Hence, all closed-loop system matrices H_k have $n_p + n_c$ eigenvectors in common. This fact can be exploited to derive simple conditions for stability as we shall discuss in the following section.

IV. Stability

Assume that we are given N matrices of the form (5) and that the poles of the individual systems have been placed so that Lemma 3.1 is applicable.

Let the columns of $V_t \in \mathbb{C}^{n\times(n_p+n_c)}$ form a basis of the common subspace of all matrices $H_k \in \mathcal{H}$ and consider the matrix

$$T \;:=\; \begin{pmatrix} V_t & e_{n_p+n_c+1} & \cdots & e_n \end{pmatrix}. \tag{8}$$

Note that T is invertible as the vectors $e_{(n_p+n_c+1)}, \ldots, e_n$ form a basis of an invariant subspace of H_1, which does not intersect $\operatorname{span} V_t$ as $\Lambda_t \cap \sigma(K_k) = \emptyset \;\forall\, k \in \mathcal{I}$.

Applying the similarity transformation T we obtain

$$T^{-1}H_1T \;=\; \operatorname{diag}(D_t, K_2, \ldots, K_N),$$

$$T^{-1}H_2T \;=\; \operatorname{diag}(D_t, K_2, \ldots, K_N) + T^{-1}e_n\tilde{h}_2^T T,$$

up to

$$T^{-1}H_NT \;=\; \operatorname{diag}(D_t, K_2, \ldots, K_N) + T^{-1}e_n\tilde{h}_N^T T,$$

where $\sigma(D_t) = \Lambda_t$ and $\tilde{h}_m := h_{mn_p} - h_{1n_p}$ denotes the differences between the n_p'th rows of H_m and H_1. As implied by our construction the differences between the matrices are all multiples of the same columns. Furthermore inspection of the n_p'th rows of the matrices H_k shows that $\tilde{h}_m$ can only have nonzero entries in its first n_p+n_c positions and in the positions $n_p+(m-1)n_c+1, \ldots, n_p+mn_c$. Hence, in the lower block corresponding to the controllers only the controller K_m is perturbed. So that for $m = 2, \ldots, N$ the matrices after similarity transformation are of the form

$$T^{-1}H_mT = \begin{pmatrix} D_t & 0 & \ldots & U_{1m} & 0 \\ 0 & K_2 & 0 & U_{2m} & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ & & & K_m + U_{mm} & \\ & & & & 0 \\ & & & U_{Nm} & K_N \end{pmatrix}.$$

where $U_m = \begin{pmatrix} U_{1m}^T & U_{2m}^T & \ldots & U_{Nm}^T \end{pmatrix}^T \in \mathbb{R}^{n\times n_c}$ denotes the perturbation term of the m'th system. Since $rank\{H_j - H_k\} = 1$ for all $j \neq k$ and $j,k \in \mathcal{I}$ the perturbation term U_m has rank 1. We denote

$$R_1 := \operatorname{diag}(K_2, \ldots, K_N),$$

and for $m = 2, \ldots, N$ the lower right $(N-1)n_c \times (N-1)n_c$-block of $T^{-1}H_mT$ by

$$R_m := \begin{pmatrix} K_2 & 0 & U_{2m} & 0 \\ & \ddots & \vdots & \vdots \\ & & K_m + U_{mm} & \\ & & \ldots & \ddots \quad 0 \\ 0 & & U_{Nm} & K_N \end{pmatrix}.$$

It follows that the closed-loop system is exponentially stable if and only if the switched system formed by the matrices R_m, $m = 1, \ldots, N$ is exponentially stable.

Theorem 4.1: The following statements are equivalent:

1) The switched linear system (4) with $H(t) \in \mathcal{H}$ is exponentially stable.
2) The switched linear system $\dot{x} = R(t)x$ with $R : \mathbb{R} \to \{R_1, R_2, \ldots, R_N\}$ is exponentially stable.

Proof: This follows from the preceding discussion. ■

The above theorem reduces the stability analysis of the switched system of dimension $n_p + Nn_c$ to the stability of a system of dimension $(N-1)n_c$. In the following we consider two special cases and show that Theorem 4.1 can be used to obtain very elegant stability conditions.

A. N first order controllers

We begin our analysis with the case where the controllers C_k are of first order. Thus for Assumption 3.1 to hold, the plant have to be of order strictly less than three.

We now employ Theorem 3.1 in [8]. Essentially, the theorem establishes asymptotic stability of the class of switched systems (4) with the following properties:

- every matrix in $\mathcal{H}$ is Hurwitz and diagonalisable;
- the eigenvectors of any matrix in $\mathcal{H}$ are real;
- every pair of matrices in $\mathcal{H}$ share at least $n-1$ linearly independent common eigenvectors.

Let the target poles Λ_t be distinct and real. With the assumption that the pole-placement is feasible for all modes $k \in \mathcal{I}$, the resulting closed-loop system matrices H_k have pairwise $n-1$ real distinct eigenvalues. By Lemma 3.1 the matrices H_k, $k \in \mathcal{I}$, have $n_p + 1$ common eigenvectors. Moreover, since each pair of closed-loop system matrices H_k share $N-2$ of the remaining inactive controllers they have pairwise $n-1$ common eigenvectors.

Thus the requirements for Theorem 3.1 in [8] are met and the closed-loop system is exponentially stable for arbitrary switching sequences. In other words, the switched system (4) is stable for arbitrary switching if we choose arbitrary real negative target-poles Λ_t such that the design-law in Assumption 3.1 is satisfied by first-order controllers [9].

Theorem 4.1 can be used to extend this result for systems with non-real target poles Λ_t. Choosing a modal-basis for V_t in (8) we obtain a transformation matrix T with real entries. It follows that the system matrices R_k of the reduced system are in $\mathbb{R}^{N-1\times N-1}$. Further, $\sigma(R_k) = \cup_{l\neq k}\sigma(K_l)$. Since the controllers are of first order, it follows that the matrices R_k also satisfy the requirement of Theorem 3.1 in [8].

Corollary 4.1: The switched system (4) with system matrices (5) where Assumption 3.1 is satisfied using N stable first-order controllers is asymptotically stable.

B. Two subsystems of arbitrary order

Consider now the special case where $N = 2$ and the controllers are of arbitrary order n_c. Due to the pole-placement requirement (Assumption 3.1) we obtain for the respective spectra

$$\begin{aligned} \sigma(H_1) &= \Lambda_t \cup \sigma(K_2) \\ \sigma(H_2) &= \Lambda_t \cup \sigma(K_1). \end{aligned}$$

Applying the similarity transformation T of (8) to our two system matrices we obtain

$$T^{-1}H_1T \;=\; \begin{pmatrix} D_t & 0 \\ 0 & K_2 \end{pmatrix} \tag{9a}$$

$$T^{-1}H_2T \;=\; \begin{pmatrix} D_t & 0 \\ 0 & K_2 \end{pmatrix} + \begin{pmatrix} 0 & U_1 \\ 0 & U_2 \end{pmatrix} \tag{9b}$$

where $\begin{pmatrix} U_1^T & U_2^T \end{pmatrix}^T \in \mathbb{R}^{2n_c \times n_c}$ and $\sigma(D_t) = \Lambda_t$. Note that $\operatorname{rank}\{U_2\} = 1$ as we have $\operatorname{rank}\{H_1 - H_2\} = 1$. Further it follows from the spectrum of H_2 that $\sigma(K_2+U_2) = \sigma(K_1)$.

The following theorem reduces the stability problem of the switched system defined by $\{H_1, H_2\}$ to a stability problem only involving the controllers.

Theorem 4.2: Consider the matrices H_1, H_2 in (5) and let Assumption 3.1 be satisfied such that $\sigma(H_k) = \Lambda_t \cup \sigma(K_l)$ for $k, l = 1, 2, \ k \neq l$. Assume furthermore that $\Lambda_t \cap \sigma(K_k) = \emptyset$, $k = 1, 2$. Then the following statements are equivalent:

1) The switched system given by the set of matrices $\{H_1, H_2\}$ is asymptotically stable for arbitrary switching signals;
2) The switched system given by the set of matrices $\{K_2, K_2 + U_2\}$ is asymptotically stable for arbitrary switching signals;
3) The switched system given by the set of matrices $\{K_1, K_2\}$ is asymptotically stable for arbitrary switching signals.

Proof: The equivalence of (i) and (ii) can be seen as follows. Firstly, the matrices in (5) and (9) are obtained from one another by similarity. Thus the set $\{H_1, H_2\}$ defines an asymptotically stable switched system if and only if $\{T_1^{-1}H_1T_1,\ T_1^{-1}H_2T_1\}$ does. On the other hand $\Lambda_t \subset \mathbb{C}_-$, so that the exponential stability of $\{T_1^{-1}H_1T_1, T_1^{-1}H_2T_1\}$ is equivalent to that of the lower diagonal block $\{K_2, K_2+U_2\}$.

The equivalence (ii) $\Leftrightarrow$ (iii) follows if we find a similarity transformation that transforms K_2 and K_2+U_2 into K_2 and K_1 respectively. Note first, that since $\operatorname{rank}\{H_2 - H_1\} = 1$, the perturbation $(U_1^T, U_2^T)^T$ is also of rank one. Further, the block $K_2 + U_2$ is similar to K_1 since the eigenvalues in Λ_t are generated by the closed loop system of A_2 and K_2.

Consider now the matrices K_2^T and $K_2^T + U_2^T$. If we can find a vector x such that

$$x_m := (K_2^T)^m x = (K_2^T+U_2^T)^m x\,,\ m = 0, \ldots, n_c-1 \quad (10)$$

and so that the sequence $x_m, m = 0, \ldots, n_c - 1$ is linearly independent, then the similarity transformation

$$S = \begin{pmatrix} x_0 & \ldots & x_{m-1} \end{pmatrix}$$

yields

$$S^{-1}K_2^TS = K_2^T\,,\ \text{and}\ S^{-1}(K_2^T + U_2^T)S = K_1^T\,,$$

as the assumption (10) guarantees that both matrices are brought simultaneously in transposed companion form (sometimes also known as second companion form) and because the companion form of K_2+U_2 is K_1 by similarity. By taking transposes of the previous equations we have found the desired transformation that concludes the proof in case that (10) holds. Now by induction the conditions in (10) require that

$$U_2^T(K_2^T)^m x = 0\,,\ \text{for}\ m = 0, \ldots, n_c - 2\,.$$

As $\operatorname{rank}\{U_2^T\} = 1$ the kernel of $U_2^T(K_2^T)^m$ has dimension $n-1$ for $m = 0, \ldots, n_c-2$ and so by dimensionality reasons the intersection of these kernels satisfies

$$V \ := \bigcap_{m=0}^{n_c-2} \ker U_2^T(K_2^T)^m\,, \qquad \dim V \ \geq \ 1$$

Choose an $x \in V$, $x \neq 0$. If the set of vectors $\{x_m, m = 0, \ldots, n_c - 1\}$ is linearly independent, then (10) holds and we are done. If this is not the case this means that the lower-dimensional K_2^T-invariant subspace defined by

$$W := \operatorname{span}\{x_m \mid m = 0, \ldots, n_c - 1\}$$

is by definition contained in the kernel of U_2^T. Hence on this lower dimensional subspace K_2^T is not perturbed by U_2^T. We may then repeat the argument on the restriction of K_2^T to a complementary invariant subspace and repeat the argument until (10) holds on one of this lower dimensional complementary subspaces. The procedure terminates for reasons of dimensionality and the assertion follows. ■

Comment: Theorem 4.2 reduces the complexity of the stability analysis of the switched system considerably. To guarantee asymptotic stability of the switched system (4) with $N = 2$ we only need to consider the asymptotic stability of the switched system given by

$$\dot{x} = K(t)x, \quad K(t) \in \{K_1, \ldots, K_N\} \subset \mathbb{R}^{n_c \times n_c} \quad (11)$$

for arbitrary switching signals. Thus, the stability problem of the switched system (4) of order $n_p + 2n_c$ is reduced to the stability problem of a switched system of order n_c.

Comment: It should be emphasised that the proof of Theorem 4.2 relies on the fact that the controller-matrices are in companion form. At this point it is not clear what role the specific realisation chosen for the controllers plays for the result. However, it is obvious that the equivalence $(ii) \Leftrightarrow (iii)$ can only be true when $\operatorname{rank}\{K_1 - K_2\} = \operatorname{rank}\{U_2\} = 1$.

Comment: The equivalence of the asymptotic stability of the system (4) and (11) is less obvious than intuition might suggest. As we shall see in the next section, the result does not generalise for systems with more than two subsystems. In this context it is worth noting that the switched system (11) is not explicitly part of the closed-loop system (4). For the switched system (11) the controller dynamics K_k act on the same state-space; however the controllers in the closed-loop system (4) are realised as individual LTI systems and therefore do not share the states.

C. The case of $N = 3$

The above findings suggest that the switched closed loop system (4) is stable if and only if the switched system (11) consisting of the controllers form a stable system. Unfortunately that is not true as the following example shows.

Example 4.1: Consider the switched plant (2) with $N = 3$, where

$$A_1 = \begin{pmatrix} 0 & 1 \\ -11.84 & -2.4 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 1 \\ -34.28 & -11.6 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 \\ -29.7 & -11 \end{pmatrix},$$

and $b_k = \begin{pmatrix}0 & 1\end{pmatrix}^T$, $c_k^T = \begin{pmatrix}1 & 0\end{pmatrix}$ for $k = 1, 2, 3$, and let the requested target-poles be given by $\Lambda_t = \{-1\pm 3i, -1.8, -8\}$.

It can be verified that the pole-placement requirement is satisfied by the following set of controllers (3) with

$$K_1 = \begin{pmatrix} 0 & 1 \\ -9.6 & -9.4 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 1 \\ -7.4 & -0.2 \end{pmatrix},$$
$$K_3 = \begin{pmatrix} 0 & 1 \\ -5.5 & -0.8 \end{pmatrix}$$

and $m_1^T = \begin{pmatrix}30.34 & -7.536\end{pmatrix}$, $m_2^T = \begin{pmatrix}-109.7 & 34.1\end{pmatrix}$, $m_3^T = \begin{pmatrix}-19.35 & 42.54\end{pmatrix}$, and $l_k = \begin{pmatrix}0 & 1\end{pmatrix}^T$, $j_k = 0$ for $k = 1, 2, 3$.

It can be numerically verified that $V(x) = x^T P x$ with

$$P = \begin{pmatrix} 3.0745 & 0.0671 \\ 0.0671 & 0.4356 \end{pmatrix}$$

is a common quadratic Lyapunov function for the switched system (11) with $K(t) \in \{K_1, K_2, K_3\}$. Hence, the switched system (11) consisting of the controllers is asymptotically stable for arbitrary switching.

However we can find a switching sequence for which the closed-loop switched system (4) is unstable. The spectral radius

$$\varrho\left(e^{0.22H_3} e^{0.32H_2} e^{0.72H_1}\right) = 1.024$$

where H_k are the closed-loop system matrices (5). Hence, there exists a switching sequence for which the closed loop system is unstable [10].

The above example shows that Theorem 4.2 cannot be generalised for systems with N subsystems. Thus we have to resort to Theorem 4.1 for the analysis of switched systems (5) with $N \geq 3$.

V. Controllers with integrators

The stability results derived in the previous sections require that the constituent closed-loop systems $\dot{x} = H_k x$, $k \in \mathcal{I}$ are stable LTI systems. Since the eigenvalues of the non-active controllers are part of the spectrum of H_k, we cannot apply the stability results when the controllers have integrators. In this section we show that this problem can be resolved by choosing a variation of the local-state controller-architecture.

For this step we shall assume that the controllers have the same number of integrators for each mode $k \in \mathcal{I}$. Then we can choose a controller-architecture such that these integrators are shared by the controllers and therefore are always active in the closed-loop. For this purpose we choose a controller architecture with a joint integrator in front of the controller bank as shown in Figure 2 such that the local controllers $C_k(s)$ have no pure integrator.

Choosing the state-vector of the closed-loop system as $x = \begin{pmatrix}x_p^T & v^T & x_k^T & \cdots & x_N^T\end{pmatrix}^T$ yields the system matrices

$$H_k = \begin{pmatrix} A_k & b_k j_k & b_1 m_1^T \delta_{k1} & \cdots & b_N m_N^T \delta_{kN} \\ -c^T & 0 & 0 & \cdots & 0 \\ 0 & l_1 & K_1 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & \\ 0 & l_N & & 0 & K_N \end{pmatrix}$$

for all $k \in \mathcal{I}$.

Positioning the joint integrator *in front* of the controller bank preserves the property of the system matrices $\operatorname{rank}\{H_k - H_l\} = 1$ for $k \neq l \ \forall\, k, l \in \mathcal{I}$, since the matrices H_k again only differ in the n_p'th row. Since the integrator is constantly active in the closed loop the eigenvalues of H_k $\forall\, k \in \mathcal{I}$ lie in the open left half-plane when Assumption 3.1 is satisfied. Hence, the results of the previous section are applicable to controllers with integrators.

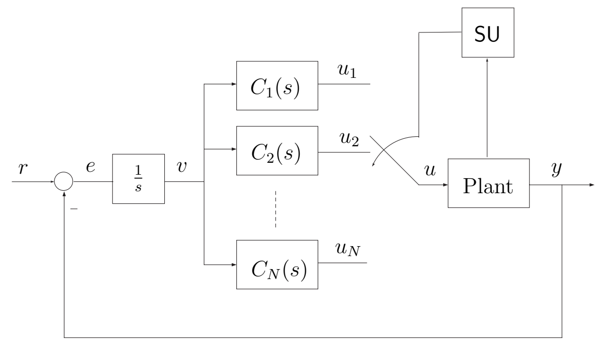

Fig. 2. Controller architecture with joint integrator.

VI. Transient-free switching

It is well known that switching can induce undesirable transients in control systems. In this section we show that these transients can be avoided using the local-state controller architecture described above.

Loosely speaking, transients occur due to the controller state-transition after the switch. While controller $C_k(s)$ is active in the loop the control-output of $C_l(s)$ evolves according to

$$U_l(s) = \frac{C_l(s)}{1 + C_k(s)P_k(s)} R(s).$$

Assume now that switching only occurs when the closed-loop system (for practical purposes) reached steady-state, i.e. $\dot{x} = 0$. Thus the steady-state $\hat{x}_k$ for each mode is given by $\hat{x}_k = -H_k^{-1} b_k$. We can eliminate transients at steady-state switching if $\hat{x}_k$ is constant for every mode $k \in \mathcal{I}$.

Since we consider systems in control-canonical form, this requirement is equivalent to demanding that the controller-outputs at steady-state is constant for each mode. Thus,

$$\lim_{s\to 0} \frac{C_l(s)}{1 + C_k(s)P_k(s)} = \gamma_l \quad \forall\, k \in \mathcal{I}. \tag{12}$$

Condition (12) is formulated in the frequency-domain such that it can be easily incorporated into the control-design procedure (Assumption 3.1). In fact, by adding one extra degree of freedom we can achieve both, stability of the switched system and transient-free switching when the system is in steady-state (see [**?**] for details). The following example illustrates this.

Example 6.1: Given the plant (2) with transfer functions

$$P_1 = \frac{2}{s+9}, \quad P_2 = \frac{1}{s+8}$$

and let the target poles be $\Lambda_t = \{-0.5 \pm i, -10\}$. We shall require a controller with integrator. Using the standard pole-placing method yields for the two controllers

$$C_1(s) = \frac{-12.75s + 12.5}{s(s+8)}, \quad C_2(s) = \frac{-3.38s + 6.25}{s(s+2)}.$$

Using Theorem 4.2 we can show that the resulting switched system is exponentially stable, since there exists a common quadratic Lyapunov function for (11). The steady-state in mode 1 is $\hat{x}_1 = \begin{pmatrix}1 & 1.4 & 0.7 & 0.18\end{pmatrix}^T$ and $\hat{x}_2 = \begin{pmatrix}1 & 1.92 & 0.96 & 0.24\end{pmatrix}^T$ in mode 2.

Figure 3a shows the step-response of the switched closed-loop system when the plant mode switches ever 20 time-units. Even though the reference value is reached in every switching interval, we can observe considerable transients at the switching instances.

In order to meet the additional condition (12) for transient-free switching we need a controller with an extra degree of freedom. As additional target-pole we choose $\lambda_t = -20$. We then obtain the controllers

$$C_1 = \frac{2.7s^2 - 6.25s + 125}{s^3 + 22s^2 + 27.78s}, \quad C_2 = \frac{63.9s^2 - 12.5s + 250}{s^3 + 28s^2 + 83.33s}.$$

This switched system is also exponentially stable for arbitrary switching and its steady-state is $\hat{x}_k = \begin{pmatrix}1 & 1 & 0.036 & 0 & 0.012 & 0\end{pmatrix}^T$, $k \in \mathcal{I}$.

The step-response of the closed-loop system shows no transients at the switching instances (Figure 3b).

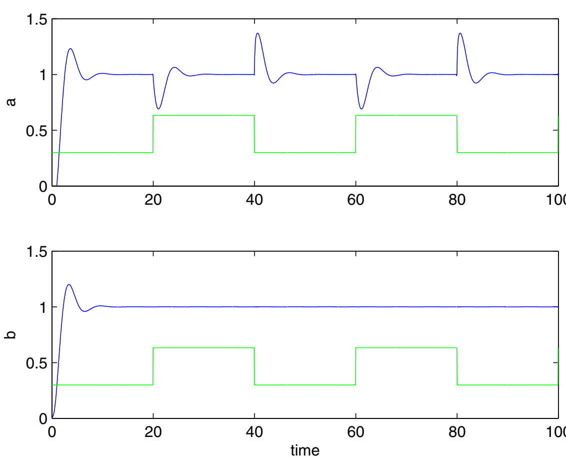

Fig. 3. (a) stable switched system with transients, (b) stable switched system without transients using condition (12).

VII. Conclusions and Discussion

In this paper we considered a typical control problem for switched linear systems. It is shown that the stability analysis can be considerably simplified by using the proposed design-law and the local-state controller architecture. Further, for systems with first-order controllers we have shown that stability for arbitrary switching is always guaranteed. In the case that the switched system has only two modes, the stability of the switched closed-loop system is equivalent to the stability of the switched system defined by the controller-matrices. Thus the stability analysis degenerates from a switched system of order $n_p + 2n_c$ to that of a switched system of order n_c.

Furthermore, we have shown that transients at the switching instances can be avoided when satisfying condition (12). By adding an extra degree of freedom both stability of the closed-loop system and transient-free switching at steady-state can be achieved.

The stability analysis in this paper depends fundamentally on the assumption that the poles of the closed-loop transfer function are invariant while switching. This requires that the respective controller outputs are instantaneously activated whenever the plant-mode changes. From a practical point of view this is an unrealistic assumption. In most applications there will be a certain time-delay between the mode-switch of the plant and the switching of the control signal. The impact of such delays on the stability of the closed-loop system are an important problem and are subject of future research.

An open question is also how the realisations of the transfer functions effect the results in this paper (*c.f.* [12]). Throughout this paper we assume that the individual controllers are realised in control canonical form. While this is a realistic approach a different choice of the realisation might provide better performance or stability properties. Since we can choose the controller realisations independently of each other, it might be possible to find conditions on the realisations that simplify the stability analysis.

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